Malaya Journal of Matematik, Vol. 8, No. 3, 822-826, 2020

https://doi.org/10.26637/MJM0803/0015

β_λ-closed spaces

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Abstract
We introduce and study β_λ-closed spaces in generalized topological spaces (GTS) as a generalization of β-closed spaces [2] in topological spaces. Several characterizations and mapping properties of such spaces are obtained.

Keywords
Generalized topology, λ-space, λ-β-open, λ-β-regular, λ-β-θ-open, β_λ-closed, β_λ-θ-converge, β_λ-θ-accumulate, β_λ-θ-c.a.p.

AMS Subject Classification
54A05, 54A10, 54D20.

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Article History: Received **17** December **2019**; Accepted **22** May **2020**

Contents

1. Introduction

Concept of generalized topological spaces (GTS) has been introduced by A. Cs*á*sz*á*r [4, 6, 8] in 2002. Since then, several research works have been done to generalize the existing notions of topological spaces to generalized topological spaces. Recently, the concept of covering properties in generalized topological spaces have been studied by some authors [15–18]. On the other hand, β-open sets [1] plays a significant role in the theory of generalized form of open sets in topological spaces. Basu and Ghosh [2] introduced the concept of β-closed spaces in topological spaces and gave several characterizations of β-closed spaces.

In this paper, we have introduced and studied a new kind of covering properties in a generalized topological space (X,λ) known as β_λ-closed spaces via λ-β-open sets [8].

2. Preliminaries

A collection λ of subsets of X is called a generalized topology (briefly GT) on X [6] if and only if $\emptyset \in \lambda$ and $G_i \in \lambda$ for $i \in I \neq \emptyset$ implies $G = \bigcup_{i\in I} G_i \in \lambda$. A set X with a GT λ on X is called a generalized topological space (GTS) and is denoted by (X,λ). By a space X or (X,λ), we will always mean a GTS. A space (X,λ) is called a λ-space [14] if $X \in \lambda$. For a space (X,λ), the elements of λ are called λ-open sets and the complements of λ-open sets are called λ-closed sets. A GT λ on X is said to be a quasi-topology [9] if and only if $A, B \in \lambda$ implies $A \cap B \in \lambda$. A set X with a quasi topology λ on X is called a quasi topological space.

For $A \subset X$, the λ-closure of A, denoted by cA is the intersection of all λ-closed sets containing A and the λ-interior of A, denoted by iA is the union of all λ-open sets contained in A. It was pointed out in [8] that each of the operations iA and cA are monotonic [10] i.e. if $A \subset B \subset X$, then $iA \subset iB$ and $cA \subset cB$, idempotent [10], i.e. if $A \subset X$, then $i(iA) = iA$ and $c(cA) = cA$, iA is restricting [10], i.e. if $A \subset X$, then $iA \subset A$, cA is enlarging [10], i.e., if $A \subset X$, then $A \subset cA$. In a space (X,λ), for $A \subset X$, $x \in iA$ if and only if there exists an λ-open set V containing x such that $V \subset A$ and $x \in cA$ if and only if $V \cap A \neq \emptyset$ for every λ-open set V containing x [5]. In a space (X,λ), $A \subset X$ is λ-open if and only if $A = iA$ and is λ-closed if and only if $A = cA$ [4] and $cA = X \setminus i(X \setminus A)$.

A subset A of a topological space is called β-open [1] if $A \subset cl(int(cl(A)))$. The complement of a β-open set is called β-closed. For a subset A of a topological space (X,τ), the β-closure of A, denoted by $\beta cl(A)$ is the intersection of all β-open sets containing A and the β-interior of A, denoted by $\beta int(A)$ is the union of all β-open sets contained in A. A topological space (X,τ) is said to be β-closed [2] if every cover of X by β-open sets has a finite subfamily whose β-closures cover X.

A set $A \subset X$ is said to be λ-semi-open (resp. λ-preopen, λ-α-open, λ-β-open) [8] if $A \subset ciA$ (resp. $A \subset icA$, $A \subset iciA$, $A \subset cicA$). We denote by $\sigma(\lambda)$ (resp. $\pi(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$) the class of all λ-semi-open sets (resp. λ-preopen sets, λ-α-open sets, λ-β-open sets). From [8], it is clear that $\lambda \subset \alpha(\lambda) \subset \sigma(\lambda) \subset \beta(\lambda)$, $\alpha(\lambda) \subset \pi(\lambda) \subset \beta(\lambda)$ and each of the collections $\sigma(\lambda)$, $\pi(\lambda)$, $\alpha(\lambda)$, $\beta(\lambda)$ forms a GTS. The complements of λ-semi-open sets (resp. λ-preopen sets, λ-α-open sets, λ-β-open sets) are called λ-semi-closed sets (resp. λ-preclosed sets, λ-α-closed sets, λ-β-closed sets). For $A \subset X$, we denote by scA (resp. pcA, αcA, βcA) the intersection of all λ-semi-closed sets (resp. λ-preclosed sets, λ-α-closed sets, λ-β-closed sets) containing A and by siA (resp. piA, αiA, βiA) the union of all λ-semi-open sets (resp. λ-preopen sets, λ-α-open sets, λ-β-open sets) contained in A. A subset A of a λ-space (X,λ) is called λ-compact [16] if any cover of A by λ-open subsets of X has a finite subcover. A λ-space (X,λ) is weakly λ-compact [17] if any cover of X by λ-open sets has finite subfamily, the union of the λ-closures of whose members covers X.

3. β_λ-closed spaces

We first state a lemma which will be used in the sequel. Proofs can be checked easily and therefore omitted.

Lemma 3.1. *The following hold for a subset A of GTS X:*
(i) $\beta iA = A \cap cicA$
(ii) $\beta cA = A \cup iciA$
(iii) $x \in \beta cA$ *if* $A \cap U \neq \emptyset$ *for every* λ-β-*open sets* U *of* X *containing* x
(iv) $\beta c(X \setminus A) = X \setminus \beta iA$
(v) A is λ-β-*closed if and only if* $A = \beta cA$.

Definition 3.2. *A subset A of a space X is said to be* λ-β-*regular if it is both* λ-β-*open and* λ-β-*closed. The family of all* λ-β-*regular sets of a space X is denoted by* $\beta r(X)$ *and that of containing a point x of X by* $\beta r(X,x)$.

Lemma 3.3. *For a subset A of a space X,* $A \in \beta(\lambda)$ *if and only if* $\beta cA \in \beta r(X)$.

Proof. First suppose, $A \in \beta(\lambda)$. Then $A \subset cicA$ and therefore, $\beta c(cA) \subset \beta c(cicA) = cicA \subset cic(\beta cA)$ i.e. βcA is λ-β-open. Since βcA is λ-β-open and λ-β-closed, $\beta cA \in \beta r(X)$. Next suppose, $\beta cA \in \beta r(X)$. Then $A \subset \beta cA \subset cic(\beta cA) \subset cic(cA) = cicA$. Hence $A \in \beta(\lambda)$. □

Definition 3.4. *A point* $x \in X$ *is said to be in the* λ-β-θ-*closure of A, denoted by* β-θ-cA, *if* $A \cap \beta cV \neq \emptyset$ *for every* λ-β-*open set V of X containing x.*

If β-θ-$cA = A$, *then A is said to be* λ-β-θ-*closed. The complement of a* λ-β-θ-*closed set is said to be* λ-β-θ-*open.*

Lemma 3.5. *For a subset A of a space X,*
β-θ-$cA = \cap\{V : A \subset V$ *and V is* λ-β-θ-*closed* $\} = \cap\{V : A \subset V$ *and* $V \in \beta r(X)\}$

Proof. We give proof of the first equality. Other is quite similar. Suppose that, $x \notin \beta$-θ-cA. Then there exists, λ-β-open set V containing x such that $\beta cV \cap A = \emptyset$. Therefore by Lemma 3.3, $X \setminus \beta cV$ is λ-β-regular and so λ-β-θ-closed set containing A such that $x \notin X \setminus \beta cV$. Hence, $x \notin \cap\{V : A \subset V$ and V is λ-β-θ-closed set$\}$. Conversely, suppose that, $x \notin \cap\{V : A \subset V$ and V is λ-β-θ-closed set$\}$. Then there exist, a λ-β-θ-closed set V containing A and $x \notin V$. Also, there exists an $U \in \beta(\lambda)$ such that $x \in U \subset \beta cU \subset X \setminus V$. Then we have, $\beta cU \cap A \subset \beta cU \cap V = \emptyset$ and so $x \notin \beta$-θ-cA. □

Lemma 3.6. *Let A and B be any subset of a space X. Then the following properties hold:*
(i) $x \in \beta$-θ-cA *if and only if* $A \cap V \neq \emptyset$ *for every* $V \in \beta r(X,x)$.
(ii) If $A \subset B$ *then* β-θ-$cA \subset \beta$-θ-cB.
(iii) β-θ-$c(\beta$-θ-$cA) = \beta$-θ-cA.
(iv) intersection of an arbitrary family of λ-β-θ-*closed sets in X is* λ-β-θ-*closed in X.*
(v) A is λ-β-θ-*open if and only if for each* $x \in A$, *there exists* $V \in \beta r(X,x)$, *such that* $x \in V \subset A$.
(vi) If $A \in \beta(\lambda)$ *then* $\beta cA = \beta$-θ-cA.
(vii) If $A \in \beta r(X)$ *then A is* λ-β-θ-*closed.*
(viii) $A = \beta r(X)$ *if and only if A is* λ-β-θ-*open and* λ-β-θ-*closed.*

Proof. We give only proof of (iv). Others proofs are obvious. (iv) Let A_α be a λ-β-θ-closed for each $\alpha \in \Delta$. Then for each $\alpha \in \Delta$, we have $A = \beta$-θ-cA_α. Therefore, β-θ-$c(\cap_{\alpha\in\Delta}A_\alpha) \subset \cap_{\alpha\in\Delta}\beta$-$\theta$-$cA_\alpha$=$\cap_{\alpha\in\Delta}A_\alpha \subset \beta$-$\theta$-$c(\cap_{\alpha\in\Delta}A_\alpha)$. Hence, β-θ-$c(\cap_{\alpha\in\Delta}A_\alpha) = \cap_{\alpha\in\Delta}A_\alpha$. Therefore, $\cap_{\alpha\in\Delta}A_\alpha$ is λ-β-θ-closed. □

Remark 3.7. *If A be a* λ-β-*regular set in a GTS* (X,λ), *then A is* λ-β-θ-*open and* λ-β-*open.*

We now introduce β_λ-closed subset A of a λ-space (X,λ). As a special case, we obtain β_λ-closed spaces when $A = X$. Several characterizations in terms of filter bases and generalized complete accumulation point are obtained.

Definition 3.8. *A subset A of a* λ-*space is called* β_λ-*closed in X if any cover of A by* λ-β-*open subsets of X has a finite subfamily, the union of* λ-β-*closures of whose members covers A.*

A λ-*space* (X,λ) *is called* β_λ-*closed if any cover of X by* λ-β-*open sets has a finite subfamily, the union of* λ-β-*closures of whose members covers X.*

Remark 3.9. *We observe that the concept of* β_λ-*closed subset of a* λ-*space generalizes the concept of* β-*closed subset [3] of a topological space. Also, if A is a subset of a topological space* (X,τ) *and* $\lambda = \tau$, *then the concepts of* β_λ-*closedness and* β-*closedness are equivalent.*

Theorem 3.10. *Every* β_λ-*closed space* (X,λ) *is weakly* λ-*compact [17].*

Proof. Proof follows from the fact that every λ-open set is λ-β-open in a space (X,λ). □

Theorem 3.11. *For a λ-space X, the following are equivalent:*
(i) A is β_λ-closed ;
(ii) every cover of A by λ-β-regular sets has a subcover;
(iii) for each family $[\{U_\alpha \in \beta r(X) : \alpha \in I\}] \cap A = \emptyset$, there exist a finite subset I_0 of I such that $[\{U_\alpha \in \beta r(X) : \alpha \in I_0\}] \cap A = \emptyset$;
(iv) every cover of A by λ-β-θ-open has a finite subcover.

Proof. Straightforward. □

Proposition 3.12. *For a subset A of a λ-space X, the following are equivalent:*
(i) A is β_λ-closed in X.
(ii) for any family $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ of λ-β-closed subsets of X such that $[\{U_\alpha : \alpha \in \Lambda\}] \cap A = \emptyset$, there exist a finite subset Λ_0 of Λ such that $[\{\beta i U_\alpha : \alpha \in \Lambda_0\}] \cap A = \emptyset$;

Proof. Straightforward. □

Definition 3.13. *A filter base $\mathscr{F}$ on a λ-space X is said to be β_λ-θ-converge to a point $x \in X$, if for each λ-β-open subset V of X containing x, there exists $F \in \mathscr{F}$ such that $F \subset \beta c V$.*

A filter base $\mathscr{F}$ is said to be β_λ-θ-accumulate at $x \in X$, if $F \cap \beta c V \neq \emptyset$ for every $F \in \mathscr{F}$ and for every λ-β-open subset V of X containing x.

Definition 3.14. *A net $\{x_\lambda\}_{\lambda \in D}$ on a λ-space X, where D is a directed set, is said to be β_λ-θ-converge to a point $x \in X$, if for each λ-β-open subset V of X containing x, there exists $n_0 \in D$ such that $\forall\ n \succ n_0$, $x_n \in \beta c V$.*

A net $\{x_\lambda\}_{\lambda \in D}$ on a λ-space X, where D is a directed set, is said to be β_λ-θ-accumulate at $x \in X$, if for each λ-β-open subset V of X containing x and $\forall\ n_0 \in D$ there exist $n \succ n_0$ such that $x_n \in \beta c V$.

Definition 3.15. *A point x in a space X is called β_λ-θ-complete accumulation point (β_λ-θ-c.a.p, for short) of a subset S of X, if $|S| = |S \cap V|$ for each $V \in \beta r(X, x)$, where $|S|$ denotes the cardinality of the subset S.*

Theorem 3.16. *The following conditions are equivalent for a λ-space X:*
(i) X is β_λ-closed;
(ii) every infinite subset of X has a β_λ-θ-c.a.p in X;
(iii) each net with a well ordered directed set as its domain β_λ-θ-accumulate to a point in X.

Proof. (i) $\Rightarrow$: Let I be an infinite subset in a β_λ-closed space and $A = \{x \in X : x \text{ is not a } \beta_\lambda\text{-}\theta\text{-c.a.p. of } I\}$. Then for each $x \in A$, there exists a $B_x \in \beta r(X, x)$ such that $|B_x \cap I| < |I|$. If A is the whole space, then it follows from the Theorem 3.11, that the cover $\{B_x : x \in A\}$ has a finite subcover, say $\{B_{x_1}, B_{x_2}, ..., B_{x_k}\}$. Then, $I \subset \cup\{B_{x_i} \cap I : i = 1, 2, ..., k\}$ and $|I| = max\{|B_{x_i} \cap I| : i = 1, 2, ..., k\}$ – a contradiction. Hence, A has a β_λ-θ-c.a.p. in X.
(ii) $\Rightarrow$ (i): Suppose X is not β_λ-closed. Then by the Theorem 3.11, there exists a cover $\mathscr{U}$ of X by λ-β-regular sets having no finite subcover. We consider $\wp = min\{|\mathscr{U}^\star| : \mathscr{U}^\star \subset \mathscr{U}$ and $\mathscr{U}^\star$ is a cover of $X\}$, where $|.|$ denotes the cardinality. Let $\mathscr{U}_0 \subset \mathscr{U}$ be a cover of X for which $|\mathscr{U}_0| = \wp$. Clearly $\wp \geq \aleph_0$ and then by the well ordering of $\mathscr{U}_0$, by some minimal well ordering $\prec$, we have $|\{U : U \in \mathscr{U}_0 \text{ and } U \prec U_0\}| < |\{U : U \in \mathscr{U}_0\}|$ for each $U_0 \in \mathscr{U}_0$. It is clear that, X can not have any subcover with cardinality less than $\wp$ and hence for each $U \in \mathscr{U}_0$, there exists a point $x_U \in X \setminus \cup\{U_0 \cup \{x_{U_0}\} : U_0 \in \mathscr{U}_0$ and $U_0 \prec U\}$. This can be done always, otherwise one can choose from $\mathscr{U}_0$ a cover of smaller cardinality. For any point $x \in X$, suppose $W = \{x_U : U \in \mathscr{U}_0\}$. Since $\mathscr{U}$ is a cover of X, $x \in U^*$ for some $U^\star \in \mathscr{U}_0$. But by the choice of x_U, $x_U \in \mathscr{U}^\star$ implies $U \prec U^\star$. Therefore, $Z = \{U \in \mathscr{U}_0 \text{ and } x_U \in U^\star\} \subset \{U \in \mathscr{U}_0 : U \prec U^\star\}$. By the minimality of $\prec$, $|Z| < \wp$ and so $|W \cap U^\star| < \wp$. Now, since for $U_1, U_2 \in \mathscr{U}_0$ with $U_1 \neq U_2$, we have $x_{U_1} \neq x_{U_2}$, then $|W| = \wp \geq \aleph_0$. Therefore, the infinite set W has no β_λ-θ-c.a.p. in X – a contradiction. Hence, X is β_λ-closed.
(iii) $\Rightarrow$ (ii): Let I be an infinite subset of X. Then by Zorn's lemma, I can be assumed to be a net with a well ordered directed set as its domain. So, it has a β-θ-adherent point say, x and then clearly x is an β_λ-θ-c.a.p. of I.
(i) $\Rightarrow$ (iii): Let $\{x_\lambda\}_{\lambda \in D}$ be a net with well ordered directed set D and it has no β_λ-θ-adherent point in X. Then, for each $x \in X$, there exists $U_x \in \beta r(X, x)$ and a $\lambda_x \in D$ such that $x_\lambda \in X \setminus U_x\ \forall\ \lambda \geq \lambda_x$. Now, since X is β_λ-closed, the cover $\{U_x : x \in X\}$ has a finite subcover $\{U_{x_1}, U_{x_2}, ..., U_{x_k}\}$ (say). Suppose $\{\lambda_{x_1}, \lambda_{x_2}, ..., \lambda_{x_k}\}$ be the corresponding elements in D and since it is finite, by the well orderedness of D, there exists a largest element say, λ_{x_k} in D. Therefore, we have $x_\lambda \in \cap_{i=1}^k (X \setminus U_{x_i}) = X \setminus \cup_{i=1}^k U_{x_i} = \emptyset$ for $\lambda > \lambda_{x_k}$ – a contradiction. Hence, the net $\{x_\lambda\}_{\lambda \in D}$ has a β_λ-θ-c.a.p. in X. □

The proof of the following proposition is straightforward and thus omitted.

Proposition 3.17. *Let $\mathscr{F}$ be a filter base on a λ-space X and $x \in X$. Then*
(i) If $\mathscr{F}$ is β_λ-θ-converge to x, then $\mathscr{F}$ β_λ-θ-accumulates at x.
(ii) If $\mathscr{F}$ is a maximal filter base, then $\mathscr{F}$ β_λ-θ-converges if and only if β_λ-θ-accumulates at x.

Theorem 3.18. *For a subset A of a λ-space X, the following are equivalent:*
(i) A is β_λ-closed in X;
(ii) every maximal filter base on X, each of whose members meet A, β_λ-θ-converges to some point of A;
(iii) every filter base on X, each of whose members meet A, β_λ-θ-accumulate to a point of A.

Proof. (i) $\Rightarrow$ (ii): Let $\mathscr{F}$ be a maximal filter base on X, each of whose members meet A, such that $\mathscr{F}$ does not β_λ-θ-converge to any point of A. Now, since $\mathscr{F}$ is maximal, by above Proposition 3.17, $\mathscr{F}$ does not β_λ-θ-accumulates at any point of A. So, for each $x \in A$, there exists $F_x \in \mathscr{F}$ and λ-β-open set U_x of X containing x and $F_x \cap \beta c U_x = \emptyset$.

But, A being β_λ-closed in X, there exists $x_1, x_2, ..., x_n \in X$ such that $A \subset \cup_{i=1}^n \beta c U_{x_i}$. Again, since $\mathscr{F}$ is filter base on X, there exists $F \in \mathscr{F}$ such that $F \subset \cap_{i=1}^n F_{x_i}$, but $F_{x_i} \cap \beta c U_{x_i} = \emptyset$ for each $i \in \{1,2,...,n\}$. Therefore, $F \cap \beta c U_{x_i} = \emptyset$ for each $i \in \{1,2,...,n\}$ i.e. $\emptyset = (\cup_{i=1}^n \beta c U_{x_i}) \cap F \supset A \cap F$ - a contradiction.
(ii) $\Rightarrow$ (iii): Let $\mathscr{F}$ be a filter base on X, each of whose members meet A. Then $\mathscr{F}^A = \{F \cap A : F \in \mathscr{F}\}$ is a filter base on X. Therefore, $\mathscr{F}^A$ is contained in a maximal filter base $\mathscr{H}$ on X, each of whose members meet A. Hence, $\mathscr{H}$ β_λ-θ-converges to some y of A and so by Proposition 3.17 (ii), $\mathscr{H}$ β_λ-θ-accumulates at y. But since $\mathscr{F}^A \subset \mathscr{H}$, so $\mathscr{F}^A$ β_λ-θ-accumulates at y. Hence, $\mathscr{F}$ β_λ-θ-accumulates at y.
(iii) $\Rightarrow$ (i): If possible, suppose that, A is not β_λ-closed. Then by Proposition 3.12, there exist a cover $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ of A by λ-β-open subsets of X such that for any finite subset Λ_0 of Λ such that $[\beta i(X \setminus U_\alpha) : \alpha \in \Lambda_0\}] \cap A \neq \emptyset$. For each finite subset Λ_0 of Λ, let $F_{\Lambda_0} = [\cap\{\beta i(X \setminus U_\alpha) : \alpha \in \lambda_0\}] \cap A$. Then $\mathscr{F} = \{F_{\Lambda_0} : \Lambda_0$ is a finite subset of $\Lambda\}$ is a filter base on X, each of whose members meet A. Therefore by (iii), $\mathscr{F}$ β_λ-θ-accumulates at some point x of A. Since $\mathscr{U}$ is a cover of A, there exists $\alpha_0 \in \Lambda$ such that $x \in U_{\alpha_0}$, but $\mathscr{F}$ β_λ-θ-accumulates at x and U_{α_0} being a λ-β-open set, $F \cap \beta c U_{\alpha_0} \neq \emptyset$ for every $F \in \mathscr{F}$. Let $F = \beta i(X \setminus U_{\alpha_0}) \cap A$, then $F \in \mathscr{F}$. Thus $\beta i(X \setminus U_{\alpha_0}) \cap A \cap \beta c U_{\alpha_0} = \emptyset$ – a contradiction. Hence, A is β_λ-closed in X. □

Proposition 3.19. *Let X be a λ-space. If A is λ-β-θ-closed subset of X and B is a β_λ-closed in X, then $A \cap B$ is β_λ-closed in X.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of $A \cap B$ by λ-β-open subsets of X. Then $\mathscr{U} \cup \{X \setminus A\}$ is a cover of B. Now, since $X \setminus A$ is λ-β-θ-open, for each $x \notin A$, there exists a λ-β-open set U_x such that $x \in U_x \subset \beta c U_x \subset (X \setminus A)$. Then $\mathscr{U} \cup \{U_x : x \in X \setminus A\}$ is a cover of B by λ-β-open subsets of X. Since B is β_λ-closed in X, so there exists $\alpha_1, \alpha_2, ..., \alpha_n \in \Lambda$ and there exist $x_1, x_2, ..., x_m \in X \setminus A$ such that $B \subset [\cup_{i=1}^n \beta c U_{\alpha_i}] \cup [\cup_{i=1}^m \beta c U_{x_i}]$. But, $\beta c U_{x_i} \subset X \setminus A$ and so $A \cap B \subset \cup_{i=1}^n \beta c U_{\alpha_i}$. Hence, $A \cap B$ is β_λ-closed in X. □

Theorem 3.20. *For a λ-space X, the following are equivalent:*
(i) X is β_λ-closed;
(ii) every proper λ-β-θ-closed set is β_λ-closed relative to X;
(iii) every proper λ-β-regular set is β_λ-closed relative to X.

Proof. (i) $\Rightarrow$ (ii): Let $\{U_\alpha : \alpha \in I\}$ be a cover of proper λ-β-θ-closed set F by λ-β-regular sets of X. Then $X \setminus F$ is λ-β-θ-open, so for each $x \in (X \setminus F)$, there exists a $V_x \in \beta r(X,x)$ such that $x \in V_x \subset (X \setminus F)$. Hence, the family $\{V_x : x \in X \setminus F\} \cup \{U_\alpha : \alpha \in I\}$ is a cover of X by λ-β-regular sets of X. Now, since X is β_λ-closed, there is a finite subset I_0 of I such that $F \subset \cup\{U_\alpha : \alpha \in I_0\}$. Hence by Theorem 3.11, F is β_λ-closed relative to X.
(i) $\Rightarrow$ (ii): Follow from Theorem 3.6 (vii).
(iii) $\Rightarrow$ (i): Let F be a λ-β-regular set. Then, $X = F \cup (X \setminus F)$ and since F and $X \setminus F$ are both λ-β-regular, X is β_λ-closed. □

4. Mapping Properties

Definition 4.1. *A function $f : (X,\lambda) \to (Y,\lambda')$ is called (λ,λ')-continuous [6] if the inverse image of each λ'-open set is λ-open.*

Definition 4.2. *A function $f : (X,\lambda) \to (Y,\lambda')$ is called $\beta_{(\lambda,\lambda')}$-irresolute if the inverse image of each λ'-β-open set is λ-β-open.*

Proof of the following lemma is quite straightforward and thus omitted.

Lemma 4.3. *Let $f : (X,\lambda) \to (Y,\lambda')$ be a function. Then the following are equivalent:*
(i) f is $\beta_{(\lambda,\lambda')}$-irresolute ;
(ii) for every $x \in X$ and for every λ'-open set V containing $f(x)$, there exists a λ-open set U containing x such that $f(U) \subset V$;
(iii) $f(\beta c_\lambda A) \subset \beta c_{\lambda'} f(A)$ for every subset A of X (where $c_\lambda A$ denotes λ-closure of A in (X,λ));
(iv) $\beta c_\lambda f^{-1}(B) \subset f^{-1}(\beta c_{\lambda'} B)$ for every subset B of Y.

Theorem 4.4. *Let $f : (X,\lambda) \to (Y,\lambda')$ be a $\beta_{(\lambda,\lambda')}$-irresolute function, where (X,λ) is λ-space and (Y,λ') is a λ'-space. If A is β_λ-closed in X, then $f(A)$ is $\beta_{\lambda'}$-closed in Y.*

Proof. Let $\mathscr{U} = \{U_\alpha : \alpha \in \Lambda\}$ be a cover of $f(A)$ by λ'-β-open subsets of Y. Since f is $\beta_{(\lambda,\lambda')}$-irresolute, $\mathscr{V} = \{f^{-1}(U_\alpha) : \alpha \in \Lambda\}$ is a cover of A by λ-β-open subsets of X. But, since A is β_λ-closed in X, there exists $\alpha_1, \alpha_2, ..., \alpha_n \in \Lambda$ such that $A \subset \cup_{i=1}^n \beta c_\lambda f^{-1}(U_{\alpha_i})$. This implies $f(A) \subset \cup_{i=1}^n f(\beta c_\lambda f^{-1}(U_{\alpha_i}))$. Again, since f is $\beta_{(\lambda,\lambda')}$-irresolute, by above Lemma 4.3, it follows that $f(\beta c_\lambda f^{-1}(U_{\alpha_i})) \subset \beta c_{\lambda'} f(f^{-1}(U_{\alpha_i})) \subset \beta c_{\lambda'} U_{\alpha_i}$. Therefore, $f(A)$ is $\beta_{\lambda'}$-closed in Y. □

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ISSN(P):2319 − 3786

Malaya Journal of Matematik

ISSN(O):2321 − 5666

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