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On 2 generated c-spaces

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Abstract
2-generated c-spaces are closely related with finite connective spaces, closure operators and graphs[5, 6]. In this article, we considered the question of whether arbitrary product, sum and quotient of 2-generated c-spaces are 2-generated or not. Further, a connected 2-generated c-space is characterized using $\mathscr{S}_Y$-connectedness.

Keywords
c-space, 2-generated c-spaces, connective spaces.

AMS Subject Classification
54A05, 54D05,05C10, 05C40.

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Contents

1. Introduction

The concept of connectedness plays a vital role in many branches of mathematics, in particular for topology and graph theory. In digital topology, it can be noted that there is no topology on $\mathbb{Z}^2$ for which connected sets are precisely 8-connected sets(graphical) in $\mathbb{Z}^2$ [7] and there is no compatible topology on C_n for $n \geq 5$[2]. Thus there are graphs with no compatible topology. Similarly there are topological spaces whose connectedness cannot be inducted from any graph. For example, $\mathbb{R}$ with discrete topology. Thus there is a lack of compatibility between these two mathematical structures. But connectedness of these two structures must be compatible as any continuous figures can alternately be studied by discretizing them. Thus connected sets and functions that maps connected sets to connected sets comes into the frontier of applied mathematics. For example, the function $f: \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} sin(\frac{1}{x}) & ; \quad \text{If } x \neq 0 \\ 0 & ; \quad \text{Otherwise} \end{cases}$$

maps connected sets to connected sets[5]. But we may note that f is not continuous. In 1983, R. Bö̈rger developed an axiomatic approach to connectedness, so called theory of c-spaces or connectivity classes. Later this theory came into the frontier areas like Link Theory, Knot Theory, Pattern Recognition, Signal Processing and Image Analysis [3, 4, 7, 11, 12]. In this article, we focus on one of the structural aspects of this space, so called 2-generatedness.

2. Preliminaries

All definitions and terminologies in this section are taken from [1, 5, 6, 8, 9] unless specified otherwise. A nonempty set X together with a collection $\mathscr{C}$ of subsets of X which satisfies the following axioms is called a c-space.

(1) $\phi \in \mathscr{C}$ and $\{x\} \in \mathscr{C}$ for every $x \in X$.

(ii) If $\{C_i : i \in I\}$ be a non empty collection of subsets in $\mathscr{C}$ with $\bigcap_{i \in I} C_i \neq \phi$, then $\bigcup_{i \in I} C_i \in \mathscr{C}$.

The collection $\mathscr{C}$ of subsets X which satisfies the above axioms is called a *c-structure*[5] or a *connectivity class* [3, 4, 12] of X. Elements of a c-structure are called *connected sets*.

One trivial example is *indiscrete c-space* where *indiscrete c-structure* is given by $\mathscr{I}_X = \mathscr{P}(X)$, the power set of X. Another example is *discrete c-space*, where discrete c-structure is given by $\mathscr{D}_X = \{\phi\} \cup \{\{x\} : x \in X\}$. Unless otherwise specified, the c-space (X, C_X) is represented by X. For example, in $\mathbb{R}$ as a c-space, connected sets are precisely intervals in $\mathbb{R}$.

Definition 2.1. *Generated c-structure*

Let X be any set and $\mathscr{B} \subseteq \mathscr{P}(X)$. Then the intersection of all c-structures on X containing $\mathscr{B}$ is a c-structure on X and is called the c-structure generated by *$\mathscr{B}$ and is denoted by $<\mathscr{B}>$. Obviously it is the smallest c-structure on X containing $\mathscr{B}$. For any cardinal α with $\alpha \leq |X|$, a c-structure $\mathscr{C}$ on X is said to be α-*generated *if there is a sub collection $\mathscr{B} \subseteq \{A \in \mathscr{C} : |A| \leq \alpha\}$ such that $\mathscr{C} =< \mathscr{B} >$. A c-space is said to be α-generated if its c-structure is α-generated.*

The nontrivial connected sets in a generated structure can be characterized as below.

Proposition 2.2. *Nontrivial connected sets of a c-structure generated by $\mathscr{B}$ are characterized by the condition that any two points of such a connected set C can be joined by a finite chain of basic connected sets(That is, elements of $\mathscr{B}$) in C. That is, for all $x, y \in C$, we can find elements B_i, $i = 0$ to n in $\mathscr{B}$ such that $B_i \subseteq C$, $B_i \cap B_{i+1} \neq \phi$ for $i = 0$ to $n-1$ and $x \in B_0$, $y \in B_n$.*

We also need an alternate formulation of the above proposition.

Theorem 2.3. *Let X be a c-space such that $\mathscr{C}_X =< \mathscr{B} >$, where $\mathscr{B} \subset \mathscr{P}(X)$. Then A is connected in X if and only if $A = \bigcup\limits_{x \in A} E_{x_0 x}$, where $x_0 \in A$ and $E_{x_0 x} = \bigcup\limits_{i=1}^{n_x} A_{x_i}$ such that $x_0 \in A_{x_1}$, $x \in A_{x_{n_x}}$, $A_{x_i} \in \mathscr{B}$ for each i with $A_{x_i} \subset A$ and $A_{x_i} \cap A_{x_{i+1}} \neq \phi$ for $1 \leq i \leq n_x - 1$.*

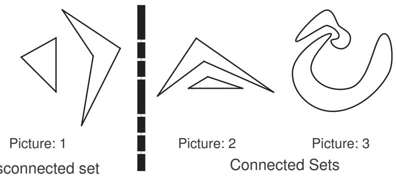
For an illustration, the following are some of the 2-generated c-structures on the set $X = \{1,2,3,4\}$.

a. If $\mathscr{B} = \{\{1,2\},\{3,4\}\}$, then $< \mathscr{B} >= \mathscr{D}_X \cup \{\{1,2\},\{3,4\}\}$.

b. If $\mathscr{B} = \{\{1,2\},\{1,4\}\}$, then $< \mathscr{B} >= \mathscr{D}_X \cup \{\{1,2\}, \{1,4\},\{1,2,4\}\}$

c. If $\mathscr{C}_X = \mathscr{P}(X)$.

A function $f : X \to Y$ is called a *c-continuous* function if it maps connected sets of X to connected sets of Y. Let $\{X_i : i \in I\}$ be a family of c-spaces and $\{f_i : X \to X_i : i \in I\}$ be a family of functions defined on a set X. Let $X = \prod X_i$. Then $\mathscr{C} = \{A \subset X : \pi_i(A) \in \mathscr{C}_{X_i}$ for every $i\}$ is a c-structure on X and X with this c-structure is called the product space of $\{X_i : i \in I\}$. Obviously it is the largest c-structure on X which make each π_i c-continuous. To make the concept more clear, an example of connected sets and a disconnected set from $\mathbb{R}^2$ is given below. A function $f : X \to Y$ is said to be a quotient map if $\mathscr{C}_Y$ is the smallest c-structure on Y which make f c-continuous. In this case, $\mathscr{C}_Y =< \{f(C) : C \in \mathscr{C}_X\} >$. In this case we say that Y is a quotient space of X. Many properties of quotient space can be seen in [9]. Further, if $(X, \mathscr{C}_X)$ be the sum of the the family of c-spaces $\{(X_i, \mathscr{C}_{X_i}) : i \in I\}$, then $\mathscr{C}_X = \bigcup\limits_{i \in I} \{C \times \{i\} : C \in \mathscr{C}_{X_i}\}$.



3. On the Product, Quotients and Sum of 2-generated c-spaces

Relevance of 2-generated c-spaces follows from the results like any finite connective spaces are precisely 2-generated c-spaces, any finite topological c-space is 2-generated and any finite c-space induced from a closure operator is 2-generated [5, 6]. Hence study of 2-generated c-spaces compasses many branches like topology, graph theory and closure spaces. Here we explore some more properties of 2-generated c-spaces.

Theorem 3.1. *[8] If product of a family of c-spaces is non empty, then each co-ordinate space is embeddable in it.*

Proposition 3.2. *2-generatedness is a hereditary property.*

Proof. Let X be a 2-generated c-space and Y a sub c-space of it. Let $x, y \in C$ where C is any connected set in Y. Then C is connected in X. As X is 2-generated, there exists a finite sequence of two element connected sets C_i, $i = 1$ to n in X such that $x \in C_1$, $y \in C_n$, $C_i \cap C_{i+1} \neq \phi$ for $i = 1$ to $n-1$ and $C_i \subseteq C$ for every i. Since $C_i \subseteq C \subseteq Y$ for every i, x and y can be joined by a finite sequence of two element connected sets in Y which are contained in C. Hence C is a 2-generated connected set in Y. Since C is arbitrary, Y is 2-generated. □

Proposition 3.3. *Let $\{X_i : i \in I\}$ be a family of c-spaces and $X = \prod\limits_{i \in I} X_i$. If X is 2-generated, so is each X_i. Converse is not true.*

Proof. Since each X_i can be embedded in X (by Theorem 3.1) and since 2-generatedness is a hereditary property (by Proposition 3.2), each X_i is 2-generated.

For the converse, consider the following 2-generated c-spaces X and Y, where $X = \{1,2,3\}$, $\mathscr{C}_X = \mathscr{D}_X \cup \{\{1,2\},\{2,3\}$, X$\}$, Y=$\{4,5,6\}$ and $\mathscr{C}_Y = \mathscr{D}_Y \cup \{\{4,6\},\{5,6\},Y\}$. We claim that $X \times Y$ is not a 2-generated c-space.

Suppose there exists a collection of 2-element connected sets in $X \times Y$ that generate the product structure on $X \times Y$. Consider a subset $C = \{(3,4),(1,5),(2,4),(3,6)\}$ of $X \times Y$. As $\pi_1(C) = X$ and $\pi_2(C) = Y$, C is connected in $X \times Y$. Then C should be a 2-generated connected set. By definition, there exists a finite chain of 2-element connected sets joining $(1,5)$ and $(3,6)$, that are contained in C. A two element connected set containing $(1,5)$ and contained in C must be one of $\{(1,5),(2,4)\}$, $\{(1,5),(3,4)\}$ and $\{(1,5),(3,6)\}$. But it can be easily checked that none of these are connected in $X \times Y$, a contradiction. Thus C cannot be a 2-generated connected set and hence $X \times Y$ is not a 2-generated c-space. □

Theorem 3.4. *Quotient space of a 2-generated c-space is a 2-generated c-space.*

Proof. Let X be a 2-generated c-space and Y a quotient space of it. Let $f : X \to Y$ be the quotient map. Then $\mathscr{C}_Y =< \{f(C) : C \in \mathscr{C}_X\} >$. Let C be a connected set in Y and let $x, y \in C$. Then there exists a finite sequence of connected sets $C_1, C_2, \dots C_n$ in X such that $x \in f(C_1)$, $y \in f(C_n)$, $f(C_i) \cap f(C_{i+1}) \neq \phi$ for $i = 1$ to $(n-1)$ and $f(C_i) \subseteq C$ for every i. Since $f(C_i) \cap f(C_{i+1}) \neq \phi$, let $k_i \in f(C_i) \cap f(C_{i+1})$ for each $i = 1$ to (n-1).

Since $x, k_1 \in f(C_1)$, there exist elements $c_1, d_1 \in C_1$ such that $f(c_1) = x$ and $f(d_1) = k_1$. Since X is 2-generated, there exists a path $P_1 : c_1 = x_0, x_1, \dots, x_n = d_1$ from c_1 to d_1 in C_1 such that x_i touches x_{i+1} for $i = 0$ to $(n-1)$. Since f is c-continuous, $f(x_i)$ touches $f(x_{i+1})$ for each i. Hence there exists a path $Q_1 : x = f(x_0), f(x_1), \dots, f(x_n) = k_1$ from x to k_1 in $f(C_1)$ such that $f(x_i)$ touches $f(x_{i+1})$.

Similarly there exists a path Q_i from k_{i-1} to k_i in $f(C_i)$ for $i = 2$ to $n-1$ and Q_n in $f(C_n)$ from k_{n-1} to y. Concatenation of these paths Q_i, $i = 1$ to n yield a path Q in $\overset{n}{\underset{i=1}{\cup}} f(C_i)$ from x to y. Since $\overset{n}{\underset{i=1}{\cup}} f(C_i) \subseteq C$, Q is a path in C from x to y by two element connected sets. Since x and y are arbitrary, C is a 2-generated connected set, so that Y is a 2-generated c-space. □

Remark 3.5. *Converse of the above theorem is not true.*

For example, consider the c-spaces $(X, \mathscr{C}_X)$ and $(Y, \mathscr{C}_Y)$ where $X = \{1,2,3,4\}$, $Y = \{a,b,c\}$, $\mathscr{C}_X = \mathscr{D}_X \cup \{\{1,2\},\{3,4\},X\}$ and $\mathscr{C}_Y = \mathscr{D}_Y \cup \{\{a,b\},\{b,c\},Y\}$. We can verify that X is not 2-generated and Y is 2-generated. Define $f : X \to Y$ by $1 \mapsto a$, $2 \mapsto b$, $3 \mapsto b$ and $4 \mapsto c$. A simple verification shows that f is a quotient map. Thus a non 2-generated c-space can have a 2-generated quotient space.

The proof of the following proposition directly follows from the definition of the sum space.

Proposition 3.6. *Let $\{X_i : i \in I\}$ by a family of 2-generated c-spaces. Then the sum space $\sum_{i \in I} X_i$ is also 2-generated.*

4. A characterization of a connected 2-generated c-space

We may note that a c-space is said to be C_1[5] if it contains no two element connected sets. That is, it is a non graphical c-space.

Definition 4.1. *[10]$\mathscr{S}_Y$-connected c-space*

Let Y be a C_1 c-space with at least three elements. A c-space X is said to be $\mathscr{S}_Y$-connected if every c-continuous function $f : X \to Y$ is a constant.

Theorem 4.2. *Let X be a 2-generated c-space. Then X is connected if and only if X is $\mathscr{S}_Y$-connected.*

Proof. Let X be a connected 2-generated c-space. Fix an element x_0 in X. Then by Theorem 2.3, $X = \bigcup_{x \in X} E_{x_0,x}$ where $E_{x_0,x} = \overset{n_x}{\underset{i=1}{\cup}} A_i$ where each A_i is connected, $|A_i| = 2$ for every i and $A_j \cap A_{j+1} \neq \phi$ for $1 \le j \le n_x - 1$.

Let $f : X \to Y$ be any c-continuous function. Then

$$\begin{aligned} f(X) &= \bigcup_{x \in X} f(E_{x_0,x}) \\ &= \bigcup_{x \in X} \bigcup_{i=1}^{n_x} f(A_i) \end{aligned}$$

Since $|A_i| = 2$, $|f(A_i)| \le 2$. $f(A_i)$ being a connected set in Y and since Y is a C_1 c-space, we have $f(A_i) = \{a\}$ for some $a \in Y$. Since $A_i \cap A_{i+1} \neq \phi$, we must have $f(A_{i+1}) = \{a\}$ for every i. Consequently $f(X) = \{a\}$. Hence X is $\mathscr{S}_Y$-connected.

We can easily verify that every $\mathscr{S}_Y$-connected space is connected and hence converse part follows. □

5. Conclusion

c-spaces are mainly used for application based study. Structural properties are less studied and studies in that direction is still young. This paper is an another attempt by us in that direction. Many more properties are to be introduced and studied.

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