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Oscillation criteria for third order neutral type advanced difference equation

A. Murugesan[1]* and P. Selvakumar[2]

Abstract

In this paper, we establish the oscillatory criteria for the third-order neutral type difference equation of the form

$$\Delta(a(n)(\Delta^2(x(n)+p(n)x(n-k)))^{\alpha})+q(n)f(x(n-l))=0.$$

We derive new oscillation condition that really take into account the advanced arguments.

Keywords

Oscillation, third order neutral difference equations.

AMS Subject Classification

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[1] *Department of Mathematics, Government Arts College (Autonomous), Salem-636007, Tamil Nadu, India.*

[2] *Department of Mathematics, Christ Institute of Technology (Formely Dr. S.J.S. Paul Memorial College of Engineering and Technology), Puducherry-605502, India.*

***Corresponding author**: [1] amurugesan3@gmail.com; [2]psiselva1606@gmail.com

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Contents

1. Introduction

This article concerns the oscillatory criteria of solutions for the third-order neutral type difference equation

$$\Delta(a(n)(\Delta^2(x(n)+p(n)x(n-k)))^{\alpha})+q(n)f(x(n-l))=0 \tag{1.1}$$

where $n \in \mathbb{N}=\{n_0,n_1,\dots\}$, n_0, k and l are non-negative integer, $\{a(n)\}$ is a positive real sequence with $\Sigma_{n=n_0}^{\infty}\frac{1}{a^{1/\alpha}(n)}=\infty$, $\{p(n)\}$ and $\{q(n)\}$ are positive real sequences, $f:\mathbb{R}\to\mathbb{R}$ is continuous with $uf(u)>0$ for $u\neq 0$ and α is a ratio of odd positive integers.

Let $\theta=\max\{k,l\}$. By an answer of condition (1.1), we mean a real sequence $\{x(n)\}$ characterized for all $n\geq n_0-\theta$, and fulfills condition (1.1) for all $n\geq\mathbb{N}$. A nontrivial arrangement of condition (1.1) is supposed to be oscillatory if the particulars of the sequence $\{x(n)\}$ are neither in the all positive nor in the all negative, what's more, nonoscillatory something else.

We established sufficient conditions for the oscillation and asymptotic behavior of all solutions under the condition

$$\Sigma_{n=n_0}^{\infty}\frac{1}{c^{1/\alpha}(n)}=\infty, \quad 0\leq p(n)\leq p<\infty.$$

For more details on this theory and on its applications, we suggest the reader to refer [1–6].

2. Oscillation Theorems

In this segment, we acquire some sufficient conditions for the oscillation of all solutions of (1.1). We may manage the positive solutions of equation (1.1) since the verification for the negative case. We additionally present a typical show, to be specific, for the sequence $\{f(j)\}$ and any $m\in\mathbb{N}$ we put $\Sigma_{j=m}^{m-1}f(j)=0$ and $\prod_{j=m}^{m-1}f(j)=1$. We start with certain lemmas that will be utilized to demonstrate our main results. In the accompanying, for accommodation we mean

$$z(n)=x(n)+p(n)x(n-k), \text{ and } \beta(n)=\min\{q(n),q(n-k)\}.$$

Lemma 2.1. *[7] Assume that* $\alpha\geq 1$, $x(1),x(2)\in[0,\infty)$. *Then*

$$x^{\alpha}(1)+x^{\alpha}(2)\geq\frac{1}{2^{\alpha-1}}(x(1)+x(2))^{\alpha}.$$

Lemma 2.2. *Assume that* $0<\alpha\leq 1$, $x(1),x(2)\in[0,\infty)$. *Then*

$$x^{\alpha}(1)+x^{\alpha}(2)\geq(x(1)+x(2))^{\alpha}.$$

Proof. Let $x(1) = 0$ or $x(2) = 0$. Assume $x(1) > 0, x(1) > 0$. Define $x^{\alpha}(1) + x^{\alpha}(2) - (x(1) + x(2))^{\alpha}$. Then

$$\frac{df(x(1),x(2))}{dx(2)} = \alpha x^{\alpha-1}(2) - \alpha(x(1)+x(2))^{\alpha-1}$$
$$= \alpha[x^{\alpha-1}(2) - (x(1)+x(2))^{\alpha-1}] \geq 0.$$

which yields $f(x(1),x(2)) \geq 0$. □

Lemma 2.3. *Let $\{x(n)\}$ be a positive solution of equation (1.1), then*

(a) $z(n) > 0$, $\Delta z(n) > 0$, $\Delta^2 z(n) > 0$, $\Delta(a(n)(\Delta^2 z(n))^{\alpha}) \leq 0$;

(b) $z(n) > 0$, $\Delta z(n) < 0$, $\Delta^2 z(n) > 0$, $\Delta(a(n)(\Delta^2 z(n))^{\alpha}) \leq 0$,

for all $n \geq n_0 \in \mathbb{N}$, where n_0 is large.

Lemma 2.4. *Suppose that $\{z(n)\}$ satisfies Lemma 2.3 for all $n \geq N \in \mathbb{N}$. Then*

$$z(n-l) \geq \frac{\tau(n-l,n_1)}{\tau(n,N)}\Delta z(n), \tag{2.1}$$

where $\tau(n,N) = \Sigma_{s=N}^{n-1}\frac{1}{a_s^{1/\alpha}}$ *and* $\tau(n-l,n_1) = \Sigma_{s=n_1}^{n-l-1}\left(\Sigma_{t=N}^{s-1}\frac{1}{a_t^{1/\alpha}}\right)$ *for some $n_1 > N$.*

Proof. Since $\Delta(a(n)(\Delta^2 z(n))^{\alpha}) \leq 0$, we have $a(n)(\Delta^2 z(n))^{\alpha}$ is non-increasing for all $n \geq N$. Then

$$\Delta z(n) \geq \Delta z(n) - \Delta z(N) = \Sigma_{s=N}^{n-1}\frac{(a(s)(\Delta^2 z(s))^{\alpha})^{1/\alpha}}{a^{1/\alpha}(s)}$$
$$\geq a^{1/\alpha}(n)\Delta^2 z(n)\tau(n,N).$$

That is

$$a^{-1/\alpha}(n)\Delta z(n) - \Delta^2 z(n)\tau(n,N) \geq 0,$$

which yields

$$\Delta\Big(\frac{\Delta z(n)}{\tau(n,N)}\Big) \leq 0. \tag{2.2}$$

Since $n - l \leq n$, we have

$$\frac{\Delta z(n-l)}{\Delta z(n)} \geq \frac{\tau(n-l,N)}{\tau(n,N)}, \tag{2.3}$$

from (2.2), we obtain

$$\begin{aligned} z(n) &= z(n_1) + \Sigma_{s=n_1}^{n-1}\Delta z(s) \\ &\geq \Sigma_{s=n_1}^{n-1}\frac{\Delta z(s)}{\tau(s,N)}\tau(s,N) \end{aligned} \tag{2.4}$$

from (2.3) and (2.4) is that

$$\frac{z(n-l)}{\Delta z(n)} = \frac{\Delta z(n-l)}{\Delta z(n)}\frac{z(n-l)}{\Delta z(n-l)} \geq \frac{\tau(n-l,n_1)}{\tau(n,N)}$$

for all $n \geq n_1$. This completes the proof. □

Lemma 2.5. *Assume that $\{z(n)\}$ satisfies Lemma 2.3 for all $n \geq N \in \mathbb{N}$. Then*

$$\Delta z(n) \geq (a^{1/\alpha}(n)\Delta^2 z(n))\tau(n,N),$$
$$z(n) \geq (a^{1/\alpha}(n)\Delta^2 z(n))\eta(n,N),$$

where $\eta(n,N) = \Sigma_{s=N}^{n-1}\frac{(n-1-s)}{a^{1/\alpha}(s)}$.

Proof. Since $\Delta(a(n)(\Delta^2 z(n))^{\alpha}) \leq 0$, then

$$\Delta z(n) \geq \Delta z(n) - \Delta z(N) = \Sigma_{s=N}^{n-1}\frac{(a(s)(\Delta^2 z(n))^{\alpha})^{1/\alpha}}{a^{1/\alpha}(s)}$$
$$\geq (a^{1/\alpha}(n)\Delta^2 z(n))\Sigma_{s=N}^{n-1}\frac{1}{a^{1/\alpha}(s)}.$$

Similarly

$$z(n) \geq (a^{1/\alpha}(n)\Delta^2 z(n))\Sigma_{s=N}^{n-1}\Sigma_{t=N}^{s-1}\frac{1}{a^{1/\alpha}(s)}.$$

Therefore

$$z(n) \geq (a^{1/\alpha}(n)\Delta^2 z(n))\eta(n,N).$$

□

Lemma 2.6. *Let $\alpha > 0$. If $f(n) > 0$ and $\Delta f(n) > 0$ for all $n \geq N \in \mathbb{N}$, then*

$$\Delta f^{\alpha}(n) \geq \alpha f^{\alpha-1}(n)\Delta f(n) \quad \text{if } \alpha \geq 1,$$
$$\Delta f^{\alpha}(n) \geq \alpha f^{\alpha-1}(n)\Delta f(n) \quad \text{if } 0 < \alpha \leq 1$$

for all $n \geq N$.

Proof. we have $n \geq n_0$
$\Delta f^{\alpha}(n) = f^{\alpha}(n+1) + f^{\alpha}(n) = \alpha t^{\alpha-1}\Delta f(n)$ where $f(n) < t < f(n+1)$. □

Next, we state and prove our main results.

Theorem 2.7. *Consider the sequences $\tau(n-l,n_1)$ and $\tau(n,N)$. Let $\alpha \geq 1$ and $l \geq k$. Assume that there exist a positive non-decreasing real sequence $\{\rho(n)\}$ and a nonnegative real sequence $\{\delta(n)\}$ such that*

$$\limsup_{n\to\infty}\Sigma_{s=n_2}^{n-1}\Big[2^{1-\alpha}M\rho(s)\beta(s)\Big(\frac{\tau(s-l,n_1)}{\tau(s,N)}\Big)^{\alpha} - \mu(s)\Big] = \infty \tag{2.5}$$

for a sufficiently large $N \in \mathbb{N}$, and for some $n_2 > n_1 > N$, where

$$\mu(n) = \frac{(\Delta(\rho(n))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha}(n)}(a(n) + p^{\alpha}a(n-k))$$
$$+\Delta(\rho(n)a(n)\delta(n) + p^{\alpha}\rho(n)a(n-k)\delta(n-k).$$

If

$$\limsup_{n\to\infty}\Sigma_{t=n+k}^{n+l}\Big(\Sigma_{s=n}^{t}\Big(\frac{1}{a_{s-k}}\Sigma_{i=s}^{t}\beta(i)\Big)^{1/\alpha}\Big) > \Big(\frac{2^{\alpha-1}(1+p^{\alpha})}{M}\Big)^{1/\alpha} \tag{2.6}$$

for all $n \geq N \in \mathbb{N}$, then equation (1.1) is oscillatory.

Proof. Expect the opposite that equation (1.1) has an in the long run positive solution $\{x(n)\}$, that is, there exists a $n_1 \in \mathbb{N}$ with the end goal that $x(n) > 0$, $x_{n-k} > 0$ furthermore, $x_{n-l} > 0$ for all $n \geq n_1$. From the definition of $z(n)$, we have $z(n) > 0$ for all $n \geq N \in \mathbb{N}(n_1)$, where N is is picked with the goal that two instances of Lemma 2.3 hold for all $n \geq N$. We will show that for each situation we are directed to an inconsistency.
Case (i): By Lemma 2.1. To see,

$$\Delta(a(n)(\Delta^2 z(n))^\alpha) + p^\alpha \Delta(a(n-k)(\Delta^2 z(n-k))^\alpha) + M\frac{\beta(n)}{2^{\alpha-1}}(z(n-l))^\alpha \leq 0, \quad n \geq N. \tag{2.7}$$

Define

$$w(n) = \rho(n)\Big(\frac{a(n)(\Delta^2 z(n))^\alpha}{(\Delta z(n))^\alpha} + a(n)\delta(n)\Big), \quad n \geq N. \tag{2.8}$$

Then $w(n) > 0$ for all $n \geq N$, and from (2.8) and Lemma 2.6, we have

$$\begin{aligned}\Delta w(n) &= \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) + \rho(n)\Delta(a(n)\delta(n)) \\ &\quad + \rho(n)\frac{\Delta(a(n)(\Delta^2 z(n))^\alpha)}{(\Delta z(n))^\alpha} \\ &\quad - \rho(n)\frac{a(n+1)(\Delta^2 z(n+1))^\alpha}{(\Delta z(n+1))^\alpha(\Delta z(n))^\alpha}\Delta((\Delta z(n))^\alpha) \\ &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) \\ &\quad + \rho(n)\Delta(a(n)\delta(n)) + \rho(n)\frac{\Delta(a(n)(\Delta^2 z(n))^\alpha)}{(\Delta z(n))^\alpha} \\ &\quad - \alpha\rho(n)\frac{a(n+1)(\Delta^2 z(n+1))^\alpha}{(\Delta z(n+1))^\alpha}\frac{\Delta^2 z(n)}{\Delta z(n)}.\end{aligned} \tag{2.9}$$

From (2.8) and (2.9), we sustain

$$\begin{aligned}\Delta w(n) &\leq \frac{\Delta\rho(n)}{\rho(n+1)}w(n+1) - \frac{\alpha\rho(n)}{a^{1/\alpha}(n)}\Big(\frac{w(n+1)}{\rho(n+1)} \\ &\quad - a(n+1)\delta(n+1)\Big)^{1+1/\alpha} + \rho(n)\Delta(a(n)\delta(n)) \\ &\quad + \rho(n)\frac{\Delta(a(n)(\Delta^2 z(n))^\alpha)}{(\Delta z(n))^\alpha}, \quad n \geq N.\end{aligned} \tag{2.10}$$

From (2.10) and (2.8), we have

$$\begin{aligned}\Delta w(n) &\leq \Delta\rho(n)u(n) - \frac{\alpha\rho(n)}{a^{1/\alpha}(n)}u^{1+1/\alpha}(n) \\ &\quad + \Delta(\rho(n)a(n)\delta(n)) \\ &\quad + \rho(n)\frac{\Delta(a(n)(\Delta^2 z(n))^\alpha)}{(\Delta z(n))^\alpha},\end{aligned} \tag{2.11}$$

where $u(n) = \frac{w(n+1)}{\rho(n+1)} - a(n+1)\delta(n+1) > 0$. Now using the inequality

$$Cu - Du^{1+1/\alpha} \leq \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}}\frac{C^{\alpha+1}}{D^\alpha}, \quad D > 0 \tag{2.12}$$

In (2.11), put $C = \Delta\rho(n)$ and $D = \frac{\alpha\rho(n)}{a(n)^{1/\alpha}}$, we obtain

$$\begin{aligned}\Delta w(n) &\leq \frac{a(n)(\Delta\rho(n))^{\alpha+1}}{(\alpha+1)^{\alpha+1}(\rho(n))^\alpha} + \Delta(\rho(n)a(n)\delta(n)) \\ &\quad + \rho(n)\frac{\Delta(a(n)(\Delta^2 z(n))^\alpha)}{(\Delta z(n))^\alpha}.\end{aligned} \tag{2.13}$$

Define another function $v(n)$ by

$$v(n) = \rho(n)\Big(\frac{a(n-k)(\Delta^2 z(n-k))^\alpha}{(\Delta z(n-k))^\alpha} + a(n-k)\delta(n-k)\Big). \tag{2.14}$$

Similarly, we get,

$$\begin{aligned}\Delta v(n) &\leq \frac{a(n-k)(\Delta\rho(n))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^{\alpha+1}(n)} + \Delta(\rho(n)a(n-k)\delta(n-k)) \\ &\quad + \rho(n)\frac{\Delta(a(n-k)(\Delta^2 z(n-k))^\alpha)}{(\Delta z(n-k))^\alpha}.\end{aligned} \tag{2.15}$$

from (2.13), (2.15) and (2.7) is that

$$\begin{aligned}&\Delta w(n) + p^\alpha \Delta v(n) \\ &\leq \rho(n)\Big\{\frac{\Delta(a(n)(\Delta^2 z(n))^\alpha) + p^\alpha\Delta(a(n-k)(\Delta^2 z(n-k))^\alpha)}{(\Delta z(n))^\alpha}\Big\} \\ &\quad + \Delta(\rho(n)a(n)\delta(n) + p^\alpha\rho(n)a(n-k)\delta(n-k)) \\ &\quad + \frac{(\Delta\rho(n))^{\alpha+1}}{(\alpha+1)^{\alpha+1}\rho^\alpha(n)}(a(n) + p^\alpha a(n-k)) \\ &\leq \frac{-M}{2^{\alpha-1}}\rho(n)\beta(n)\frac{z^\alpha(n-l)}{(\Delta z(n))^\alpha} + G(n), \quad n \geq n_1 \geq N.\end{aligned}$$

Using Lemma 2.4, we obtain

$$\begin{aligned}&\Sigma_{s=n_2}^{n-1}\Big[M2^{1-\alpha}\rho(s)\beta(s)\Big(\frac{\tau(s-l,n_1)}{\tau(s,N)}\Big)^\alpha - G(s)\Big] \\ &\leq w(n_2) + p^\alpha v(n_2) < \infty\end{aligned}$$

which contradicts (2.5).
Case (ii): Let $n \geq N \in \mathbb{N}$, and adding the inequality (2.7) from n to j, we have

$$\begin{aligned}&a(j+1)(\Delta^2 z(j+1))^\alpha - a(n)(\Delta^2 z(n))^\alpha \\ &\quad + p^\alpha a(j+1-k)(\Delta^2 z(j+1-k))^\alpha \\ &- p^\alpha a(n-k)(\Delta^2 z(n-k))^\alpha \\ &\quad + \frac{M}{2^{\alpha-1}}\Sigma_{t=n}^{j}\beta(t)z)^\alpha(t-l \leq 0.\end{aligned}$$

Put $j \to \infty$, since $\{a(j)(\Delta^2 z(j))^\alpha\}$.

$$\begin{aligned}&-\Delta^2 z(n-k) + \Big(\frac{M}{2^{\alpha-1}(1+p^\alpha)}\Big)^{1/\alpha} \\ &(\times)\Big(\frac{1}{a(n-k)}\Sigma_{t=n}^{\infty}\beta(t)z(t-l)^\alpha\Big)^{1/\alpha} \leq 0.\end{aligned}$$

Adding again from n to j, we obtain

$$-\Delta z(j+1-k)+\Delta z(n-k)+\Big(\frac{M}{2^{\alpha-1}(1+p^{\alpha})}\Big)^{1/\alpha}$$
$$(\times)\Sigma_{t=n}^{j}\Big(\frac{1}{a_{t-k}}\Sigma_{s=n}^{t}\beta(s)\Big)^{1/\alpha}z(t-l)\leq 0.$$

Put $j\to\infty$, since $\{\Delta z(j)\}$, we have

$$\Delta z(n-k)+\Big(\frac{M}{2^{\alpha-1}(1+p^{\alpha})}\Big)^{1/\alpha}$$
$$(\times)\Sigma_{t=n}^{\infty}\Big(\frac{1}{a(t-k)}\Sigma_{s=n}^{t}\beta(s)\Big)^{1/\alpha}z(t-l)\leq 0.$$

Adding the above inequality from $n+k$ to j, we obtain

$$z(j+1-k)-z(n)+\Big(\frac{M}{2^{\alpha-1}(1+p^{\alpha})}\Big)^{1/\alpha}$$
$$(\times)\Sigma_{t=n}^{j}\Big[\Sigma_{s=n}^{t}\Big(\frac{1}{a(s-k)}\Sigma_{i=s}^{t}\beta(i)\Big)^{1/\alpha}z(t-l)\Big]\leq 0.$$

Put $j\to\infty$, since $\{z(n)\}$, we have

$$\Sigma_{t=n+k}^{n+l}\Big[\Sigma_{s=n}^{t}\Big(\frac{1}{a(s-k)}\Sigma_{i=s}^{t}\beta(i)\Big)^{1/\alpha}\Big]\leq\Big(\frac{2^{\alpha-1}(1+p^{\alpha})}{M}\Big)^{1/\alpha}$$

which contradicts (2.6) as $n\to\infty$. This completes the proof. □

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