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An annotation on the prime graph of an integral domain

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Abstract
We introduce the prime graph of the product ring $R_1 \times R_2$ where R_1, R_2 are integral domains, which is an extension of study on prime graph of an integral domain. We prove that, if R_1, R_2 are two integral domains, the graph obtained by removing the isolated vertices from PG($R_1 \times R_2$) is a bipartite graph. We obtain some consequences.

Keywords
Associative ring, Integral Domain, Graph, Prime Graph.

AMS Subject Classification
05C20, 05C25, 13E15, 68R10, 05C99.

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Contents

1. Introduction

The prime graph of an associative ring, a concept from algebraic graph theory was introduced by Satyanarayana et.al [11] has shown a new path for the researchers to explore and extend the study in their fields of interest. Satyanarayana et. al [4, 5], studied prime graphs related to a ring of integers modulo n. The complement of a prime graph of a ring was studied by Power and Joshi [2]. These studies motivated us to derive few results in the prime graph of an integral domain which is an extension to the work of Satyanarayana et. al [5].

Our study is presented in three small sections. Section 1, is a collection of necessary definitions, and results from the literature. Section 2 and 3 contains new findings.

Definition 1.1. *[7] An algebraic system with a non-empty set R together with two binary operations addition and multiplication is said to be a ring (or an associative ring) if $(R,+)$ is an abelian group; $(R,\cdot)$ is a semigroup and multiplication is distributive over the addition among the elements of R. If in addition R satisfies commutative property with multiplication, then it is called a commutative ring. Further ring containing multiplicative identity is called a ring with unity.*

Definition 1.2. *[7] A non-empty subset I of a ring R is called a left ideal if $(I,+)$ is subgroup of $(R,+)$ and for any element r of R and i of I, $ri \in I$. It is called right ideal if $ir \in I$ for all elements r of R and i of I with $(I,+)$ being subgroup of $(R,+)$.*

Definition 1.3. *[7] (i) An ideal P of a ring R is said to be prime for any two ideals A, B of R, and $AB \subseteq P$ imply $A \subseteq P$ or $B \subseteq P$ (equivalently, $a, b \in R$ and $aRb \subseteq P \Rightarrow a \in P$ or $b \in P$).*
(ii) Let I, J be two ideals of R such that $I \subseteq J$. We say that I is essential (or ideal essential) in J if it satisfies the following condition: $K \trianglelefteq R, K \subseteq J, I \cap K = (0)$ imply $K = (0)$.
(iii) Given two distinct ideals I and J of R, if I is essential in J, then we say that J is proper essential extension of I. We use $I \leq_e J$ to represent I is essential in J.

Definition 1.4. *[7] (i) A non-zero ideal I of R is said to be uniform if for any other non-zero ideal J or R contained in I imply $J \leq_e I$.*
(ii). A non- zero ideal K of R is said to have finite dimension on ideals of R (FDIR, in short) if K does not contain an infinite number of non-zero ideals of R whose sum is direct. It is clear that if R has FDI, then every non-zero ideal of R has FDIR.

Definition 1.5. *A commutative ring with unity is said to be an integral domain if for any two element a and b, $ab = 0$ implies* *either $a = 0$ or $b = 0$.*

Theorem 1.6. *[7] Suppose H is a non-zero ideal of a ring R and H has finite dimension on ideals of R. Then there exist ideals $U_1, U_2, \ldots, U_n$ of R which are uniform whose sum is direct and essential in H and further these are unique in number.*

Corollary 1.7. *[7] If R is a ring with FDI, then there exist uniform ideals $U_1, U_2, \ldots U_n$ in R whose sum is direct and essential in R; and if $V_1, 1 \le i \le k$, possessing the same property as of $U_j, 1 \le j \le n$ mentioned above, then $k = n$.*

Definition 1.8. *The number n, obtained above, is called the dimension of H, and is denoted by dim H.*

For further developments in this dimension concept in ring theory, we refer [3, 7, 9].

Now we present some Graph theoretic concepts: A graph is a system $G(V,E,\varphi)$ consist of non-empty set V of elements called vertices; another set E of elements called edges and incidence relation φ from E to v_i, v_j of V. If in G, both $|V|$ and |E| are finite, then G is called a *finite graph*. If edge set in graph becomes empty then G is called an empty graph or a null graph. A simple graph is a graph in which no edge incident to same end vertices and no two edges share the same end vertices. A complete graph is a simple graph in which every vertex is adjacent to every other vertex in the graph. We use K_n to denote a complete graph with n vertices. The degree of a vertex $d(v)$ is the count of number edges incident to it. A component of a graph is a subgraph which is maximally connected. The distance between any two vertices u and v of a graph G is denoted by $d(u,v)$. In this paper we study only simple graphs. For a graph $G(V,E,\varphi)$ if there is graph G_1 with vertex set X which is a non-empty subset of V and edge set which are exclusively connecting the vertices of X is called the subgraph generated by X or the maximal subgraph with vertex set X.

A star graph is a graph having a fixed vertex v and edge set containing only edges which are incident with v and are not forming loop with the fixed vertex. An n-star graph is a star graph having n vertices in it.

We refer Herstein [1], and Satyanarayana and Syam Prasad [10] for further readings in ring theory and graph theory.

Definition 1.9. *[11] A prime graph of a ring R is a graph $G(V,E)$ having the vertex set as R and edge set contains only edges which satisfied either $xRy = 0$ or $yRx = 0$ for all distinct x, y from V. It is denoted by $PG(R)$.*

Example 1.10. *The prime graph of a ring of integers modulo 6 is given in following diagram 1.1.*

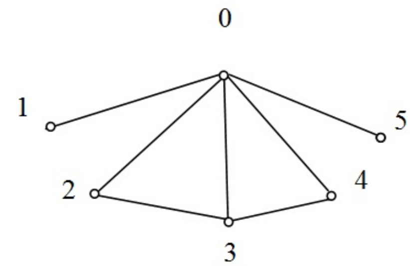

Figure 1. $PG(\mathbb{Z}_6)$

Observation 1.11. *[11] (i) Every prime graph of a ring is a simple graph. (ii) The degree of an additive identity element of a ring is always one less than number of elements of the ring. We can find a n-star graph as a sub graph of it as there always an edge between fixed vertex 0 to any other non-zero vertex of V together with edge connecting any two non-zero vertices satisfying the property mentioned in the definition. It is always a connected graph with distance from a vertex 0 to any other vertex is 1 and maximum distance from any two vertices 2. (iii) The distance between any two vertices of $PG(R)$ becomes 2 if and only if when $xRy \neq 0$. (iv) The domination number of a prime graphs is 1 as $\{0\}$ is a dominating set.*

For further developments in prime graphs of a ring, we refer [2, 4–6, 9].

2. PG(R) where R is an integral domain

Lemma 2.1. *[6] If the ring R becomes an integral domain, then prime graph of it is a star graph with number of vertices $|R|$.*

Theorem 2.2. *[6] Given a prime number p, the set of integers modulo $p, \mathbb{Z}_p$ is a field and hence it is an integral domain. PG($\mathbb{Z}_p$) is a star graph with number of vertices p and centre '0'. Conversely any star graph with p vertices is isomorphic to the graph PG($\mathbb{Z}_p$).*

Example 2.3. *[6]**(Prime graph of** $R\times \mathbb{Z}_2$**)** Suppose R is an integral domain and $\mathbb{Z}_2$ is a ring of integers modulo 2. For $(a,b),(c,d) \in R\times\mathbb{Z}_2$, we define addition and multiplication component wise. Then $R\times \mathbb{Z}_2$ becomes the product ring, and the zero element of $R\times \mathbb{Z}_2$ in $(0,0).(0,0)\times(1,0)$ and $(0,0)\times(0,1)$ are two elements in $R\times\mathbb{Z}_2$ with $(1,0) \neq (0,1) \neq (0,0)$. So $R\times \mathbb{Z}_2$ is not an integral domain.*

Theorem 2.4. *[6] Let R contains n elements. Then PG($R\times \mathbb{Z}_22$) contains two particular elements $(0,0) = a$, (say), $(0,1) = b$ (say) such that $|(V(PG(R\times \mathbb{Z}_22))| = 2n$ and PG($R\times \mathbb{Z}_2$) = [the $2n$-star graph with $R\times \mathbb{Z}_22$ as vertex set and centre a] $\cup$ [the n-star graph with vertex set $\{(x,0)/0 \neq x \in R\}$ with centre b].*

Note 2.5. *In the proof of this theorem we arrived at two subgraphs H and K of PG($R\times \mathbb{Z}_22$). We can state that $E(H)\cap E(K) = \phi$ and $a \notin V(K)$.*

Remark 2.6 (6)**.** *The graph PG($R\times \mathbb{Z}_2 2$) where R an integral domain, satisfy the following properties:*
(i) $|V(G)|= 2n$ where $n =|R|$.
(ii) It contains two particular vertices $a,b \in V(G)$ with $a \neq b$.
(iii) There exists a subgraph H of G such that H is a 2n-star graph (with centre a).
(iv) There exists a subgraph K of G such that K is a n-star graph (with centre b).
(v) $G = H \cup K$.

Theorem 2.7. *[6] Suppose G is a graph satisfying the following conditions:*
(i) $|V(G)| = 2p$, where p is a prime number.
(ii) G contains two particular vertices a^*, b^* with $a^* \neq b^*$.
(iii) H^* is a 2p-star graph (with centre a^*) which is a subgraph of G.
(iv) K^* is a p-star graph of G (with center b^*) and $a^* \notin V(K)$.
(v) $G = H^* \cup K^*$. Then G is isomorphic to PG($\mathbb{Z}_p \times \mathbb{Z}_2 2$).

Now we obtain the following new results:

Theorem 2.8. *If R is an integral domain, then*
(i) R is a uniform ideal and (ii) dim$(R) = 1$.

Proof. Let I be a non-zero ideal of R. We wish to prove that I is essential in R. In a contrary way, suppose that I is not essential in R. Then there exists a non-zero ideal J of R such that $I \cap J = (0)$. Let $0 \neq x \in I$ and $0 \neq y \in J$. Now $xy \in I \cap J = (0)$. We proved that x, y are two non-zero elements such $xy = 0$, a contradiction (to the fact that R is an integral domain). This shows that I is essential in R. Therefore every non-zero ideal of R is essential in R. By Theorem 7[8], we have that R is Uniform and hence dim $R = 1$. □

The proof of the following corollary from the fact that every field is an integral domain.

Corollary 2.9. *If R is a field, then R is uniform and dim $R = 1$.*

We denote the set of all isolated points of graph G by Iso(G).

Theorem 2.10. *If R_1, R_2 are two integral domains, then PG($R_1 \times R_2$) Iso(PG($R_1 \times R_2$)) is a bipartite graph.*

Proof. Write $R_1^* = \{(a,0)/0 \neq a \in R_1\}$ and $R_2^* = \{(0,b)/0 \neq b \in R_2\}$. Write $S = (R_1 \times R_1)$ $(R_1^* \cup R_2^*)$. We wish to show that (i) $S = Iso(PG(R_1 \times R_2))$ and (ii) subgraph of PG($R_1 \times R_2$) generated by $R_1^* \cup R_2^*$ is a complete bipartite graph. It is clear that $S \subseteq (R_1 \times R_2) = V(PG(R_1 \times R_2))$.
Proof for (i): Let $(a,b) \in S$. If $(a,b) = (0,0)$ then it is isolated. Suppose $(a,b) \neq (0,0)$. We show that $d(a,b) = 0$ where $d(a,b)$ is the degree of the vertex (a,b). Since $(0,0) \neq (a,b) \notin R_1^* \cup R_2^*$ we have that a $\neq 0 \neq b$. In a contrary way, suppose that $d(a,b) \neq 0$. Then there exists $(0,0) \neq (x,y) \in V(PG(R_1 \times R_2))$ such that (a,b) and (x,y) are adjacent. By the definition of prime graph $(a,b)(x,y) = (0,0)$ that implies $ax = 0$ and by $= 0$ implies that $x = 0$ and $y = 0$. (Since $0 \neq a \in R_1, 0 \neq b \in R_2, R_1$ and R_2 are integral domains). Implies that $(x,y) = (0,0)$, a contradiction. Hence $d(a,b) = 0$ and so (a,b) is an isolated point. Hence $S \subseteq$ Iso(PG($R_1 \times R_2$)).
Let $(a,b) \subseteq$ Iso PG($R_1 \times R_2$). If $(a,b) = (0,0)$ then $(a,b) \in S$. If $0 \neq a$ and $0 \neq b$ then $(a,b) \notin R_1^* \cup R_2^*$ and so $(a,b) \in S$. If $0 \neq a$ and $b = 0$ then $(a,b) = (a,0) \in R_1^*$ and $(a,0)(0,1) = 0$, so there is an edge between (a,b) and $(0,1)$ hence (a,b) is not an isolated point, a contradiction. (So the case a $\neq 0$ and $b = 0$ do not arise).
If $a = 0$ and $b \neq 0$, then $(a,b) = (0,b) \in R_2^*$ and $(1,0)(0,b) = 0$, so there is an edge between $(1,0)$ and (a,b), hence (a,b) is not an isolated point, a contradiction (so the case $a = 0$ and $b \neq 0$ do not arise). Now we proved that Iso (PG($R_1 \times$ R$_2$)) $\subseteq S$. Therefore, Iso (PG($R_1 \times R_2$)) $= S = (R_1 \times R_2)$ $(R_1^* \cup R_2^*)$. □

Proof of (ii): To show that the subgraph generated by $R_1^* \cup R_2^*$ is a complete bipartite graph we show the following four conditions. (i) $R_1^* \cap R_2^* = \phi$. (ii) there is no edge between two vertices belonging to R_1^*. (iii) There is no edge between two vertices belonging to R_2^*. (iv) (a, b) $\in R_1^*, (c,d) \in R_2^*$ implies there is an edge between (a,b) and (c,d). $R_1^* \cap R_2^* = \{(a,0)/0 \neq a \in R_1\} \cap \{(0,b)/0 \neq b \in R_2\} = \phi$. Let $(u,0)(v,0) = (0,0)$ and so $uv = 0$. That implies $u = 0$ or $v = 0$ (since R_1 is an integral domain) and hence $(0,0) \in R_1^*$, a contradiction. So we verified that there is no edge between any two vertices in R_1^*. A similar valid argument shows that there is no edge between any two vertices of R_2^*. Let $(a,0) \in R_1^*$ and $(0,b) \in R_2^*$. Then $(a,0) \neq (0, 0) \neq (0,b)$ and $(a,0)(0,b) = (0,0)$ and so there is an edge between $(a,0)$ and $(0,b)$. Hence one can conclude that the graph generated by $R_1^* \cup R_2^*$ is a complete bipartite graph.
Proof of (iii) By Part(i), we have that $R_1^* \cup R_2^* = R_1 \times R_2$ Iso (PG($R_1 \times R_2$)). So vertex set of the subgrph generated by $R_1^* \cup R_2^* = V(PG(R_1 \times R_2))$ Iso (PG($R_1 \times R_2$)). By; part (ii), the subgraph generated by $(R_1 \times \cup R_2)$ is a complete bipartite graph. This shows that $PG(R_1 \times R_2) Iso(PG(R_1 \times R_2)$

3. An application to Z_p, ring of integers modulo a prime number p

Let p, q be two prime numbers. Then Z_p, Z_q are two integral domains.

Lemma 3.1. *PG($Z_p \times Z_q$)) Iso (PG($Z_p \times Z_q$) forms a complete bipartite graph ($K_{(p-1)(q-1)}$).*

Proof. Write $R_1 = Z_p$ and $R_2 = Zq$. Then the proof follows from Theorem 2.9. □

Theorem 3.2. *Suppose that p, q are prime numbers. Then the subgraph PG($Z_p \times Z_q$)) Iso(PG($Z_p \times Z_q$) is complete bipartite graph ($K_{(p-1)(q-1)}$). Conversely any complete bipartite graph ($K_{(p-1)(q-1)}$) (where p, q are primes) is isomorphic to a subgraph of PG($R_1 \times R_2$) that is generated by $R_1^* \cup R_2^*$ where $R_1 = Z_p$ and $R_2 = Z_q$.*

Proof. Write $R_1 = Z_p$ and $R_2 = Z_q$. Then the first part is Lemma 3.1.
Converse: Consider the complete bipartite graph $(K_{(p-1)(q-1)})$ with p, q are prime. Suppose the set of vertices of $(K_{(p-1)(q-1)})$ are divided into the partition $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$. Now $V(K_{(p-1)(q-1)}) = \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\}$. Write $R_1 = Z_p$, the integral domain of integer modulo p, and $R_2 = Z_q$ the integral domain of integer modulo q. Now $R_1^* = \{(i,0)/1 \leq i \leq p-1\}$ and $R_2^* = \{(0,j)/1 \leq j \leq q-1\}$. Define $f : R_1^* \cup R_2^* \to V(K_{(p-1)(q-1)})$ by $f((i,0)) = x_i$ for all $1 \leq i \leq p-1$ and $f((0,j)) = y_j$ for all $1 \leq j \leq q-1$. Also $f(\overline{((i,0)(j,0)}) = \overline{x_i y_j} = \overline{f(i,0) f(j,0)}$. We proved that $K_{(p-1)(q-1)}$ is isomorphic to the subgraph PG($Z_p \times Z_q$) Iso PG($Z_p \times Z_q$) of PG($Z_p \times Z_q$). □

Example 3.3. $Z_p \times Z_q$

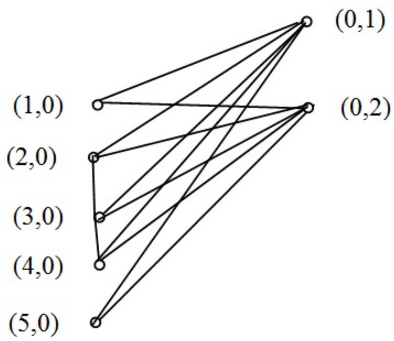

Figure 2. $PG(\mathbb{Z}_6 \times \mathbb{Z}_3)$

Observation 3.4. *PG($\mathbb{Z}_6 \times \mathbb{Z}_3$) is not a complete graph because there is no edge between $(1,0)$ and $(2,0)$. PG($\mathbb{Z}_6 \times \mathbb{Z}_3$) is not bipartite graph because it contains a triangle $\{(2,0),(0,2),(3,0)\}$.*

Note 3.5. *Example 3.3. shows that Theorem 3.2 fails if p is not a prime number. So our main result 3.2 of this section is not true if both p, q are not prime numbers.*

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