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A discrete time $Geo/Geo/1$ inventory system with modified N-policy

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Abstract

This paper considers a discrete-time (s,S) inventory system with positive service and N-policy. The arrival of customers constitutes a Bernoulli process. The system will be on vacation up to N customers and it starts batch service of size N with geometrically distributed service time. The subsequent customers who arrive during the batch service period are served in a single with geometrically distributed service time. The maximum storage of inventory is S. Whenever the on-hand inventory level drops to prefixed level s, an order for replenishment is placed. Lead time is also geometrically distributed. The system is analysed and the stability condition is derived using Matrix Analytic Method. Busy period, waiting-time distribution, reorder time distribution and inter-replenishment time are obtained. Numerical experiments are also incorporated to study system variation of system parameters.

Keywords

Bernoulli process, Discrete time inventory, Geometric distribution, Matrix Analytic Method, N-Policy.

AMS Subject Classification

60K25, 90B05, 91B70.

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1. Introduction

The concept of discrete-time is mainly introduced by Meisling [1] in which time is considered as a discrete variable. Neuts et al. [2] analysed time-dependent behaviour of a discrete-time queue in terms of a bivariate Markov chain and proved continuous-time queue can be considered as the limiting case of the discrete-time queue. Noted inventory work on the discrete-time queue is of Lian et al. [3] in which inventory system with a common life is considered. queueing systems with inventory have studied by several researchers in recent years. A detailed review and direction for some future work on continuous-time inventory models involving positive service time are given in Krishnamoorthy et al. [4]. Krishnamoorthy and Jose [5] studied queues with varying buffer sizes depending on the inventory level. The analysis of the optimal value of queue length for starting service is done by Yadin and Naor [6]. They optimized the queue size for turning the server on, assuming that the server is turned off when the queue is empty. Heyman [7] studied economic behavior of

M/G/1 queues by considering start-up cost, server shutdown cost, holding cost of customer and cost per unit time when the service is going on. The author proved that there is a stationary optimal value for the number of customers to be present to start service. An expression for the exact rate of cost as a function of N and a closed-form expression for the optimum value of N is derived.

Balachandran [8] analysed both N and D policies in a single server queue. These two optimum policies are compared under linear cost and showed that for constant service, these two policies are equivalent and for exponential service, D policy is not superior to N policy. Artelejo [9] analyzed N, T and D policies and compared optimum values with different cost functions based on workload and average queue length. By considering both cost function simultaneously, the author concluded that T policy is the worst policy compared to N and D. Gakis et al. [10] derived an expression for first two moments of busy and vacation periods of M/G/1 queues operating under different policies. In all the policies, they observed that the probability that the server is busy is equal to the traffic intensity of the queueing system. Krishnamoorthy et al. [11] analysed an (s, S) inventory system with N-policy, positive service time and lead time in which local purchase is made when the inventory level falls to some predetermined level. The work in this paper is an extension of the modified N policy of Krishnamoorthy and Deepak [12] in a discrete-time queueing inventory system. In this paper, we use the discrete version of the Matrix Analytic Method (MAM)used in Alfa [13, 14]. For details of MAM, one can refer to Neuts [15].

The rest of the paper is organized as follows. Section 2 provides mathematical modeling and analysis. The stability condition is derived in section 3. Steady-state probability vector and algorithmic analysis are discussed in section 4 and 5 respectively. Some relevant performance measures are included in section 6. Analysis of the busy period is included in section 7. Waiting time distribution is Incorporated in section 8. Section 9 and 10 provide different cases of reordering time distributions and inter replenishment time respectively. Finally, section 11 contains numerical experiments.

2. Mathematical Modelling and Analysis

The following are the assumptions and notations used in this model.

Assumptions

1. Inter-arrival times of customers are geometrically distributed with parameter p.
2. The server will be on vacation until N units accumulate for service. As soon as N unit is arrived the system starts batch service of size N. Arrivals during batch service receive a single service.
3. The single service time and batch service time are geometrically distributed with parameters q_1 and q_2 respectively.
4. No customer is allowed to enter the system when the inventory level is zero.
5. Lead time is geometrically distributed with parameter r.
6. All inventories in the system are removed when the server is on vacation and maximum inventory S is filled up when the system starts batch service.
7. Demand requires at least one unit of service time (late arrival with delayed access).
8. Replenishments take place at the end of slot boundaries.

Notations

$X(n)$: Number of customers in queue at an epoch n

$$Y(n) = \begin{cases} 0; & \text{server is in vacation} \\ 1; & \text{server busy with single customer} \\ 2; & \text{server busy with batch} \end{cases}$$

$I(n)$: Inventory level at the epoch n

Then $\{(X(n),Y(n),I(n)); n = 0,1,2,3,..\}$ is in the structure of a $GI/M/1$ queueing system with state space $\{(0,0,0)\} \cup \{(i,1,k); i \geq 1, 0 \leq k \leq S\} \cup \{(i,0,0); 1 \leq i \leq N-1\} \cup \{(i,2,S); i \geq N\}$. We arrange the state space as $(0,0,0),(1,0,0),(1,1,0),(1,1,1),\ldots,(1,1,S),\ldots,(N-1,0,0),(N-1,1,0),\ldots,(N-1,1,S),(N,2,0),(N,1,0),\ldots,(N,1,S),\ldots$ Now the transition probability matrix of the process is given by,

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ \vdots \\ N-1 \\ N \\ N+1 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccccccc} 0 & 1 & 2 & 3 & \ldots & N-2 & N-1 & N & N+1 & \ldots & \ldots \end{array} \\ \begin{pmatrix} D_1 & D_0 & 0 & & & & & & & & \\ B_2 & B_1 & B_0 & & & & & & & & \\ 0 & A_2 & B_1 & B_0 & & & & & & & \\ & & & & & & & & & & \\ & \vdots & \vdots & \ddots & \ddots & \ddots & & & & & \\ & & & & & & & & & & \\ & & & & . & A_2 & B_1 & B_0 & & & \\ K & A_N & & & & & A_2 & A_1 & A_0 & & \\ & A_{N+1} & A_N & & & & & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & & & & & \ddots & \ddots & \ddots \end{pmatrix} \end{array}$$

where the blocks $B_0, A_0, A_{1,i}$ and $A_{2,i} (i \geq 0)$ are given by

$$D_0 = \begin{pmatrix} p & 0 & \ldots & 0 \end{pmatrix}$$

$$D_1 = \begin{pmatrix} \bar{p} \end{pmatrix}$$

$$B_0 = \begin{pmatrix} p & & & & & & 0 \\ & 0 & & & & & 0 \\ & & p\bar{q}_1\bar{r} & & & & p\bar{q}_1 r \\ & & & \ddots & & & \vdots \\ & & & & p\bar{q}_1\bar{r} & & p\bar{q}_1 r \\ & & & & & p\bar{q}_1 & \\ & & & & & \ddots & \\ & & & & & & p\bar{q}_1 \end{pmatrix},$$

$$K = \begin{pmatrix} \bar{p}q_2 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 \\ 0 \\ \bar{p}q_1 \\ \bar{p}q_1 \\ \vdots \\ \bar{p}q_1 \end{pmatrix},$$

$$B_1 = \begin{pmatrix} \bar{p} & & & & & & & 0 \\ & \bar{r} & & & & & & r \\ & pq_1\bar{r} & \bar{p}\bar{q}_1\bar{r} & & & & & \bar{p}\bar{q}_1 r + pq_1 r \\ & & \ddots & \ddots & & & & \vdots \\ & & & pq_1\bar{r} & p\bar{q}_1\bar{r} & & & p\bar{q}_1 r + pq_1 r \\ & & & & pq_1 & p\bar{q}_1 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & pq_1 & p\bar{q}_1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & & & & & & & 0 \\ & 0 & & & & & & \\ & \bar{p}q_1\bar{r} & 0 & & & & & \bar{p}q_1 r \\ & & \ddots & \ddots & & & & \vdots \\ & & & \bar{p}q_1\bar{r} & 0 & & & \bar{p}q_1 r \\ & & & & \bar{p}q_1 & 0 & & \\ & & & & & \ddots & \ddots & \\ & & & & & & \bar{p}q_1 & 0 \end{pmatrix}$$

$$[A_0]_{ij} = \begin{cases} p\bar{q}_2, & \text{for } i = j = 1 \\ [B_0]_{ij}, & \text{otherwise} \end{cases}$$

$$[A_1]_{ij} = \begin{cases} \bar{p}\bar{q}_2 & \text{for i=j=1} \\ [B_1]_{ij} & otherwise \end{cases}$$

$$[A_N]_{ij} = \begin{cases} pq_2 & \text{for i=1,j=S-N+2} \\ 0 & otherwise \end{cases}$$

$$[A_{N+1}]_{ij} = \begin{cases} \bar{p}q_2 & \text{for i=1,j=S-N+2} \\ 0 & otherwise \end{cases}$$

3. Stability Condition

Theorem 3.1. *The system under consideration is stable if and only if $p < q_1$.*

Proof. Consider the matrix $A = A_0 + A_1 + A_2 + A_N + A_{N+1}$. Then

$$[A]_{ij} = \begin{cases} \bar{q}_2 & \text{for i=j=1} \\ q_2 & \text{for i=1,j=S-N+2} \\ \bar{r} & \text{for i=j=2} \\ r & \text{for } 2 \le i \le s+2, j = S+2 \\ \bar{q}\bar{r} & \text{for i = j and } 2 \le i \le s+2 \\ q\bar{r} & \text{for j = i-1 and} 2 \le i \le s+2 \\ \bar{q} & \text{for i=j and s+2} \le i \le S+2 \\ q & \text{for j=i-1 and } s+2 \le i \le S+2 \end{cases}$$

Let π be the steady state probability vector of A. Then, the system is stable if and only if $\pi A_0 e < \pi A_2 e$. On simplification we get the condition $p < q_1$. □

4. Steady-State Analysis

Let $\mathbf{x} = (x_0, x_1, \dots, x_N, x_{N+1}, \dots)$ be the steady state probability vector of Q. Under the stability condition, $x_i' s$ are given by

$$x_{N+r} = x_N R^r (r \ge 1)$$

where R is the minimal non-negative solution of the equation

$$R^{N+1}A_{N+1} + R^N A_N + R^2 A_2 + RA_1 + A_0 = R$$

for which the spectral radius is less than 1 and the vectors $x_0, x_1, \dots, x_N$ are obtained by solving

$$\left.\begin{aligned} & x_0 D_1 + x_1 B_2 + X_N K = x_0 \\ & x_0 D_0 + x_1 B_1 + x_2 A_2 + x_N (A_N + RA_{N+1}) = x_1 \\ & \text{for } 2 \le i \le N-1 \\ & x_{i-1}B_0 + x_i B_1 + x_{i+1}A_2 + x_N R^{i-1}(A_N + RA_{N+1}) = x_i \\ & x_{N-1}B_0 + x_N (A_1 + RA_2 + R^{N-1}A_N + R^N A_{N+1}) = x_N \end{aligned}\right\} \quad (4.1)$$

subject to the normalizing condition

$$\left[\sum_{i=0}^{N-1} x_i + x_N (I-R)^{-1}\right] \mathbf{e} = 1 \qquad (4.2)$$

5. Algorithmic Analysis

5.1 Evaluation of the Rate Matrix R

The rate matrix R is given by $R = \lim_{n\to\infty} R_n$, where $R_{n+1} = (R_n^{N+1}A_{N+1} + R_n^N A_N + R_n^2 A_2 + A_0)(I - A_1)^{-1}$ and $R_0 = \mathbf{0}$, The iteration is usually terminated when $|\ (R_{n+1} - R_n)\ |_{ij} < \varepsilon \forall\ i, j$.

5.2 Computation of Boundary Probabilities

Now the system (1) can be solved using the block Gauss-Seidel iterative method. The vectors $x_0, x_1, \dots, x_N$ in the $(n+1)$th iteration are given by

$$\begin{aligned} x_0(n+1) =& (x_1(n)B_2 + x_N(n)K)(I - C_1)^{-1} \\ x_1(n+1) =& [x_0(n+1)D_0 + x_2(n-1)A_2 + x_N(n)(A_N \\ & + RA_{N+1})](I - B_1)^{-1} \\ & \text{for} 2 \le i \le N-1 \\ x_i(n+1) =& [x_{i-1}(n+1)B_0 + x_{i+1}(n)A_2 + x_N(n)(A_N \\ & + R^{i-1}A_{N+1})](I - B_1)^{-1} \\ x_N(n+1) =& x_{N-1}(n+1)B_0(I - A_1 - RA_2 + R^{N-1}A_N \\ & + R^N A_{N+1})^{-1} \end{aligned}$$

Each iteration is subject to the normalizing condition 4.2.

6. Performance Measures

Assume $x_i = (y_i^*, y_{i,0}, y_{i,1}, \ldots, y_{i,S}); i \geq 1$
Where

$$y_i^* = \begin{cases} P(X = i, Y = 0, I = 0), & \text{if } 1 \leq i < N \\ P(X = i, Y = 2, I = S), & \text{otherwise} \end{cases} \tag{6.1}$$

$y_{i,j} = P(X = i, Y = 1, I = j)$ Some important performance measures of the system under steady state are the following.

1. Expected inventory level, EIL, is given by

$$EIL = \sum_{j=1}^{S} \sum_{i=1}^{\infty} j y_{i,j} + S \sum_{i=1}^{\infty} y_i^*$$

2. Expected number of customers in system, EC, is given by

$$EC = \sum_{j=1}^{S} \sum_{i=0}^{\infty} i y_{i,j} + \sum_{i=1}^{\infty} i y_i^*$$

3. Expected balance inventory, EBI, is given by

$$EBI = q_1 \sum_{j=1}^{S} (j-1) y_{1,j}$$

4. Expected loss rate of customers, ELC, is given by

$$ELC = p \sum_{i=1}^{\infty} y_{i,0}$$

5. Expected Reorder rate,ERR,is given by

$$ERR = q_1 \sum_{i=1}^{\infty} y_{i,s+1}$$

6. Expected rate of departure after completing service, EDS, is given by

$$EDS = q_1 \sum_{i=1}^{\infty} \sum_{j=1}^{S} y_{i,j} + Nq \sum_{i=N}^{\infty} y_i^*$$

7. Expected replenishment rate, ERR, is given by

$$ERR = r \sum_{i=1}^{\infty} \sum_{j=0}^{s} y_{i,j}$$

8. Probability that the inventory level is zero is

$$x_0 + \sum_{i=1}^{\infty} y_{i,0} + \sum_{i=1}^{N-1} y_i^*$$

9. Expected number of customers waiting in the system for single service when the inventory level is zero, $EW0$, is given by

$$EW0 = \sum_{i=1}^{\infty} i y_{i,0}$$

10. Expected number of customers waiting for batch service, ECB, is given by

$$ECB = \sum_{i=1}^{N-1} i y_i^*$$

11. Expected number of customer waiting in the system for single service, ECQ, is given by

$$ECQ = \sum_{i=1}^{\infty} \sum_{j=0}^{S} (i-1) y_{i,j} + \sum_{i=N}^{\infty} (i-N) y_i^*$$

7. Busy Period

The busy period is initiated when N^{th} customer arrives and the system starts batch service. The period ends when the system becomes empty. Let B denote the duration of busy period, then it can be studied using matrix analytic method.

Consider the discrete time GI/M/1 system obtained by modifying it at boundary state $(0,0,0)$ as the absorbing state. Here, Markov chain starts at the state $(N,2,S)$. The transition probability matrix P' is defined as

$$P' = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \\ N-1 \\ N \\ N+1 \\ \vdots \\ 2N-2 \\ 2N-1 \\ 2N \\ \vdots \end{array} \begin{array}{ccccccccccc} 0 & 1 & 2 & 3 & \ldots & N-2 & N-1 & N & N+1 & \ldots & . \\ 1 & & & & & & & & & & \\ B_2^* & B_1^* & B_0^* & & & & & & & & \\ 0 & A_2^* & B_1^* & B_0^* & & & & & & & \\ & & & & & & & & & & \\ & & & \ddots & \ddots & \ddots & & & & & \\ & & & & . & A_2^* & B_1^* & B_0^{**} & & & \\ K & A_N^{**} & & & & & A_2^{**} & A_1 & A_0 & & \\ & A_{N+1}^{**} & A_N^{**} & & & & & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & & & & & \ddots & \ddots & \ddots \\ & & & & & A_{N+1}^{**} & A_N^{**} & & & & \\ & & & & & & A_{N+1}^{**} & A_N & & & \\ & & & & & & & A_{N+1} & A_N & & \\ & & & & & & & & \ddots & \ddots & \end{array}$$

where B_2^* is obtained from B_2 by deleting the first row, B_i^* for $i = 0, 1$ and A_2^* are respectively obtained from B_i and A_2 by deleting the first row and column. A_i^{**} and B_0^{**} are obtained from A_i and B_0 by deleting first column and row respectively. If $x^{(n)} = (x_0^{(n)}, x_1^{(n)}, x_2^{(n)}, \ldots)$ is the state of the system at time n, $x^{(0)} = (0, ..., 0, 1, 0, ..)$ with 1 in the $[2 + N(S+1)]^{th}$ position and remaining entries are zeros. Using the recursion formula,

$$x^{(n+1)} = x^{(n)} P'$$

$x_0^{(n)}$ is the probability that the busy period last for n or less units of time.
Let $F(n) = x_0^{(n)} = Pro(B \leq n)$. This can be solved by rectangular iteration method introduced by Shi et al. [16]. This is illustrated as follows.

Let P_n be the top left sub matrix of P' of order $[n(S+1)+1] \times [(n+1)(S+1)+1]$ for $0 \leq n \leq N-2$ and of order $[1 + (N-1)(S+1) + (n-N+1)(S+2)] \times [1 + (N-1)(S+1) +$

$(n-N+2)(S+2)]$ for $n \geq N-1$. Let $x^{*(n)} = (x_0^{(n)}, \dots, x_{n+1}^{(n)})$, then

$$x^{*(n+1)} = x^{*(n)}P', n \geq 0.$$

From this, $x^n_{(0)}$ can be calculated. Expected busy period, EBP, can be calculated using the formula

$$EBP = \sum_n n(F(n) - F(n-1))$$

Expected busy cycle, EBC, is given by,

$$EBC = EBP + N/p$$

8. Waiting-Time distribution

Case 1: Suppose the arriving customer find the V_n customers in the system (including himself) waiting for batch service. Then, he has to wait for service until the arrival of N-V_n customers. This waiting time follows a discrete phase-type distribution with transition probability matrix

$$P(W) = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & \dots & N-1 & N \end{array} \\ \begin{pmatrix} \bar{p} & p & & & & \\ & \bar{p} & p & & & \\ & & & \ddots & \ddots & \\ & & & & \bar{p} & p \\ & & & & & 1 \end{pmatrix} \end{array}$$

Case 2: Suppose the arriving customers finds that server is doing the batch service. Consider the absorbing Markov chain $(W_n, Y(n), I(n)), n \geq 0$ having state space $\{(0,0,0)\} \cup \{(i,1,k), 1 \leq i \leq N-1, 0 \leq k \leq S\} \cup \{(i,j,k), i \geq N, j = 1,2, 0 \leq k \leq S\}$ in which W_i represent number of customers ahead of the arriving customer. Here, we order the state space as $(0,0,0),(1,1,0),(1,1,1),\dots,(1,1,S),\dots,(N-1,1,0),\dots,$ $(N-1,1,S),(N,2,0),(N,1,0),\dots,(N,1,S),\dots$. We assume that the queue follows the discipline FCFS . The transition probability matrix $P(W)$ is given by

$$P(W) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ N-1 \\ N \\ N+1 \\ \vdots \\ 2N-1 \\ 2N \\ \vdots \end{array}\begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & \dots & N-1 & N & N+1 & \dots \end{array} \\ \begin{pmatrix} 1 & & & & & & & & & & \\ t_2 & T_1^* & & & & & & & & & \\ & T_2^* & T_1^* & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & T_2^* & T_1 & & & & & & \\ t_n & & & & T_2^{**} & T_1 & & & & & \\ & T_n^{**} & & & & T_2 & T_1 & & & & \\ & & \ddots & & & & & & & & \\ & & & & T_n^{**} & & & T_2 & T_1 & & \\ & & & & & T_n & & & T_2 & T_1 & \\ & & & & & & \ddots & & & \ddots & \ddots \end{pmatrix} \end{array}$$

where

$$T_1 = \begin{pmatrix} \bar{q}_2 & & & & & & & 0 \\ & \bar{r} & & & & & & r \\ & & \bar{q}_1\bar{r} & & & & & \bar{q}_1 r \\ & & & \ddots & & & & \vdots \\ & & & & \bar{q}_1\bar{r} & & & \bar{q}_1 r \\ & & & & & \bar{q}_1 & & \\ & & & & & & \ddots & \\ & & & & & & & \bar{q}_1 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} 0 & & & & & & & 0 \\ 0 & & & & & & & 0 \\ & q_1\bar{r} & & & & & & q_1 r \\ & & \ddots & & & & & \vdots \\ & & & q_1\bar{r} & & & & \bar{q}_1 r \\ & & & & q_1 & & & \\ & & & & & \ddots & & \\ & & & & & & q_1 & 0 \end{pmatrix}$$

$$t_n = \begin{pmatrix} q_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{S+2\times 1}, t_2 = \begin{pmatrix} 0 \\ q_1 \\ \vdots \\ q_1 \end{pmatrix}_{S+1\times 1} \quad \text{and}$$

$$[T_N]_{ij} = \begin{cases} q_2, & \text{for } i = 1, j = S-N+2 \\ 0, & \text{otherwise} \end{cases}$$

T_i^* is obtained by deleting first row and column from T_i and T_i^{**} is obtained by deleting first column from T_i.

Let $y^{(0)}$ be the state of the system when a customer arrives. Consider the recursive formula

$$y^{(n+1)} = y^{(n)}P(W), n \geq 0$$

Then, we have

$$P(W \leq n) = y_0^{(n)}, \text{ the first co-ordinate of } y^{(n)}$$

9. Reorder Time Distribution

Case:1. Reorder time just after vacation Period

After the vacation period, the system starts service only after accumulating N customers in the system. During the batch service of these customers, if no arrival occurs the system will become a vacation and subsequently, all the items in the inventory are removed from the system. Therefore, no reorder takes place in this situation. On the other hand, if there is at least one customer in the system during batch service, then the system starts a single service. Consequently, a replenishment order is placed when the inventory level is s. Consider the discrete-time Markov chain$\{I(n)); n \geq 0\}$ having space $\{0', s, s+1, s+2, \dots, S-N, S\}$ with $\{0', s\}$ as an absorbing states, where $0'$ represent state of vacation. Then transition probability matrix P_1 is given by

$$P_1 = \begin{array}{c} \\ 0' \\ s \\ s+1 \\ \vdots \\ S-N-1 \\ S-N \\ S \end{array} \begin{array}{c} \begin{array}{ccccccc} 0' & s & s+1 & \ldots & S-N-1 & S-N & S \end{array} \\ \left(\begin{array}{ccccccc} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ q_1 y_0^{(s)} & q_1 y_1^{(s)} & \bar{q}_1 & & & & \\ & & & \ddots & & & \\ & & & & & & \\ q_1 y_0^{(S-N-1)} & & & & q_1 y_1^{(S-N-1)} & \bar{q}_1 & \\ q_2 y_0^{(S-N)} & & & & & q_2 y_1^{(S-N)} & \bar{q}_2 \end{array} \right) \end{array}$$

Denote $P_1 = \begin{pmatrix} I & 0 \\ H_1 & Q_1 \end{pmatrix}$, where H_1 and Q_1 are respective block matrices. Let $\mathbf{a} = (0,0,\ldots,1)$, where 1 in $(S-N+2)^{th}$ position and remaining entries are zeros. Let T' be the reorder time, then

$$P(T'=n) = \mathbf{a}Q_1^{n-1}H_1(2), \text{ second coordinate of } \mathbf{a}Q_1^{n-1}H_1 .$$

On assuming the occurrence of reorder, expected reorder time, $E(T') = \mathbf{a}D(2)$, where

$$\begin{aligned} D &= (I + 2Q_1 + 3Q_1^2 + 4Q_1^3 + \cdots)H_1 \\ &= (1-Q_1)^{-2}H_1 \end{aligned}$$

Case:2. Reorder time just after the subsequent replenishment

In this case we consider the reorder time after the first replenishment. During the single service of the arriving customers, whenever the queue becomes empty, the balance items in the inventory are removed from the system and hence system will be in vacation; otherwise the single service will continue. Consider the discrete time Markov chain$\{I(n));n \geq 0\}$ having space $\{0,s,s+1,s+2,...,S\}$ with $\{0',s\}$ as a set of absorbing states, transition probability matrix P_2 is given by

$$P_2 = \begin{array}{c} \\ 0' \\ s \\ s+1 \\ \vdots \\ S-2 \\ S-1 \\ S \end{array} \begin{array}{c} \begin{array}{ccccccc} 0' & s & s+1 & \ldots & S-2 & S-1 & S \end{array} \\ \left(\begin{array}{ccccccc} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ q_1 y_0^{(s)} & q_1 y_1^{(s)} & \bar{q}_1 & & & & \\ & & & \ddots & & & \\ & & & & & & \\ q_1 y_0^{(S-N-1)} & & & & q_1 y_1^{(S-N-1)} & \bar{q}_1 & \\ q_2 y_0^{(S-N)} & & & & & q_1 y_1^{(S-N)} & \bar{q}_1 \end{array} \right) \end{array}$$

Denote $P_2 = \begin{pmatrix} I & 0 \\ H_2 & Q_2 \end{pmatrix}$, as in the previous case.

Then $P(T'=n) = aQ_2^{n-1}H_2(2)$

On assuming the occurrence of reorder, we have $E(T') = \mathbf{a}(1-Q_2)^{-2}H_2(2)$, where $\mathbf{a}$ is a $S+1$ row vector with the last entry one and remaining entries are zeros.

10. Inter Replenishment Time

Consider the time for replenishment after placing an order. This can be defined as the time taken to become the inventory level S from any of the inventory levels $\{0,1,2\ldots,s\}$, without the system become in vacation. Consider the Markov chain $\{I(n);n \geq 0\}$ having space $\{0',0,1,2,\ldots,s\}$ with $\{0',S\}$ as absorbing state. Here $0'$ represents the state at which the system is in vacation. The transition probability matrix, P_3, is given by

$$P_3 = \begin{array}{c} \\ 0' \\ S \\ 0 \\ 1 \\ \vdots \\ s-1 \\ s \end{array} \begin{array}{c} \begin{array}{ccccccc} 0' & S & 0 & 1 & \ldots & s-1 & s \end{array} \\ \left(\begin{array}{ccccccc} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ 0 & r & \bar{r} & & & & \\ q_1 y_0^{(0)} & q_1 r y_1^{(0)} + \bar{q}_1 r & q_1 \bar{r} y_1^{(0)} & \bar{q}_1 \bar{r} & & & \\ & & & & \ddots & & \\ q_1 y_0^{(s-2)} & q_1 r y_1^{(s-2)} + \bar{q}_1 r & & & q_1 \bar{r} y_1^{(s-2)} & \bar{q}_1 \bar{r} & \\ q_1 y_0^{(s-1)} & q_1 r y_1^{(s-1)} + \bar{q}_1 r & & & & q_1 \bar{r} y_1^{(s-1)} & \bar{q}_1 \bar{r} \end{array} \right) \end{array}$$

where $y_1^{(j)}$ and $y_0^{(j)}$ are respectively the probabilities that the system contains at least one customer and no customer when the inventory level is j given that single service is going on in the system.

Now, P_3 has the block form given by, $P_3 = \begin{pmatrix} I & 0 \\ H_3 & Q_3 \end{pmatrix}$. Let T'' be the replenishment time after reorder time, then $P(T''=n) = aQ_3^{n-1}H_3(2)$.

On assuming the happening of replenishment, we have $E(T'') = \mathbf{a}(1-Q_3)^{-2}H_3(2)$, $\mathbf{a} = (0,0,\ldots,1)_{1\times(s+2)}$, as in the earlier case.

Now from the above cases, $T' + T''$ will give the inter replenishment time.

11. Numerical Experiments

11.1 Cost Function

Define the expected total cost of the system per unit time as

$$\begin{aligned} ETC =& c_0 ERR + \sum_{j=1}^{s} r(S-j) y_j c_1 + c_2 EIL + c_3 ELC \\ &+ c_4 EW0 + c_5 ECQ + \\ &\frac{c_6}{EBC}[(N-1)/p + (N-1)/p ... + 1/p] + c_7 EDS \end{aligned}$$

where, y_j is the probability that the inventory level is j during single service mode and c_0= The setup cost/order, c_1= Procurement cost/ unit/unit time, c_2= The inventory holding cost/unit/unit time, c_3=Cost due to loss of customers/unit/unit time, c_4= Holding cost of a customer /unit time when inventory level is zero, c_5= Holding cost of a customer /unit time when the server is busy, c_6= Cost towards waiting/unit/unit time until server starts after vacation period, c_7= Service cost/unit/unit time.

11.2 Table/Graphical Illustrations

11.3 Interpretations

Table(1) shows that as q_1 increases expected inter replenishment time decreases when we keep all other parameters constant. Figure 1, 2 and 3 represent the variation of ETC with p, r and q_1 respectively by considering all individual costs as unity and the other parameters fixed. From these graphs, the value of the minimum ETC can be calculated and the corresponding value of the parameter can also be evaluated. In figure 3, ETC has some sort of stagnation after $q = 0.7$. Hence, in this case, the optimum value of ETC is attained at

$S = 10; s = 3; N = 5; c_i = 1, \text{for } 0 \leq i \leq 7; q_1 = 0.7; q_2 = 0.3; r = 0.2$

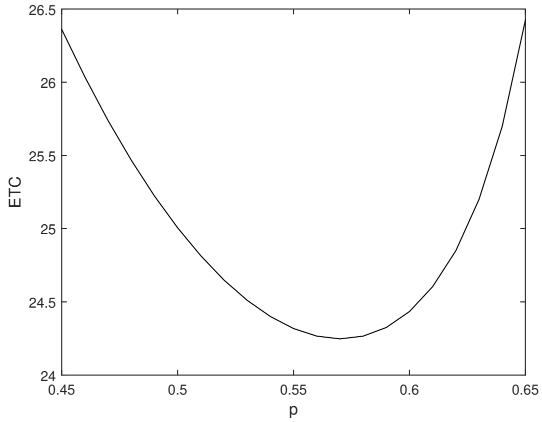

Figure 1. Variation of expected total cost with p

$S = 10; s = 3; N = 5; c_i = 1, \text{for } 0 \leq i \leq 7; q_1 = 0.7; q_2 = 0.3; p = 0.57$

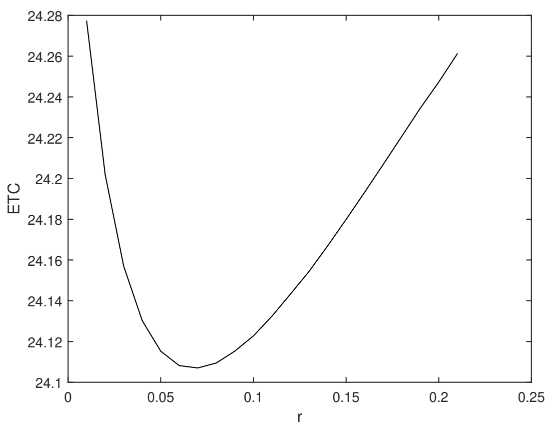

Figure 2. Variation of expected total cost with r

$S = 10; s = 3; N = 5; c_i = 1, \text{for } 0 \leq i \leq 7; p = 0.2; q_2 = 0.3; r = 0.2$

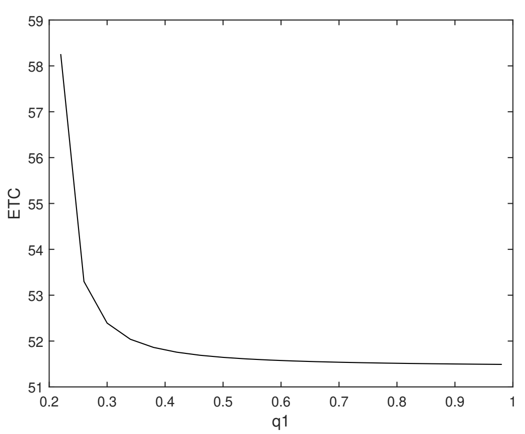

Figure 3. Variation of expected total cost with q_1

$S = 6; s = 1; N = 5; q_1 = 0.7; q_2 = .3; r = 0.1; p = 0.3$

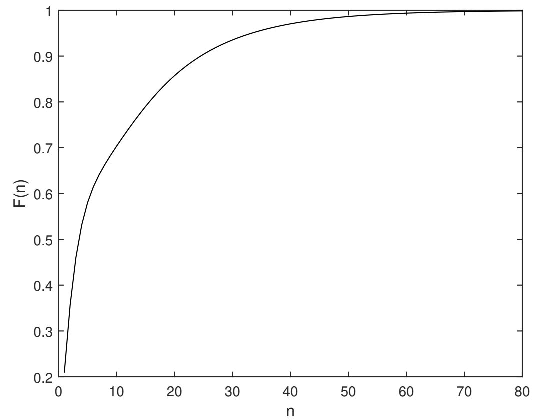

Figure 4. The distribution of Busy Period

$S = 6; s = 1; N = 5; q_1 = 0.7; q_2 = .3; r = .1$

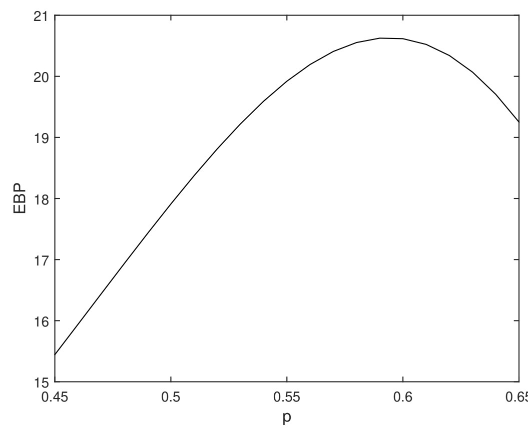

Figure 5. Variation of Expected Busy Period with p

$S = 6; s = 1; N = 5; q_1 = 0.7; q_2 = 0.3; p = 0.3$

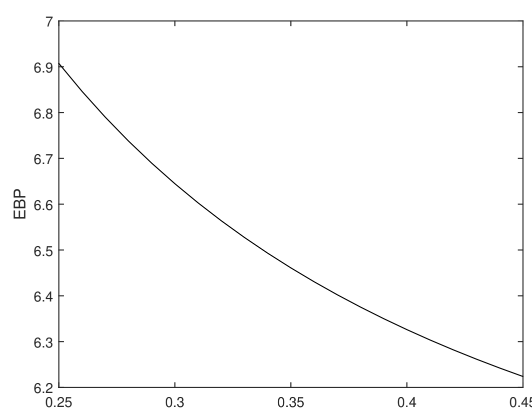

Figure 6. Variation of Expected Busy Period with r

q_1	F_{IRT}	S_{IRT}
0.70	5.5110	11.4040
0.72	5.3980	11.0920
0.74	5.2906	10.7970
0.76	5.1884	10.5170
0.78	5.0910	10.2510
0.80	4.9981	9.9981
0.82	4.9093	9.7569
0.84	4.8244	9.5268

Table 1. Effect of q_1 on Expected Inter-Replenishment times when $S = 10; s = 3; N = 5; p = 0.5; q_2 = 0.8; r = 0.5$, F_{IRT}= First Inter Replenishment Time, S_{IRT}= Subsequent Inter Replenishment Time

$S = 6; s = 1; N = 5; q_2 = 0.3; r = 0.1; p = 0.3$

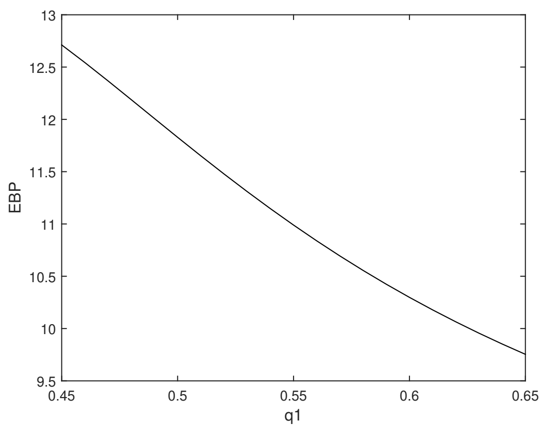

Figure 7. Variation of Expected Busy Period with q_1

this point. The figures 4, 5, 6 and 7 illustrate the distribution of busy period and the changes of expected length of busy period with p, r and q_1 respectively. From the graph, we can observe that the expected busy period decreases with an increase of r and decreases with an increase of q_1. But the expected busy period initially increases with p to a certain extent then it decreases with a further increase of p. This is due to the loss of customers in the absence of inventory.

Concluding Remarks

In this article, we studied a discrete-time (s,S) inventory system with an N policy. The primary demand constituted a geometric process. Assumptions made on service time, lead-time are geometric distribution. Stability conditions and important system performance measures are derived. Numerical experiments are carried out based on a suitable cost function. Busy periods, waiting-time distribution, re-order time distribution and inter replenishment time are obtained. For future study, one can extend the present study to another one by considering Phase type distribution instead of geometric distribution and T policy along with modified N-policy.

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