



Existence of solutions to discrete boundary value problem of fractional difference equations

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Abstract

In this paper, we obtained the sufficient conditions for the existence of solutions to the discrete boundary value problems of fractional difference equation depending on parameters. We use Krasnosel'skii fixed point theorem to establish the existence results.

Keywords

Fractional difference equation, fixed point theorem, existence, boundary value problem.

AMS Subject Classification

26A33, 39A10, 34A08.

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1. Introduction

The theory of discrete fractional calculus has started to receive desirable attention in last two decades. On the other hand, the theory of continuous fractional calculus has seen tremendous development. However, there is huge scope for the development of discrete fractional calculus.

Initially, Atici and Eloe [2–4] published some important results on discrete fractional calculus with delta operator during last few years. Atici and Sengul [5] provided some initial attempts at using the discrete fractional difference equations to model tumor growth. Goodrich [8–11] explores the theory of fractional difference equations in his work. He deduced some important existence and uniqueness theorems for fractional order discrete boundary value problems. In his research he used Krasnosel'skii fixed point theorem to prove existence results. However, H. Chen, et al. [15, 16] followed this trend to provide some results about positive solutions to boundary

value problems of fractional difference equations under certain conditions. H. Chen, et al. [6] provided multiple solutions to fractional difference boundary value problem using Krasnosel'skii and Schauder's fixed point theorem. D. B. Pachpatte et al. [17] and Jinhua Wang, et al. [18] established the existence results for a discrete fractional boundary value problem using Krasnosel'skii and Schaefer fixed point theorem.

Getting inspiration by all this work, in this paper, we consider a discrete boundary value problem of fractional difference equation of the form,

$$-\Delta^\mu y(t) = \lambda h(t + \mu - 1) f(y(t + \mu - 1)), \quad (1.1)$$

$$y(\mu - 2) = 0, \quad \Delta y(\mu - 2) = \Delta y(\mu + b - 1), \quad (1.2)$$

where $t \in [0, b]_{\mathbb{N}_0}$, $f : [0, \infty) \rightarrow [0, \infty)$ is continuous.

$h : [v - 1, v + b]_{\mathbb{N}_{v-1}} \rightarrow [0, \infty)$, $1 < v \leq 2$ and λ is a positive parameter. We established the existence results for this problem using Krasnosel'skii fixed point theorem and to illustrate our main result we provide two examples at the end of this paper.

The present paper is organized in 3 sections. In section 2, together with some important basic definitions, we will demonstrate some important lemmas and theorem in order to prove our main result. In section 3, we establish the main results for existence of solutions to the boundary value problem (1.1)–(1.2). We conclude the paper with some examples illustrating the results which are proved here.

2. Preliminaries

In this section, let us first collect some basic definitions and lemmas that are very much important to us in the sequel.

Definition 2.1 ([4, 12]). *We define*

$$t^v = \frac{\Gamma(t+1)}{\Gamma(t+1-v)}, \tag{2.1}$$

for any t and v for which right hand side is defined. We also appeal to the convention that if $t+v-1$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^v = 0$.

Definition 2.2 ([12, 14]). *The v -th fractional sum of a function f , for $v > 0$ is defined as*

$$\Delta_a^{-v} f(t) = \Delta_a^{-v} f(t, a) := \frac{1}{\Gamma(v)} \sum_{s=a}^{t-v} (t-s-1)^{v-1} f(s), \tag{2.2}$$

for $t \in \mathbb{N}_{a+N-v}$. We also define the v -th fractional difference of f by

$$\Delta_a^v f(t) := \Delta^N \Delta_a^{v-N} f(t),$$

where $t \in \mathbb{N}_{a+N-v}$ and $N \in \mathbb{N}$ is chosen so that $0 \leq N-1 < v \leq N$.

Now, we give some important lemmas.

Lemma 2.3 ([4, 10]). *Let t and v be any numbers for which t^v and t^{v-1} are defined. Then $\Delta t^v = vt^{v-1}$.*

Lemma 2.4 ([4, 10]). *Let $0 \leq N-1 < v \leq N$. Then*

$$\Delta_a^{-v} \Delta_a^v y(t) = y(t) + C_1 t^{v-1} + C_2 t^{v-2} + \dots + C_N t^{v-N}, \tag{2.3}$$

for some $C_i \in \mathbb{R}$ with $1 \leq i \leq N$.

Lemma 2.5 ([18]). *Let $h : [v-1, v+b]_{\mathbb{N}_{v-1}} \rightarrow [0, \infty)$ be given. Then the unique solution of discrete fractional boundary value problem*

$$-\Delta_0^v y(t) = h(t+v-1),$$

$$y(v-2) = 0, \quad \Delta y(v-2) = \Delta y(v+b-1),$$

where $t \in [0, b]_{\mathbb{N}_0}$, is

$$y(t) = -\frac{1}{\Gamma(v)} \sum_{s=0}^b G(t, s) h(s+v-1), \tag{2.4}$$

where $G : [v-2, v+b]_{\mathbb{N}_{v-2}} \times [0, b]_{\mathbb{N}_0} \rightarrow \mathbb{R}$ is defined as

$$G(t, s) = \begin{cases} \frac{t^{v-1}(v+b-s-2)^{v-2}}{\Gamma(v-1)-(v+b-1)^{v-2}} + (t-s-1)^{v-1}, & 0 \leq s < t-v \leq b \\ \frac{t^{v-1}(v+b-s-2)^{v-2}}{\Gamma(v-1)-(v+b-1)^{v-2}}, & 0 \leq t-v < s \leq b. \end{cases} \tag{2.5}$$

Theorem 2.6. *Let $f : [0, \infty) \rightarrow [0, \infty)$ and $h : [v-1, v+b]_{\mathbb{N}_{v-1}} \rightarrow [0, \infty)$ be given. A function $y(t)$ is a solution of discrete fractional boundary value problem (1.1)–(1.2) iff $y(t)$ is a fixed point of the operator $F : \mathcal{B} \rightarrow \mathcal{B}$ defined as,*

$$Fy(t) := \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b G(t, s) h(s+v-1) f(y(s+v-1)), \tag{2.6}$$

where $G(t, s)$ is given in above Lemma 2.5 and \mathcal{B} is defined after Theorem 2.8.

Proof. From Lemma 2.4, we say that a general solution to problem (1.1)–(1.2) is

$$y(t) = -\Delta_a^{-v} \lambda h(t+v-1) f(y(t+v-1)) + C_1 t^{v-1} + C_2 t^{v-2}, \tag{2.7}$$

from the boundary condition $y(v-2) = 0$, we get

$$\begin{aligned} y(v-2) &= -\Delta_a^{-v} \lambda h(t+v-1) f(y(t+v-1)) \Big|_{t=v-2} \\ &\quad + C_1 (v-2)^{v-1} + C_2 (v-2)^{v-2} \\ &= -\frac{\lambda}{\Gamma(v)} \sum_{s=0}^{t-v} \left\{ (t-s-1)^{v-1} h(s+v-1) \right. \\ &\quad \cdot f(y(s+v-1)) \Big|_{t=v-2} + C_2 \Gamma(v-1) \\ &= C_2 \Gamma(v-1) = 0. \end{aligned}$$

Since, $(v-2)^{v-1} = 0$ and $(v-2)^{v-2} = \Gamma(v-1)$. Therefore, $C_2 = 0$.

On the other hand, using boundary condition $\Delta y(v-2) = \Delta y(v+b-1)$ and Lemma 2.32 in [12], for all $t \in [0, b]_{\mathbb{N}_0}$, we get

$$\begin{aligned} \Delta y(t) &= -\frac{1}{\Gamma(v-1)} \sum_{s=0}^{t-(v-1)} (t-s-1)^{v-2} \lambda h(s+v-1) \\ &\quad \cdot f(y(s+v-1)) + C_1 (v-1) t^{v-2} + 0, \end{aligned}$$

$$\begin{aligned} \Delta y(v-2) &= -\frac{\lambda}{\Gamma(v-1)} \sum_{s=0}^{-1} (t-s-1)^{v-2} h(s+v-1) \\ &\quad \cdot f(y(s+v-1)) \Big|_{t=v-2} \\ &\quad + C_1 (v-1) (v-2)^{v-2} \\ &= C_1 (v-1) \Gamma(v-1) \\ &= C_1 \Gamma(v), \end{aligned} \tag{2.8}$$

$$\begin{aligned} \Delta y(v+b-1) &= -\frac{1}{\Gamma(v-1)} \sum_{s=0}^b (v+b-s-2)^{v-2} \\ &\quad \cdot \lambda h(s+v-1) f(y(s+v-1)) \\ &\quad + C_1 (v-1) (v+b-1)^{v-2}, \end{aligned} \tag{2.9}$$



equating equations (2.8) and (2.9) we get,

$$\begin{aligned}
 C_1 & \left(\Gamma(v) - (v-1)(v+b-1)^{v-2} \right) \\
 & = -\frac{1}{\Gamma(v-1)} \sum_{s=0}^b (v+b-s-2)^{v-2} \\
 & \quad \cdot \lambda h(s+v-1) f(y(s+v-1)), \\
 C_1 & = -\frac{1}{\Gamma(v-1) \left(\Gamma(v) - (v-1)(v+b-1)^{v-2} \right)} \\
 & \quad \cdot \sum_{s=0}^b (v+b-s-2)^{v-2} \lambda \\
 & \quad \cdot h(s+v-1) f(y(s+v-1)).
 \end{aligned}$$

Thus, from equation (2.7) we have,

$$\begin{aligned}
 y(t) & = -\frac{1}{\Gamma(v)} \sum_{s=0}^{t-v} (t-s-1)^{v-1} \lambda h(s+v-1) \\
 & \quad \cdot f(y(s+v-1)) + t^{v-1} \\
 & \quad \cdot \left\{ \frac{-1}{\Gamma(v-1) \left(\Gamma(v) - (v-1)(v+b-1)^{v-2} \right)} \right\} \\
 & \quad \cdot \sum_{s=0}^b (v+b-s-2)^{v-2} \lambda h(s+v-1) \\
 & \quad \cdot f(y(s+v-1)) \\
 & = -\frac{\lambda}{\Gamma(v)} \left\{ \sum_{s=0}^{t-v} \left\{ (t-s-1)^{v-1} \right. \right. \\
 & \quad \left. \left. + \frac{(v+b-s-2)^{v-2} \cdot t^{v-1}}{\left(\Gamma(v-1) - (v+b-1)^{v-2} \right)} \right\} \right\} \\
 & \quad + \sum_{s=t-v+1}^b \left\{ \frac{(v+b-s-2)^{v-2} \cdot t^{v-1}}{\left(\Gamma(v-1) - (v+b-1)^{v-2} \right)} \right\} \\
 & \quad \cdot h(s+v-1) f(y(s+v-1)) \\
 & = -\frac{\lambda}{\Gamma(v)} \sum_{s=0}^b G(t,s) h(s+v-1) f(y(s+v-1)).
 \end{aligned}$$

Consequently, we observe that $y(t)$ implies that whenever y is a solution of (1.1)–(1.2), y is a fixed point of (2.6), as desired. \square

Lemma 2.7 ([18]). *The function G given in Lemma 2.5 satisfies the following conditions:*

1. $0 \leq G(t,s) \leq \frac{D(v+b)^{v-1}}{(s+v-1)^{v-1}} G(s+v-1,s);$
 $(t,s) \in [v-2, v+b]_{\mathbb{N}_{v-2}} \times [0, b]_{\mathbb{N}_0}.$
2. $\min_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} G(t,s) \geq \frac{\Gamma(v)}{(s+v-1)^{v-1}} G(s+v-1,s) > 0.$

Here,

$$\begin{aligned}
 D & = \max_{s \in [0, b]} \left\{ 1 + \frac{\Gamma(v-1) - (v+b-1)^{v-2}}{(v+b-s-2)^{v-2}} \right\} \\
 & = 1 + \frac{\Gamma(v-1) - (v+b-1)^{v-2}}{(v+b-2)^{v-2}}. \tag{2.10}
 \end{aligned}$$

Theorem 2.8 ([16, 18]). *Let E be a Banach space, and let $\mathcal{K} \subset E$ be a cone in E . Assume that Ω_1 and Ω_2 are bounded open sets contained in E such that $0 \in \Omega_1$ and $\overline{\Omega_1} \subseteq \Omega_2$, and let $S : \mathcal{K} \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow \mathcal{K}$ be a completely continuous operator such that either*

1. $\|Sy\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Sy\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$; Or
2. $\|Sy\| \geq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_1$ and $\|Sy\| \leq \|y\|$ for $y \in \mathcal{K} \cap \partial\Omega_2$.

Then, S has at least one fixed point in $\mathcal{K} \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Let \mathcal{B} be the collection of all functions $y : [v-2, v+b]_{\mathbb{N}_{v-2}} \rightarrow \mathbb{R}$ with the norm

$$\|y\| = \max \left\{ |y(t)| : t \in [v-2, v+b]_{\mathbb{N}_{v-2}} \right\}.$$

Now, we define the cone $\mathcal{K} \subset \mathcal{B}$ by

$$\mathcal{K} = \left\{ y \in \mathcal{B} : \min_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} y(t) \geq \frac{\Gamma(v)}{D(v+b)^{v-1}} \|y\| \right\}.$$

Lemma 2.9 ([18]). *Let F be the operator defined in (2.6) and \mathcal{K} be the cone defined as above, then $F : \mathcal{K} \rightarrow \mathcal{K}$.*

Proof. The proof of Lemma 2.9 is similar to Lemma 3.4 in [18], hence omitted. \square

3. Main Results

In the sequel, now we present the next structural assumption that can be impose on (1.1)–(1.2) to get the existence of solution.

H1: $\lim_{y \rightarrow 0} \frac{f(y)}{y} = \infty,$

H2: $\lim_{y \rightarrow b} \frac{f(y)}{y} = 0,$

H3: $\lim_{y \rightarrow 0} \frac{f(y)}{y} = l, \quad 0 < l < \infty,$

H4: $\lim_{y \rightarrow \infty} \frac{f(y)}{y} = L, \quad 0 < L < \infty.$

Also, we consider

$$\sigma = \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} \frac{1}{\Gamma(v)} \sum_{s=0}^b G(t,s) h(s+v-1), \tag{3.1}$$

$$\tau = \min_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} \frac{1}{\Gamma(v)} \sum_{s=0}^b G(t,s) h(s+v-1). \tag{3.2}$$



Theorem 3.1. Assume that the function $f : [v - 1, v + b]_{\mathbb{N}_{v-1}} \rightarrow [0, \infty)$ is continuous and the conditions H1 and H2 holds. Then problem (1.1)–(1.2) has at least one solution.

Proof. From Lemma 2.9, we know that $F(\mathcal{K}) \subseteq \mathcal{K}$. By condition H1, we can select a number $\eta_1 > 0$ sufficiently small so that both $\min_{0 \leq y \leq r_1} f(y) > \frac{y}{\eta_1}$ and $\lambda \sum_{s=0}^b \frac{G(s+v-1, s)h(s+v-1)}{D(v+b)^{v-1}(s+v-1)^{v-1}} > \eta_1$ holds for all $t \in [v - 1, v + b]_{\mathbb{N}_{v-1}}$ and $0 \leq y \leq r_1$ where $r_1 := r_1(\eta_1)$.

Define $\Omega_1 = \{y \in \mathcal{B} : \|y\| < r_1\}$, we have

$$\begin{aligned} \|Fy\| &= \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b \{G(t, s)h(s+v-1) \\ &\quad \cdot |f(y(s+v-1))|\} \\ &\geq \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b \min_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} \{G(t, s)h(s+v-1) \\ &\quad \cdot |f(y(s+v-1))|\}, \end{aligned}$$

from the condition 2 of Lemma 4, we get

$$\|Fy\| > \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b \frac{G(s+v-1, s)}{(s+v-1)^{v-1}} h(s+v-1) \cdot \frac{y}{\eta_1},$$

using the definition of cone, we have

$$\begin{aligned} \|Fy\| &\geq \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b \frac{G(s+v-1, s)}{(s+v-1)^{v-1}} h(s+v-1) \cdot \frac{1}{\eta_1} \\ &\quad \cdot \frac{\Gamma(v)}{D(v+b)^{v-1}} \|y\| \\ &\geq \lambda \sum_{s=0}^b \frac{G(s+v-1, s)h(s+v-1)}{D(v+b)^{v-1}(s+v-1)^{v-1}} \cdot \frac{\|y\|}{\eta_1} \\ &> \|y\|. \end{aligned}$$

It implies that $\|Fy\| > \|y\|$ for any $y \in \partial\Omega_1 \cap \mathcal{K}$.

On the other hand, from condition H2, we can select $\eta_2 > 0$ sufficiently large so that both $\max_{b-r_2 \leq y \leq b+r_2} f(y) \leq \frac{y}{\eta_2}$

and $\lambda \sum_{s=0}^b \frac{D(v+b)^{v-1}G(s+v-1, s)h(s+v-1)}{\Gamma(v)(s+v-1)^{v-1}} < \eta_2$ holds for all $t \in [v - 1, v + b]_{\mathbb{N}_{v-1}}$ and $b - r_2 \leq y \leq b + r_2$ where $r_2 := r_2(\eta_2)$ and $0 < r_1 < b + r_2$.

Let $\Omega_2 = \{y \in \mathcal{B} : \|y\| < b + r_2\}$. Then for $y \in \partial\Omega_2 \cap \mathcal{K}$, from the condition 1 of Lemma 4, we have

$$\begin{aligned} \|Fy\| &= \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} \left| \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b G(t, s)h(s+v-1) \right. \\ &\quad \left. \cdot f(y(s+v-1)) \right| \\ &\leq \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b \frac{D(v+b)^{v-1}G(s+v-1, s)}{(s+v-1)^{v-1}} \\ &\quad \cdot h(s+v-1) |f(y(s+v-1))|. \end{aligned}$$

So,

$$\begin{aligned} \|Fy\| &\leq \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b \frac{D(v+b)^{v-1}G(s+v-1, s)}{(s+v-1)^{v-1}} \\ &\quad \cdot h(s+v-1) \frac{\|y\|}{\eta_2} \\ &< \|y\|. \end{aligned}$$

It implies that $\|Fy\| < \|y\|$ for any $y \in \partial\Omega_2 \cap \mathcal{K}$.

Hence, by Theorem 2.8, there exists a function $\bar{y} \in K \cap (\overline{\Omega_2/\Omega_1})$ such that $F\bar{y} = \bar{y}$, where \bar{y} is solution to problem (1.1)–(1.2), which completes the proof. \square

Theorem 3.2. Suppose H3 and H4 holds. For each λ satisfying

$$\frac{1}{\tau L} < \lambda < \frac{1}{\sigma l}, \tag{3.3}$$

or

$$\frac{1}{\tau l} < \lambda < \frac{1}{\sigma L}, \tag{3.4}$$

equation (1.1)–(1.2) has a solution.

Proof. Suppose λ satisfies the condition (3.3). Let $\alpha > 0$ be such that

$$\frac{1}{\tau(L-\alpha)} \leq \lambda \leq \frac{1}{\sigma(l+\alpha)}.$$

By condition H3 there exists $u_1 > 0$ such that $f(y) \leq (l + \alpha) \|y\|$ for $0 < \|y\| \leq u_1$. Hence, for $y \in \mathcal{K}$ and $\|y\| = u_1$, we have

$$\begin{aligned} \|Fy\| &= \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b \{G(t, s)h(s+v-1) \\ &\quad \cdot |f(y(s+v-1))|\} \\ &\leq \lambda(l + \alpha) \max_{t \in [v-1, v+b]_{\mathbb{N}_{v-1}}} \frac{1}{\Gamma(v)} \sum_{s=0}^b \{G(t, s) \\ &\quad \cdot h(s+v-1)\|y\|\} \\ &\leq \lambda(l + \alpha) \cdot \sigma \cdot \|y\| \\ &\leq \frac{1}{\sigma(l + \alpha)} (l + \alpha) \cdot \sigma \cdot u_1 \\ &= u_1 = \|y\|. \end{aligned}$$

Thus, $\|Fy\| \leq \|y\|$ for $y \in \mathcal{K}$ and $\|y\| = u_1$.

Next, by condition H4 there exist $\bar{u}_2 > 0$ such that $f(y) \geq (L - \alpha) \|y\|$ for $\|y\| \geq \bar{u}_2$. Let $u_2 = \max\{2u_1, \bar{u}_2\}$. Then for $y \in \mathcal{K}$ and $\|y\| = u_2$, we have



$$\begin{aligned} \min_{t \in [v-1, v+b]_{\mathbb{N}-1}} Fy(t) &= \min_{t \in [v-1, v+b]_{\mathbb{N}-1}} \frac{\lambda}{\Gamma(v)} \sum_{s=0}^b \{G(t, s) \\ &\quad \cdot h(s+v-1) f(y(s+v-1))\} \\ &= \lambda \min_{t \in [v-1, v+b]_{\mathbb{N}-1}} \frac{1}{\Gamma(v)} \sum_{s=0}^b \{G(t, s) \\ &\quad \cdot h(s+v-1) |f(y(s+v-1))|\} \\ &\geq \lambda (L - \alpha) \cdot \tau \cdot \|y\| \\ &\geq \frac{1}{\tau(L - \alpha)} (L - \alpha) \cdot \tau \cdot u_2 \\ &= u_2 = \|y\|. \end{aligned}$$

Thus, $\|Fy\| \geq \|y\|$ for $y \in \mathcal{K}$ and $\|y\| = u_2$.

Hence, from Theorem 2.8, we see that equation (1.1)–(1.2) have a solution.

The other case is similarly proved. □

4. Example

Example 4.1. Consider the following fractional boundary value problem,

$$\Delta_0^{\frac{13}{10}} y(t) = -\lambda \left(y^2 \left(t + \frac{3}{10} \right) - 5y \left(t + \frac{3}{10} \right) \right) \cdot \log \left(y \left(t + \frac{3}{10} \right) \right), \quad (4.1)$$

$$y \left(-\frac{7}{10} \right) = 0, \quad \Delta y \left(-\frac{7}{10} \right) = \Delta y \left(\frac{53}{10} \right), \quad (4.2)$$

where $v = \frac{13}{10}, b = 5, f(y) = (y^2 - 5y) \log y, t \in [0, 5]_{\mathbb{N}_0}$. We have

$$\lim_{y \rightarrow 0} \frac{f(y)}{y} = \lim_{y \rightarrow 0} \frac{(y^2 - 5y) \log y}{y} = \infty,$$

and

$$\lim_{y \rightarrow 5} \frac{f(y)}{y} = \lim_{y \rightarrow 5} \frac{(y^2 - 5y) \log y}{y} = 0.$$

That is, the conditions of Theorem 3.1 are satisfied, hence the fractional boundary value problem (4.1)–(4.2) has at least one solution.

Example 4.2. Consider the following fractional boundary value problem,

$$\Delta_0^{\frac{3}{2}} y(t) = -\lambda y \left(t + \frac{1}{2} \right) \left\{ 1 + e^{-y(t+\frac{1}{2})} \right\}, \quad (4.3)$$

$$y \left(-\frac{1}{2} \right) = 0, \quad \Delta y \left(-\frac{1}{2} \right) = \Delta y \left(\frac{21}{2} \right), \quad (4.4)$$

where $v = \frac{3}{2}, b = 10, f(y) = y(1 + e^{-y}), t \in [0, 10]_{\mathbb{N}_0}$. We have

$$\lim_{y \rightarrow 0} \frac{f(y)}{y} = \lim_{y \rightarrow 0} \frac{y(1 + e^{-y})}{y} = 2,$$

and

$$\lim_{y \rightarrow \infty} \frac{f(y)}{y} = \lim_{y \rightarrow \infty} \frac{y(1 + e^{-y})}{y} = 1.$$

That is, the conditions of Theorem 3.2 are satisfied, hence the fractional boundary value problem (4.3)–(4.4) has at least one solution.

5. Conclusion

In this paper, we have developed an existence results for the solutions to the class of discrete boundary value problem of fractional difference equations. We obtained the sufficient conditions for the existence of solutions to the problem (1.1)–(1.2). We developed the corresponding Green’s function for the problem (1.1)–(1.2). By means of Krasnosel’skii fixed point theorem we established our main results. Finally, two illustrative examples have given to show the applicability of our main results.

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