Malaya Journal of Matematik, Vol. 8, No. 3, 893-897, 2020

https://doi.org/10.26637/MJM0803/0027

Extension of proposed family of one-dimensional continuous wavelets to two-dimensional continuous wavelets

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Abstract

In this paper, the two-dimensional continuous wavelets are introduced by extending the proposed family of one-dimensional continuous wavelets. The efficiency in detecting singularity, edges and contours of an image are discussed using these wavelets.

Keywords

Two-dimensional continuous wavelet transform, directional wavelets, edge detection.

AMS Subject Classification

42C40, 65T60.

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Article History: Received **23** January **2020**; Accepted **14** May **2020**

Contents

1. Introduction

In general, the two-dimensional continuous wavelet is a natural extension from the one-dimensional continuous wavelet. The one-dimensional continuous wavelet transform of a signal, captures the changes occurring in a signal in a particular interval of time [1, 5, 8]. Similarly, the two-dimensional continuous wavelet transform of an image, captures different features in the image such as singularity, edges and contours [1, 2, 4, 8]. This behavior of two-dimensional continuous wavelet transform can be used in the detection of cracks in the plate and detection of edges in an image[3, 6, 7, 12].

Recently we have proposed a family of continuous one-dimensional wavelets [9]. In this paper we are extending these wavelet functions to two-dimensional continuous wavelets.

Here, two classes of wavelets are designed, where the first class of wavelet functions are effective in identifying the edges whose magnitude changes only in vertical or horizontal directions of the processed image [6, 7, 10, 11] and the second class of wavelet functions are effective in identifying contour of processed image[10, 13]. In [13], the author uses directional Gaussian wavelet which now can be replaced by the second class of wavelets in detecting a cross-like notch in a suspended aluminum plate whose images are scanned by a laser vibrometer.

2. Proposed one-dimensional continuous wavelets

The proposed family of one-dimensional continuous mother wavelets are constructed with the sequential differentiation of the product of Gaussian function and Cauchy-Lorentz distribution function [9] i.e.

$$\eta(t) = e^{-t^2}.\frac{1}{1+t^2}.$$

the k^{th} derivative of $\eta(t)$ gives rise to a mother wavelet $\xi_k(t)$

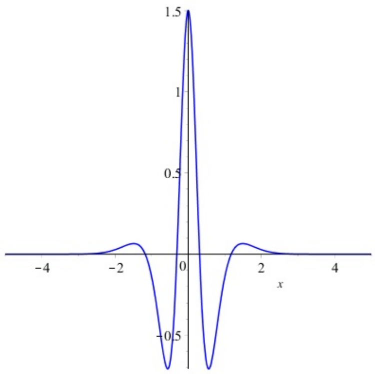

Figure 1. The fourth member of the proposed family $\xi_4(t)$.

of order k. Hence

$$\xi_k(t) = \frac{1}{\sqrt{C_k}} \frac{d^k}{dt^k}(\eta(t))$$

where C_k is the normalization constant. The fourth member of the proposed family $\xi_4(t)$ is shown in Figure 1.

3. Introduction to two-dimensional continuous wavelet transform

The two-dimensional mother wavelet is an extension of the one-dimensional mother wavelet. Analogous to 1D mother wavelet, 2D mother wavelet is a real or complex-valued, oscillatory function $\xi(\vec{x}) \in L^2(\mathbb{R}^2, d^2\vec{x})$, satisfying the admissibility condition on real plane $\vec{x} \in \mathbb{R}^2$, where $L^2(\mathbb{R}^2, d^2\vec{x})$ is a set of 2D measurable functions in a plane, which forms the Hilbert space. If ξ is regular enough as in most cases, the admissibility condition can be expressed as

$$\xi(\vec{0}) = 0 \Leftrightarrow \int_{\mathbb{R}^2} \xi(\vec{x}) d^2\vec{x} = 0$$

The 2D continuous wavelet transform of a signal $f(\vec{x}) \in L^2(\mathbb{R}^2, d^2\vec{x})$ with the mother wavelet $\xi(\vec{x})$ is defined as [1, 2, 4, 8]

$$\begin{aligned} Wf(s,\vec{\tau},\theta) &= \langle f, \xi_{s,\vec{\tau},\theta}\rangle \\ &= \frac{1}{s}\int_{\mathbb{R}^2} f(\vec{x})\xi^*\left[\mathbf{r}_{-\theta}\left(\frac{\vec{x}-\vec{\tau}}{s}\right)\right] d^2\vec{x} \end{aligned} \tag{3.1}$$

Here

$$\xi_{s,\vec{\tau},\theta}(\vec{x}) = s^{-1}\xi\left[\mathbf{r}_{-\theta}\left(\frac{\vec{x}-\vec{\tau}}{s}\right)\right], \quad s>0, \vec{\tau}, \theta \in \mathbb{R}^2$$

is the translation by a vector $\vec{\tau}$, dilation by a scaling factor s, and rotation by an angle θ of mother wavelet function $\xi(\vec{x})$, $\xi^*(.)$is the complex conjugate of $\xi(.)$ and the 2D rotation matrix, $\mathbf{r}_{-\theta}$ is defined as

$$\mathbf{r}_{-\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

The above equation(3.1) can also be computed by transforming it into Fourier domain, according to Plancherel's theorem as

$$\begin{aligned} \langle f, \xi_{s,\vec{\tau},\theta}\rangle &= \int_{\mathbb{R}^2} f(\vec{x})\xi^*_{s,\vec{\tau},\theta}(\vec{x})d^2\vec{x} \\ &= \int_{\mathbb{R}^2} \hat{f}(\vec{x})\hat{\xi}^*_{s,\vec{\tau},\theta}(\vec{x})e^{i\vec{\tau}\vec{\omega}}d^2\vec{\omega} \end{aligned} \tag{3.2}$$

and

$$\hat{\xi}_{s,\theta}(\vec{\omega}) = s\hat{\xi}[s\mathbf{r}_{\theta}^{-1}(\vec{\omega})]$$

where $\hat{\xi}$ is the fourier transform of ξ.

The mother wavelet that depends on the angle θ in the analysis is called directional(non-isotropic) wavelet, otherwise non-directional(isotropic) wavelet.

4. Extension of proposed family of one-dimensional continuous wavelets to family of two-dimensional continuous wavelets

The single variable function $\eta(t)$ can be extended to a two-variable function as

$$\phi(\vec{x}) = \frac{e^{-|\vec{x}|^2}}{1+|\vec{x}|^2}$$

where $\vec{x} = (x_1, x_2) \in \mathbb{R}^2$.

Here, we defined two classes of continuous 2D wavelets. First, we define a vertical wavelet $\xi_{x_1}(\vec{x})$ and a horizontal wavelet $\xi_{x_2}(\vec{x})$ as partial derivatives of $\phi(\vec{x})$ with respect to x_1 and x_2 respectively. i.e.

$$\xi_{x_1}(\vec{x}) = \frac{\partial}{\partial x_1}\phi(\vec{x}) = -2\frac{x_1 e^{-|\vec{x}|^2}}{(1+|\vec{x}|^2)^2}(2+|\vec{x}|^2)$$

and

$$\xi_{x_2}(\vec{x}) = \frac{\partial}{\partial x_2}\phi(\vec{x}) = -2\frac{x_2 e^{-|\vec{x}|^2}}{(1+|\vec{x}|^2)^2}(2+|\vec{x}|^2).$$

Second, we define (m,n) order directional wavelets $\xi^{m,n}(\vec{x})$ as mixed partial derivatives of $\phi(\vec{x})$ with respect to x_1 and x_2. i.e.

$$\xi^{m,n}(\vec{x}) = \frac{\partial^{m+n}}{\partial x_1^m \partial x_2^n}\phi(\vec{x}).$$

The $(2,2)$ order 2D mother wavelet is

$$\begin{aligned} \xi^{2,2}(\vec{x}) = \frac{4e^{-|\vec{x}|^2}}{(1+|\vec{x}|^2)^5}\Big[&(x_1x_2)^2(4|\vec{x}|^8+32|\vec{x}|^6+120|\vec{x}|^4+256|\vec{x}|^2+260) \\ &-(2|\vec{x}|^{10}+13|\vec{x}|^8+36|\vec{x}|^6+48|\vec{x}|^4+18|\vec{x}|^2-5)\Big] \end{aligned}$$

the 3D plot of which is shown in Figure 2 along with its planform. The plots of planforms of other three directional wavelets formed by rotating $\xi^{2,2}(\vec{x})$ through angles $\pi/8$, $2\pi/8$, $3\pi/8$ are given in Figure 3.

It can be easily verified that

$$\int_{\mathbb{R}^2} \xi^{m,n}(\vec{x}) d^2\vec{x} = 0.$$

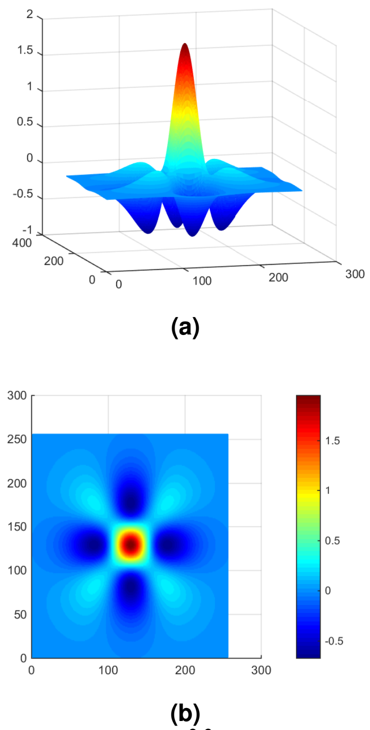

Figure 2. (a) The 3D plot of $\xi^{2,2}(\vec{x})$ at scale $s = 2$, angle $\theta = 0$ and (b) it's planform.

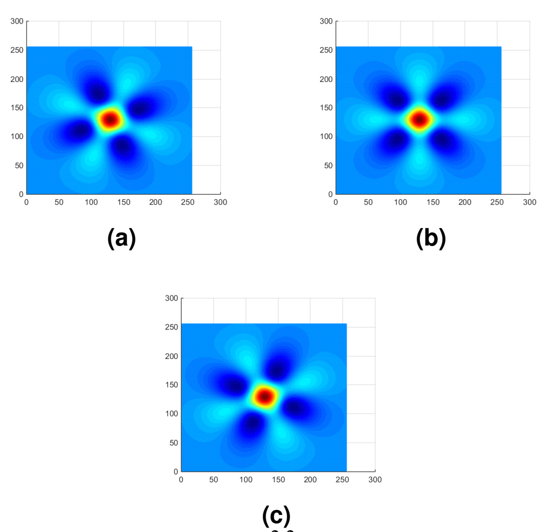

Figure 3. The planforms of $\xi^{2,2}(\vec{x})$ at angle (a) $\theta = \frac{\pi}{8}$ (b) $\theta = \frac{2\pi}{8}$ (c) $\theta = \frac{3\pi}{8}$

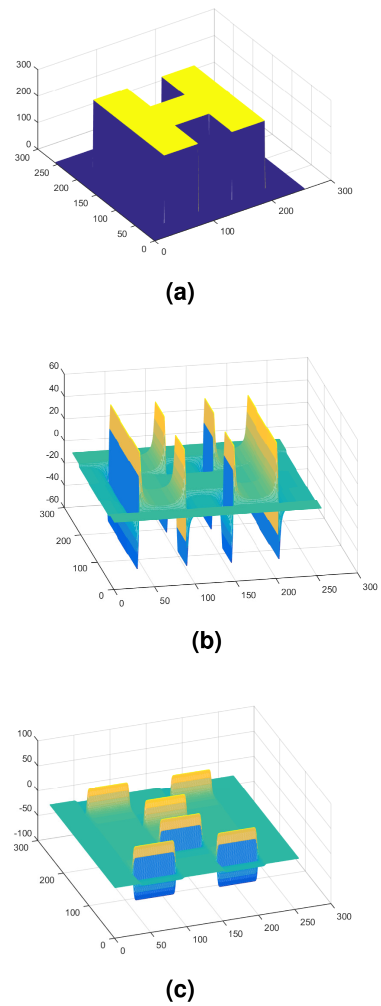

Figure 4. (a) Image H, (b) and (c) are wavelet coefficient plots of image H using $\xi_{x_1}(\vec{x})$ and $\xi_{x_2}(\vec{x})$ with at angle $\theta = 0$ respectively.

5. Implementation

In this section, we will see the efficiency of both the classes of wavelet functions in the transformation. In the present case, we will work on simple objects, such as image letters H and X and check the efficiencies of them in analysing these objects.

The first class of wavelets $\xi_{x_1}(\vec{x})$ and $\xi_{x_2}(\vec{x})$ are used in the analysis of the image H and X given in Figure 4(a) and Figure 5(a). The wavelet coefficient plot of the image H, X using $\xi_{x_1}(\vec{x})$ and $\xi_{x_2}(\vec{x})$ are shown in Figure 4(b), 5(b) and Figure 4(c), 5(c) respectively.

From Figure 4(b) and 5(b), it is clear that the wavelet coefficient obtained by using wavelet $\xi_{x_1}(\vec{x})$, detects only the edges whose magnitude changes in vertical direction in the images H and X, whereas $\xi_{x_2}(\vec{x})$ detects only the edges whose magnitude changes in horizontal directions as shown in Figure 4(c) and 5(c).

Now, if we rotate $\xi_{x_1}(\vec{x})$, $\xi_{x_2}(\vec{x})$ by an angle $\pi/4$ and

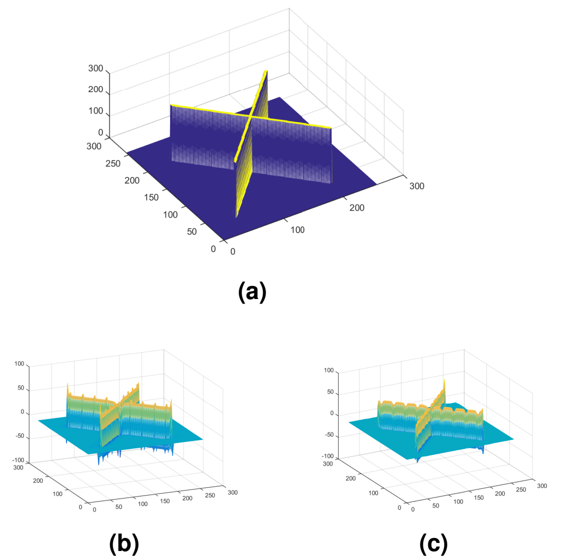

Figure 5. (a) Image X, (b) and (c) are wavelet coefficient plots of image X using $\xi_{x_1}(\vec{x})$ and $\xi_{x_2}(\vec{x})$ with at angle $\theta = 0$ respectively.

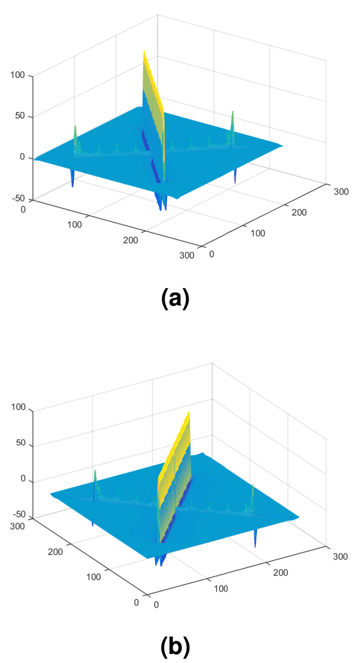

Figure 6. (a) and (b) are wavelet coefficient plots of image X using $\xi_{x_1}(\vec{x})$ and $\xi_{x_2}(\vec{x})$ with at angle $\theta = \frac{\pi}{4}$ respectively.

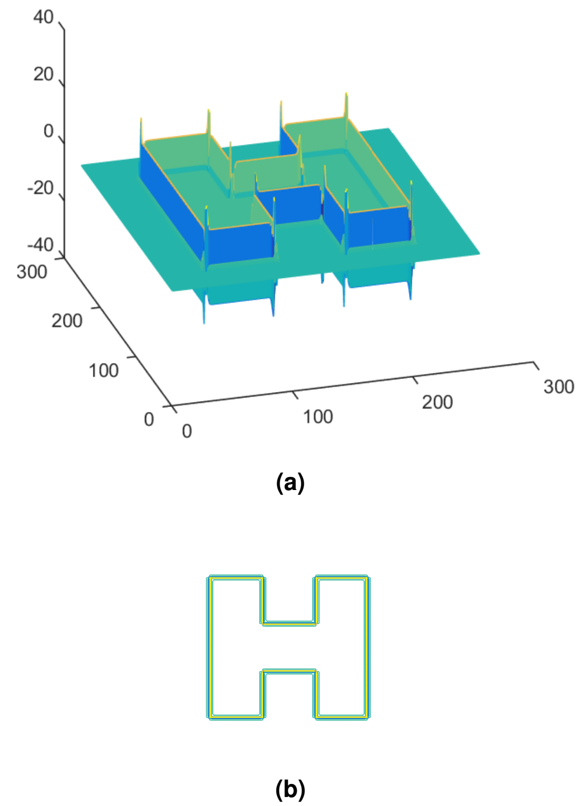

Figure 7. (a) The wavelet coefficient plot of image H using $\xi^{2,2}(\vec{x})$ at scale $s = 1/16$, angle $\theta = \pi/4$, (b) Contour plot of (a)

analyse the same image X, we obtain the wavelet coefficients shown in Figure 6(a) and 6(b).

It is clear from the figure that only the vertical variations are captured by $\xi_{x_1}(\vec{x})$ and horizontal variations are captured by $\xi_{x_2}(\vec{x})$.

This shows how efficient the wavelets $\xi_{x_1}(\vec{x})$ and $\xi_{x_1}(\vec{x})$ in detecting the vertical and horizontal variation in the edges.

On the other hand, the second class of wavelets captures variations in the image with all directions at different angles. Practically at $\theta = \pi/4$ gives very detail information of the image. The Figure 7 shows the wavelet coefficients of image H at $\theta = \pi/4$ and using $\xi^{2,2}(\vec{x})$.

It can be observed from the wavelet coefficient plot of the image H, there are three sets of values in the wavelet coefficient plot, one with very large magnitude showing the singularity in H (at corners), second with relatively lower values, yield the edge information in H and the third set with vanishing coefficients inside and outside the edges where the image is constant in magnitude.

6. Conclusion

The two different classes of 2D continuous wavelets are designed using the proposed family of 1D continuous wavelets.

We successively demonstrated the effectiveness of the family of 2D wavelets in identifying the edges of the synthetic images H and X.

Acknowledgment

The research was supported by Council of Scientific and Industrial Research (CSIR), New Delhi.

APPENDIX

Algorithm to numerically compute 2D wavelet transform:
sept1: Compute the 2D Fast Fourier transform $\hat{f}(\vec{\omega})$ of the image $f(\vec{x})$ to be processed.
sept2: Now, take the Fast Fourier transform of wavelet $\xi(\frac{\vec{x}}{s})$, at a given scale $'s'$ (which gives the function $\hat{\xi}(s\vec{\omega})$ in frequency domain).
step3: Inverse Fourier transform of the Hadamand product (element-wise multiplication) of the two functions $\hat{f}(\vec{\omega})$ and $s\hat{\xi}(s\vec{\omega})$ obtained in step1 and step2 gives the 2D wavelet transform of $f(\vec{x})$.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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