



New oscillation criteria for fourth order delay difference equations with damping

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Abstract

The main purpose of this article is to study the oscillatory behavior of solutions of the fourth-order DDE with damping,

$$\Delta^4 u(n) + p(n)\Delta u(n+1) + q(n)u(\sigma(n)) = 0$$

under the assumption that the auxiliary third order difference equation,

$$\Delta^3 z(n) + p(n)z(n+1) = 0$$

is nonoscillatory. In contrast with the existing results, the paper present oscillation of all solutions and simplify the examination process of oscillation. An example is provided to dwell upon the importance of the main result.

Keywords

Fourth-order, delay difference equation(DDE), oscillation, comparison theorem.

AMS Subject Classification

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1. Introduction

This paper concerned with fourth-order damped DDE

$$\Delta^4 u(n) + p(n)\Delta u(n+1) + q(n)u(\sigma(n)) = 0, \quad (1.1)$$

where $n \in \mathbb{N}(n_0) = \{n_0, n_0 + 1, \dots\}$, n_0 is a nonnegative integer, and Δ denotes the difference operator $\Delta u(n) = u(n+1) - u(n)$. Throughout the paper, we assume that

(H₁) $\{p(n)\}$ and $\{q(n)\}$ are positive real sequences for $n \in \mathbb{N}(n_0)$;

(H₂) $\{\sigma(n)\}$ is a monotone nondecreasing sequence of integers with $\sigma(n) \leq n+1$ and $\sigma(n) \rightarrow \infty$ as $n \rightarrow \infty$.

A real sequence $\{u(n)\}$ which satisfies (1.1). For each $n \in \mathbb{N}(n_0)$ one can term a solution of (1.1). Hereafter the term “solution” shall mean a “nontrivial solution”. By the graph of a solution $\{u(n)\}$ one will mean the polygonal path connecting the points $(n, u(n))$, $n \in \mathbb{N}(n_0)$. Any point, where the graph of $\{u(n)\}$ intersects the real axis, is called a node. A solution of (1.1) will be called “oscillatory”, if it has infinitely many nodes; otherwise, it is said to be “nonoscillatory”.

Over the last few decades, there is a great interest in studying the oscillatory and asymptotic behavior of solutions of functional difference equations. An immense body of relevant literature has been devoted to this topic, the reader can refer monographs [1, 2] for many important oscillation results. Among the higher-order difference equations, those of fourth order are generally of considerable practical importance. For example, such equations arise in the mathematical biology, bending of beams, and other areas of science and technology, and hence they are often investigated separately.

As per above statement, the study on oscillation of the fourth order difference equations received considerable portion of attention and some significant results have been ob-

tained. By establishing comparison theorems, Smith [16] Smith and Taylor Jr [17] and Taylor Jr [18] studied extensively the nature and behavior of solutions of a self-adjoint linear fourth order difference equations of the form

$$\Delta^2(a(n)\Delta^2u(n)) \pm p(n)u(n+2) = 0.$$

Therefore, the problem of finding sufficient conditions for oscillatory and asymptotic properties of different classes of two-term fourth order difference equations. For example, delay and neutral delay difference equations has been and still are receiving great attention. This paper shows the reader to [3–6, 9–15, 19–25, 27], and the references contained theorems.

The problem of the oscillation of trinomial difference equations has been studied by several authors using various techniques especially for lower order equations. The motivation for this work stem from the third order difference equation with damping of the form

$$\Delta(a_n\Delta(b_n\Delta u(n))) + p(n)\Delta u(n+1) + q(n)f(u(\sigma(n))) = 0. \tag{1.2}$$

For this equation, the authors used a generalized Riccati technique and summation averaging method established some sufficient conditions which ensure that any solution of (1.2) oscillates or converges to zero, see [7, 8]. The key assumption in the above paper is the existence of nonoscillatory solutions of the auxiliary second order difference equation

$$\Delta(a(n)\Delta v(n)) + \frac{p(n)}{b(n+1)}v(n+1) = 0.$$

Therefore in this paper, it is investigated the oscillatory and asymptotic behavior of the solutions of the fourth-order damped delay difference equation (1.1). Finding oscillation criteria for equation (1.1) is not easy, because the presence of the middle term $p(n)\Delta u(n+1)$ causes the structure of possible nonoscillatory solutions unclear. Similar to the technique used for (1.2), one may reduce the equation (1.1) without the damping term.

This paper offers a new method, which uses a positive solution of an auxiliary difference equation

$$\Delta^3z(n) + p(n)z(n+1) = 0 \tag{1.3}$$

and a positive solution (+ve s) of related second order difference equation in order to obtain its associated binomial form.

This paper is organized as follows: In section 2, the author presents some preliminary lemmas needed to prove the main results. In section 3, the researcher presents some new sufficient conditions for the oscillation of all solutions of (1.1) which involves coefficient $\{p(n)\}$ relating to a damped term and does not depend on solution of the auxiliary difference equations. Finally, an example is given to show the importance of the main result.

2. Preliminary Lemmas

From the familiar discrete Kneser’s theorem [1], the particular case of (1.1), namely

$$\Delta^4u(n) + q(n)u(\sigma(n)) = 0 \tag{2.1}$$

the set S

$$S = S_1 \cup S_3,$$

where

$$u(n) \in S_1 \iff \Delta u(n) > 0, \Delta^2 u(n) < 0, \Delta^3 u(n) > 0, \Delta^4 u(n) < 0,$$

or

$$u(n) \in S_3 \iff \Delta u(n) > 0, \Delta^2 u(n) > 0, \Delta^3 u(n) > 0, \Delta^4 u(n) < 0.$$

In view of the presence of the term $p(n)\Delta u(n+1)$, such an approach cannot be applied to the solution space of (1.1). To overcome the difficulties caused by the presence of the damping term, an associated binomial form of (1.1) is used, that allow us to deduce the result on the sign of $\Delta^i u(n)$, $i = 1, 2, 3, 4$.

The main theorem of this section, as well as the latter ones, relate properties of solutions of (1.1) to those of solutions of an auxiliary third-order linear difference equation (1.3), we summarize its asymptotic properties briefly. If (1.3) is nonoscillatory then it admits a decreasing solution $\{z(n)\}$ satisfying

$$z(n) > 0, \Delta z(n) < 0, \Delta^2 z(n) > 0, \Delta^3 z(n) < 0, \tag{2.2}$$

and increasing solution such that

$$z(n) > 0, \Delta z(n) > 0, \Delta^2 z(n) > 0, \Delta^3 z(n) < 0. \tag{2.3}$$

The formal adjoint equation to (1.3) given by

$$\Delta^3 w(n) - p(n+1)w(n+2) = 0 \tag{2.4}$$

has been important in the study of oscillatory behavior to (1.3). It is known that [16–18] all solutions of (1.3) are nonoscillatory if and only if all solutions of (2.4) so does.

Our next result is based on the linear difference operator

$$D_u = \Delta^4 u(n) + p(n)\Delta u(n+1) \tag{2.5}$$

in terms of a +ve s of (1.3).

Lemma 2.1. *Let $\{z(n)\}$ be a +ve s of (1.3). Then (2.4) can be written as*

$$D_u = \Delta \left(\frac{1}{z(n+1)} \Delta \left(z(n)z(n+1) \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) \right) + \Delta^2 z(n+1) \Delta \left(\frac{\Delta u(n+1)}{z(n+1)} \right). \tag{2.6}$$



Proof. By a simple calculation, one can easily show that the right of (2.6) equals

$$\begin{aligned} & \Delta \left(\frac{1}{z(n+1)} \Delta (z(n) \Delta^2 u(n) - \Delta z(n) \Delta u(n)) \right) \\ & + \Delta^2 z(n+1) \Delta \left(\frac{\Delta u(n+1)}{z(n+1)} \right) \\ & = \Delta \left(\Delta^3 u(n) - \Delta u(n+1) \frac{\Delta^2 z(n)}{z(n+1)} \right) \\ & + \Delta^2 z(n+1) \Delta \left(\frac{\Delta u(n+1)}{z(n+1)} \right) \\ & = \Delta^4 u(n) - \Delta u(n+1) \frac{\Delta^3 z(n)}{z(n+1)}, \end{aligned}$$

which in view of (1.3) leads to

$$D_u = \Delta^4 u(n) + p(n) \Delta u(n+1)$$

and the proof is now complete. \square

Lemma 2.2. Let $\{z(n)\}$ be a +ve s of (1.3) and let the equation

$$\Delta \left(\frac{1}{z(n+1)} \Delta v(n) \right) + \frac{\Delta^2 z(n+1)}{z(n+1)z(n+2)} v(n+1) = 0 \quad (2.7)$$

possess a +ve s. Then (2.4) can be written as

$$D_u = \frac{1}{v(n+1)} \Delta \left[\frac{v(n)v(n+1)}{z(n+1)} \Delta \left(\frac{z(n)z(n+1)}{v(n)} \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) \right] \quad (2.8)$$

Proof. It is straightforward to verify the right hand side of (2.3) equals

$$\begin{aligned} & \frac{1}{v(n+1)} \Delta \left[- \frac{\Delta v(n)}{z(n+1)} z(n)z(n+1) \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right. \\ & \left. + \frac{v(n)}{z(n+1)} \Delta \left(z(n)z(n+1) \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) \right] \\ & = \frac{1}{v(n+1)} \left[- \Delta \left(\frac{\Delta v(n)}{z(n+1)} \right) z(n+1)z(n+2) \right. \\ & \Delta \left(\frac{\Delta u(n+1)}{z(n+1)} \right) \\ & - \frac{\Delta v(n)}{z(n+1)} \Delta \left(z(n)z(n+1) \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) \\ & + \frac{\Delta v(n)}{z(n+1)} \Delta \left(z(n)z(n+1) \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) + v(n+1) \\ & \left. \Delta \left(\frac{1}{z(n+1)} \Delta \left(z(n)z(n+1) \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) \right) \right] \\ & = \Delta \left(\frac{1}{z(n+1)} \Delta \left(z(n)z(n+1) \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) \right) \\ & - \frac{z(n+1)z(n+2)}{v(n+1)} \Delta \left(\frac{\Delta v(n)}{z(n+1)} \right) \Delta \left(\frac{\Delta u(n+1)}{z(n+1)} \right). \end{aligned}$$

Applying (2.6) from Lemma 2.1, the above equality yields

$$\begin{aligned} & = \Delta^4 u(n) + p(n) \Delta u(n+1) - \Delta^2 z(n+1) \Delta \left(\frac{\Delta u(n+1)}{z(n+1)} \right) \\ & - \frac{z(n+1)z(n+2)}{v(n+1)} \Delta \left(\frac{\Delta v(n)}{z(n+1)} \right) \Delta \left(\frac{\Delta u(n+1)}{z(n+1)} \right) \\ & = \Delta^4 u(n) + p(n) \Delta u(n+1) \\ & - \frac{z(n+1)z(n+2)}{v(n+1)} \Delta \left(\frac{\Delta u(n+1)}{z(n+1)} \right) \\ & \left[\Delta \left(\frac{\Delta v(n)}{z(n+1)} \right) + \frac{\Delta^2 z(n+1)}{z(n+1)z(n+2)} v(n+1) \right]. \end{aligned}$$

Since $\{v(n)\}$ is a solution of (2.3), the previous equality yields

$$\begin{aligned} & \frac{1}{v(n+1)} \Delta \left(\frac{v(n)v(n+1)}{z(n+1)} \Delta \left(\frac{z(n)z(n+1)}{v(n)} \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) \right) \\ & = \Delta^4 u(n) + p(n) \Delta u(n+1) = D_u. \end{aligned}$$

This completes the proof. \square

In view of Lemma 2.1 and 2.2, one can rewrite the equation (1.1) in the binomial form

$$\begin{aligned} & \Delta \left(\frac{v(n)v(n+1)}{z(n+1)} \Delta \left(\frac{z(n)z(n+1)}{v(n)} \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) \right) \\ & + v(n+1)q(n)u(\sigma(n)) = 0, \quad (2.9) \end{aligned}$$

where it is assumed that $\{z(n)\}$ and $\{v(n)\}$ are +ve ss of (1.3) and (2.7), respectively. Now, in the next result the criterion for (2.7) to have a +ve s is derived.

Lemma 2.3. If $\{z(n)\}$ is a +ve s of (2.1) and $\{z_*(n)\}$ is any solution of (2.7), then

$$v(n) = z(n) \Delta z_*(n) - z_*(n) \Delta z(n) \quad (2.10)$$

is a solution of (2.7).

Proof. From (2.10), we have

$$\Delta v(n) = z(n+1) \Delta^2 z_*(n) - z_*(n+1) \Delta^2 z(n)$$

and hence

$$\begin{aligned} \Delta \left(\frac{\Delta v(n)}{z(n+1)} \right) & = \Delta^3 z_*(n) - \Delta \left(\frac{z_*(n+1)}{z(n+1)} \right) \Delta^2 z(n+1) \\ & - \frac{z_*(n+1)}{z(n+1)} \Delta^3 z(n). \end{aligned}$$

Since $\{z(n)\}$ and $\{z_*(n)\}$ are solutions of (1.3), we see that

$$\begin{aligned} \Delta \left(\frac{\Delta v(n)}{z(n+1)} \right) & = -p(n)z_*(n+1) + p(n)z_*(n+1) \\ & - \frac{v(n+1)}{z(n+1)z(n+2)} \Delta^2 z(n+1) \end{aligned}$$

or

$$\Delta \left(\frac{\Delta v(n)}{z(n+1)} \right) + \frac{\Delta^2 z(n+1)}{z(n+1)z(n+2)} v(n+1) = 0$$

and the proof is complete. \square



Lemma 2.4. Let (1.3) be nonoscillatory and $\{z(n)\}$ be its +ve decreasing solution. Then (2.7) has a nonoscillatory solution $\{v(n)\}$ such that

$$v(n) > 0, \Delta v(n) > 0, \Delta \left(\frac{1}{z(n+1)} \Delta v(n) \right) < 0. \quad (2.11)$$

Proof. Since (1.3) is nonoscillatory, so it has a +ve increasing solution $\{z_*(n)\}$. By Lemma 2.3, $\{v(n)\}$ given by (2.10) is a +ve s of (2.7). Further $\{v(n)\}$ also satisfies

$$-\Delta z(n+1) \Delta v(n) + z(n+1) \Delta^2 v(n) + v(n+1) \Delta^2 z(n+1) = 0.$$

From the last inequality, we see that

$$v(n+2) \Delta^3 z(n+1) + z(n+2) \Delta^3 v(n) = 0.$$

But from (1.3), we have $p(n+1) = -\frac{\Delta^3 z(n+1)}{z(n+2)}$, and hence $\{v(n)\}$ satisfies the adjoint equation (2.4). Therefore it follows that $\Delta^3 v(n) > 0$ and in view of discrete Kneser’s theorem [1], we conclude that $\Delta v(n) > 0$. This completes the proof. \square

Remark 2.5. One can recall from [26] that condition

$$\sum_{n=n_0}^{\infty} n^2 p(n) < \infty$$

is sufficient for (1.3) to be nonoscillatory.

For our subsequent discussion, it is essential for (2.9) to be in the canonical form, that is, the following conditions

$$\sum_{n=n_0}^{\infty} \frac{z(n+1)}{v(n)v(n+1)} = \infty \quad (2.12)$$

$$\sum_{n=n_0}^{\infty} \frac{v(n)}{z(n)z(n+1)} = \infty \quad (2.13)$$

$$\sum_{n=n_0}^{\infty} z(n) = \infty \quad (2.14)$$

are required to hold.

Lemma 2.6. Let (1.3) be nonoscillatory. Then there exist +ve s $\{z(n)\}$ and $\{v(n)\}$ of (1.3) and (2.7), respectively such that (2.12), (2.13) and (2.14) are hold.

Proof. Assume that $\{z(n)\}$ is a +ve decreasing solution of (1.3). The existence of +ve increasing solution of (2.7) follows from Lemma 2.4. Hence, the authors have $0 < z(n) < c_1$ and $v(n) > c_2 > 0$. Thus condition (2.13) is satisfied. On the other hand, if $\{v(n)\}$ does not satisfy (2.12), then it is easy to see that $v_*(n)$ given by

$$v_*(n) = v(n) \sum_{s=n}^{\infty} \frac{z(s+1)}{v(s)v(s+1)} \quad (2.15)$$

satisfies

$$\begin{aligned} \Delta \left(\frac{1}{z(n+1)} \Delta v_*(n) \right) &= \Delta \left(\frac{\Delta v(n)}{z(n+1)} \right) \sum_{s=n+1}^{\infty} \frac{z(s+1)}{v(s)v(s+1)} \\ &= -\frac{\Delta^2 z(n+1)}{z(n+1)z(n+2)} v(n+1) \\ &\quad \sum_{s=n+1}^{\infty} \frac{z(s+1)}{v(s)v(s+1)} \\ &= -\frac{\Delta^2 z(n+1)}{z(n+1)z(n+2)} v_*(n+1). \end{aligned}$$

Thus $\{v_*(n)\}$ is another +ve s of (2.7). Also, we see that $v_*(n)$ satisfies (2.12). For, let us denote

$$\eta(n) = \sum_{s=n}^{\infty} \frac{z(s+1)}{v(s)v(s+1)}$$

then $\lim_{n \rightarrow \infty} \eta(n) = 0$, and

$$\begin{aligned} \sum_{n=n_0}^{\infty} \frac{z(n+1)}{v_*(n)v_*(n+1)} &= -\sum_{n=n_0}^{\infty} \frac{\Delta \eta(n)}{\eta(n)\eta(n+1)} \\ &= \lim_{n \rightarrow \infty} \left(\frac{1}{\eta(n)} - \frac{1}{\eta(n_0)} \right) = \infty. \end{aligned}$$

Finally, from (2.11) and the last equality implies (2.14). This completes the proof. \square

In view of the canonical representation of (2.9) ensured by Lemma 2.6, one can obtain the following result.

Lemma 2.7. Let (1.3) be nonoscillatory and $\{z(n)\}$ and $\{v(n)\}$ be needed solutions of (1.3) and (2.7), respectively. Assume that $\{u(n)\}$ is a +ve s of (1.1), then either

- (I) $\Delta u(n) > 0, \Delta \left(\frac{1}{z(n)} \Delta u(n) \right) < 0, \Delta \left(\frac{z(n)z(n+1)}{v(n)} \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) > 0$ or
- (II) $\Delta u(n) > 0, \Delta \left(\frac{1}{z(n)} \Delta u(n) \right) > 0, \Delta \left(\frac{z(n)z(n+1)}{v(n)} \Delta \left(\frac{\Delta u(n)}{z(n)} \right) \right) > 0,$

eventually.

We conclude this section with the following lemma.

Lemma 2.8. Let (1.3) be nonoscillatory. Then any +ve s $\{u(n)\}$ of (1.1) satisfies either

$$u(n) \in S_1 \iff \Delta u(n) > 0, \Delta^2 u(n) < 0, \Delta^3 u(n) > 0, \Delta^4 u(n) < 0,$$

or

$$u(n) \in S_3 \iff \Delta u(n) > 0, \Delta^2 u(n) > 0, \Delta^3 u(n) > 0, \Delta^4 u(n) < 0,$$

eventually.

Proof. Let $\{u(n)\}$ be a +ve s of (1.1). Since, we have $\Delta u(n) > 0$, it follows from (1.1) that $\Delta^4 u(n) < 0$. Then the proof follows from the discrete Kneser’s theorem. The proof is now complete. \square



3. Oscillation Results

First we obtain our main results. For simplicity of notation, let us define

$$R_1(n) = (n - n_1), R_2(n) = \frac{(n - n_1)^{(2)}}{2}, \text{ and } R_3(n) = \frac{(n - n_1)^{(3)}}{6},$$

where $(n)^{(j)} = n(n - 1) \dots (n - j + 1)$.

Theorem 3.1. *Let (1.3) be nonoscillatory and let $\{u(n)\}$ be a +ve s of (1.1). If*

- (i) $u(n) \in S_1$, then $\left\{ \frac{u(n)}{R_1(n)} \right\}$ is decreasing,
- (ii) $u(n) \in S_3$, then $\left\{ \frac{u(n)}{R_3(n)} \right\}$ is decreasing and $\Delta u(n) \geq R_2(n)\Delta^3 u(n)$.

Proof. Let $\{u(n)\}$ be a +ve s of (1.1) and $u(n) \in S_1$. From the monotonicity of $\Delta u(n)$, it follows that

$$u(n) > u(n) - u(n_1) = \sum_{s=n_1}^{n-1} \Delta u(s) \geq R_1(n)\Delta u(n).$$

Therefore

$$\Delta \left(\frac{u(n)}{R_1(n)} \right) = \frac{R_1(n)\Delta u(n) - \Delta u(n)}{R_1(n)R_1(n+1)} < 0$$

and part (i) is proved. Next, assume that $u(n) \in S_3$. Since

$$\Delta^2 u(n) = \Delta^2 u(n_1) + \sum_{s=n_1}^{n-1} \Delta^3 u(s) > R_1(n)\Delta^3 u(n)$$

then as before $\frac{\Delta^2 u(n)}{R_1(n)}$ is decreasing. Also

$$\Delta u(n) = \Delta u(n_1) + \sum_{s=n_1}^{n-1} s \frac{\Delta^2 u(s)}{s} \geq \frac{\Delta^2 u(n)}{R_1(n)} R_2(n).$$

In view of the previous inequalities, we see that $\Delta u(n) \geq R_2(n)\Delta^3 u(n)$, and

$$\Delta \left(\frac{\Delta u(n)}{R_2(n)} \right) = \frac{R_2(n)\Delta^2 u(n) - R_1(n)\Delta u(n)}{R_2(n)R_2(n+1)} < 0$$

and we conclude that $\frac{\Delta u(n)}{R_2(n)}$ is decreasing. Also

$$u(n) = u(n_1) + \sum_{s=n_1}^{n-1} R_2(s) \frac{\Delta u(s)}{R_2(s)} > \frac{\Delta u(n)}{R_2(n)} R_3(n),$$

which implies

$$\Delta \left(\frac{u(n)}{R_3(n)} \right) = \frac{R_3(n)\Delta u(n) - R_2(n)u(n)}{R_3(n)R_3(n+1)} < 0.$$

Hence $\left\{ \frac{u(n)}{R_3(n)} \right\}$ is decreasing. This completes the proof. \square

Next, let us define

$$Q(n) = \frac{(\sigma^{-1}(n) - n)^{(2)}}{2} \sum_{s=\sigma^{-1}(n)}^{\infty} q(s).$$

Theorem 3.2. *Let (1.3) be nonoscillatory, and let $\{u(n)\}$ be a +ve s of (1.1). If*

- (i) $u(n) \in S_1$, then $\Delta u(n) \geq Q(n)u(n)$,
- (ii) $u(n) \in S_3$, then $\Delta u(n) \geq \frac{1}{R_1(n)}u(n)$.

Proof. Let $\{u(n)\}$ be a +ve s of (1.1) and $u(n) \in S_1$. For any $j > n$, we have

$$-\Delta^2 u(n) \geq \Delta^2 u(j) - \Delta^2 u(n) - \sum_{s=n}^{j-1} \Delta^3 u(s) \geq \Delta^3 u(j)(j - n)$$

again summing from n to $j - 1$, one obtains

$$\Delta u(n) \geq \sum_{s=n}^{j-1} (j - s)\Delta^3 u(s) \geq \Delta^3 u(j) \frac{(j - n)^{(2)}}{2}. \quad (3.1)$$

On the other hand, summing (1.1) from j to ∞ , yields

$$\begin{aligned} \Delta^3 u(j) &\geq \sum_{s=j}^{\infty} p(s)\Delta u(s+1) + \sum_{s=j}^{\infty} q(s)u(\sigma(s)) \\ &\geq u(\sigma(j)) \sum_{s=j}^{\infty} q(s). \end{aligned} \quad (3.2)$$

Combining (3.1) and (3.2) and letting $j = \sigma^{-1}(n)$, we have

$$\Delta u(n) \geq \left(\frac{(\sigma^{-1}(n) - n)^{(2)}}{2} \sum_{s=\sigma^{-1}(n)}^{\infty} q(s) \right) u(n).$$

This proves part (i). Now assume $u(n) \in S_3$. Since $\Delta u(n)$ is increasing and $\Delta u(n) \rightarrow \infty$ as $n \rightarrow \infty$, we have

$$\begin{aligned} u(n) &= u(n_1) + \sum_{s=n_1}^{n-1} \Delta u(s) \leq u(n_1) + \Delta u(n)(n - n_1) \\ &= u(n_1) - \Delta u(n)(n_1 - n_0) + \Delta u(n)(n - n_0) \\ &\leq \Delta u(n)R_1(n). \end{aligned}$$

This completes the proof. \square

Now, we present a new oscillation equation (1.1). We define

$$Q_1(n) = p(n)Q(n+1) + q(n) \frac{R_1(\sigma(n))}{R_1(n+1)},$$

and

$$Q_2(n) = \frac{p(n)}{R_1(n+1)} + q(n) \frac{R_3(\sigma(n))}{R_3(n+1)}.$$



Theorem 3.3. Let (1.3) be nonoscillatory and there existence +ve nondecreasing real sequence $\{\mu(n)\}$ such that

$$\limsup_{n \rightarrow \infty} \sum_{n=n_0}^n \left(\mu(s) \sum_{t=s}^{\infty} \sum_{j=t}^{\infty} Q_1(j) - \frac{(\Delta\mu(s))^2}{4\mu(s)} \right) = \infty \quad (3.3)$$

and a +ve nondecreasing real sequence $\{\gamma(n)\}$ such that

$$\limsup_{n \rightarrow \infty} \sum_{n=n_0}^n \left(\gamma(s) Q_2(s) - \frac{(\Delta\gamma(s))^2}{4\gamma(s)R_2(s)} \right) = \infty. \quad (3.4)$$

Then (1.1) is oscillatory.

Proof. Let $\{u(n)\}$ be a +ve s of (1.1). Then either $u(n) \in S_1$ or $u(n) \in S_3$. First assume that $u(n) \in S_1$. Then from Theorem 3.1, we have

$$u(\sigma(n)) \geq \frac{R_1(\sigma(n))}{R_1(n+1)} u(n+1).$$

Also from Theorem 3.2, we have

$$\Delta u(n) \geq Q(n)u(n).$$

Using these estimates in (1.1), one obtains

$$\Delta^4 u(n) + Q_1(n)u(n+1) \leq 0.$$

Summing the last inequality from n to ∞ , one gets

$$-\Delta^3 u(n) \geq \sum_{s=n}^{\infty} Q_1(s)u(s+1) \geq u(n+1) \sum_{s=n}^{\infty} Q_1(s). \quad (3.5)$$

Summing (3.5) again, we obtain

$$\Delta^2 u(n) + \left(\sum_{s=n}^{\infty} \sum_{t=s}^{\infty} Q_1(t) \right) u(s+1) \leq 0. \quad (3.6)$$

Now define

$$w(n) = \mu(n) \frac{\Delta u(n)}{u(n)} > 0.$$

Then we easily see that

$$\begin{aligned} \Delta w(n) &= \frac{\Delta\mu(n)}{\mu(n+1)} w(n+1) + \mu(n) \frac{\Delta^2 u(n)}{u(n+1)} \\ &\quad - \mu(n) \frac{(\Delta u(n))^2}{u(n)u(n+1)} \\ &\leq -\mu(n) \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} Q_1(t) + \frac{\Delta\mu(n)}{\mu(n+1)} w(n+1) \\ &\quad - \frac{\mu(n)}{\mu^2(n+1)} w^2(n+1) \\ &\leq -\mu(n) \sum_{s=n}^{\infty} \sum_{t=s}^{\infty} Q_1(t) + \frac{(\Delta\mu(n))^2}{4\mu(n)}. \end{aligned}$$

Summation of the previous inequality yields

$$\sum_{s=n_1}^n \left[\mu(s) \sum_{t=s}^{\infty} \sum_{j=t}^{\infty} Q_1(j) - \frac{(\Delta\mu(s))^2}{4\mu(s)} \right] \leq w(n_1),$$

which contradicts (3.3) as $n \rightarrow \infty$. Next assume $u(n) \in S_3$. From the results of Theorems 3.1 and 3.2, we have

$$y(\sigma(n)) \geq \frac{R_3(\sigma(n))}{R_3(n+1)} u(n+1), \quad \Delta u(n) \geq \frac{1}{R_1(n)} u(n),$$

$$\Delta u(n) \geq R_2(n) \Delta^3 u(n)$$

which in view of (1.1) provides

$$\Delta^4 u(n) + Q_2(n)u(n+1) \leq 0.$$

Now define

$$w_1(n) = \gamma(n) \frac{\Delta^3 u(n)}{u(n)} > 0.$$

Then,

$$\begin{aligned} \Delta w_1(n) &= \frac{\Delta\gamma(n)}{\gamma(n+1)} w_1(n+1) + \gamma(n) \frac{\Delta^4(n)}{u(n+1)} \\ &\quad - \gamma(n) \frac{\Delta^3 u(n) \Delta u(n)}{u(n)u(n+1)} \\ &\leq -\gamma(n) Q_2(n) + \frac{\Delta\gamma(n)}{\gamma(n+1)} w_1(n+1) \\ &\quad - \frac{\gamma(n) R_2(n) w_1^2(n+1)}{\gamma^2(n+1)} \\ &\leq -\gamma(n) Q_2(n) + \frac{(\Delta\gamma(n))^2}{4\gamma(n)R_2(n)}. \end{aligned}$$

Summing last inequality from n_1 to n , we get

$$\sum_{s=n_1}^n \left[\gamma(s) Q_2(s) - \frac{(\Delta\gamma(s))^2}{4\gamma(s)R_2(s)} \right] \leq w_1(n_1)$$

which contradicts with (3.4) as $n \rightarrow \infty$, and the proof is complete. \square

Finally, we conclude this section with an example.

Example 3.4. We consider the damped DDE

$$\Delta^4 u(n) + \frac{a}{n^4} \Delta u(n+1) + \frac{b}{n^4} u(n-k) = 0, \quad (3.7)$$

where $a > 0$ and $b > 0$ and k is a +ve integer. Now (1.3) reduces to

$$\Delta^3 z(n) + \frac{a}{n^4} z(n+1) = 0$$

which is nonoscillatory for all $a > 0$ (see Remark 2.5). A simple calculation shows that $Q(n) \approx \frac{k(k-1)}{6n^3}$, $Q_1(n) \approx \left(\frac{ak(k-1)}{6n^7} + \frac{b}{n^4} \right)$, and $Q_2(n) \approx \frac{a}{n^5} + \frac{b}{n^4}$. By taking $\mu(n) = n$, the condition (3.3) is satisfied if $b > \frac{3}{2}$ and by taking $\gamma(n) = n^3$, the condition (3.4) is satisfied if $b > \frac{9}{4}$. Hence by Theorem 3.3, equation (3.7) is oscillatory if $b > \frac{9}{4}$.



4. Conclusion

The present paper gives conditions for the oscillation of all solutions of the damped fourth-order delay difference equation (1.1). This is achieved by reducing the studied equation into binomial canonical representation. Also, the paper present a new oscillation criteria for (1.1) using Riccati transformation technique and integral averaging method. Thus, the results obtained in this paper are new and complement to the existing literature.

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