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A result on Banach space using E.A like property

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Abstract

The focus of this paper is to establish the existence of common fixed point result on Banach space using the conditions E.A like property and weakly compatible mappings.

Keywords

Fixed point,Metric space,Banach space, E.A like Property,weakly compatible mappings.

AMS Subject Classification

47H10, 54H25.

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Contents

1. Introduction

In the area of analysis fixed point theorems contribute major share in the development of the research.To mention a few Banach contraction principle is one such result.After this major result, fixed point theorems like [3],[7],[8] have been developed.Pathak,khan and Tiwari[1] established a fixed point theorem using the continuity and weakly compatible mappings on complete metric space. There are several theorems are being generated on metric spaces using these conditions like [2],[4],[5],[6],[9] and[10].Now the emphasis of this paper is to prove a common fixed point theorem on Banach space without using continuity condition and also adopting E.A like property.

2. Preliminaries

Now, we begin with some definitions:

Definition 2.1. *We define mappings G and H of a Banach space X as weakly commuting on X if* $\|GH\alpha - HG\alpha\| \leq \|G\alpha - H\alpha\|$ *for all* $\alpha \in X$.

Definition 2.2. *We define mappings G and H of a Banach Sapace X as compatible if* $\|GH\alpha_k - HG\alpha_k\| = 0$ *as* $k \to \infty$ *whenever* $\{\alpha_k\}$*is a sequence in X such that* $\|G\alpha_k - H\alpha_k\| = 0$ *as* $k \to \infty$ *for some* $\mu \in X$.

Definition 2.3. *We define the mappings G and H of a Banach space X in which if* $G\mu = H\mu$ *for some* $\mu \in X$ *such that* $GH\mu = HG\mu$ *holds,then G and H are known as weakly compatible mappings.*

Definition 2.4. *A Banach space X in which two self mappings G and H are said to satisfy the property E.A if there is a sequence* $\{\alpha_k\}$ *in X with* $G\alpha_k = H\alpha_k = \mu$ *as* $k \to \infty$ *for some* $\mu \in X$.

Definition 2.5. *A Banach space X in which two self mappings G and H are said to satisfy the E.A like property if there is a sequence* $\{\alpha_k\}$ *in X with* $G\alpha_k = H\alpha_k = \mu$ *as* $k \to \infty$ *for some* $\mu \in G(X) \cup H(X)$.

Now we show one example for E.A property and then show another example in which the mappings are satisfying E.A like property.

Example 2.6. *Suppose* $X = [0,1]$ *in Banach space with* $\|\alpha - \beta\| = |\alpha - \beta|$, $\forall \alpha, \beta \in X$. *We define self maps G and J as follows*

$$G(\alpha) = \begin{cases} \frac{2\alpha}{2} & \text{if } 0 \leq \alpha \leq \frac{1}{5}; \\ \frac{1-\alpha}{2} & \text{if } \frac{1}{5} < \alpha \leq 1. \end{cases}$$

$$J(\alpha)=\begin{cases}\frac{5-4\alpha}{6} & if\, 0\le\alpha\le\frac{1}{5};\\ \alpha & if\, \frac{1}{5}<\alpha\le 1.\end{cases}$$

Take a sequence $\alpha_k=\frac{1}{5}-\frac{1}{k}$ *for* $k>0$.
Then $G(\alpha_k)=G(\frac{1}{5}-\frac{1}{k})=\frac{2(\frac{1}{2}-\frac{1}{k})+1}{2}=\frac{7}{10}$
and $J(\alpha_k)=J(\frac{1}{5}-\frac{1}{k})=\frac{5-4(\frac{1}{5}-\frac{1}{k})}{6}=\frac{7}{10}$.
This gives $G\alpha_k=J\alpha_k=\frac{7}{10}$ *as* $k\to\infty$ *and* $\frac{7}{10}\in X$. *Hence G and J satisfies E.A property.*

Example 2.7. *Suppose* $X=[0,1]$ *in Banach space with* $\|\alpha-\beta\|=|\alpha-\beta|$, $\forall\alpha,\beta\in X$. *We define self maps G and J as follows*

$$G(\alpha)=\begin{cases}\frac{1+2\alpha}{3} & if\, 0\le\alpha\le\frac{1}{6};\\ \frac{\alpha+1}{2} & if\, \frac{1}{6}<\alpha\le 1.\end{cases}$$

$$J(\alpha)=\begin{cases}\frac{3-2\alpha}{6} & if\, 0\le\alpha\le\frac{1}{6};\\ 1-\alpha & if\, \frac{1}{5}<\alpha\le 1.\end{cases}$$

Here G(X)=$[\frac{1}{3},\frac{4}{9}]\cup(\frac{7}{12},1]$ *while J(X)=*$[\frac{4}{9},\frac{1}{2}]\cup(0,\frac{5}{6}]$.

Take a sequence $\alpha_k=\frac{1}{6}-\frac{1}{k}$ *for* $k>0$.

Then $G(\alpha_k)=G(\frac{1}{6}-\frac{1}{k})=\frac{1+2(\frac{1}{6}-\frac{1}{k})}{3}=\frac{4}{9}$ *and* $J(\alpha_k)=J(\frac{1}{6}-\frac{1}{k})=\frac{3-42(\frac{1}{6}-\frac{1}{k})}{6}=\frac{4}{9}$.

This gives $G\alpha_k=J\alpha_k=\frac{4}{9}$ *as* $k\to\infty$ *and* $\frac{4}{9}\in G(X)\cup J(X)$. *Hence G and J satisfies E.A like property.*

The following Theorem was proved in metric space [1].

Theorem 2.8. *Suppose X is a comlete metic space G,H,I and J are mappings defined X holding the conditions*
$(C1)$ $G(X)\subseteq H(X)$ *and* $I(X)\subseteq J(X)$
$(C2)$ $d(G\alpha,I\beta)^{2p}\le[a\phi_0(d(J\alpha,H\beta)^{2p})$
$+(1-a)max\{\phi_1(d(J\alpha,H\beta)^{2p}),$
$\phi_2(d(J\alpha,G\alpha)^q d(H\beta,I\beta)^{q'}),\phi_3(d(J\alpha,I\beta)^r d(H\beta,G\alpha)^{r'}),$
$\phi_4(d(J\alpha,G\alpha)^s d(H\beta,G\alpha)^{s'}),\phi_5(d(J\alpha,I\beta)^l d(H\beta,I\beta)^{l'})\}]$
for all $\alpha,\beta\in X$ *where* $\phi_k\in\phi, k=0,1,2,3,4,5$, $0\le a\le 1$, $0<p,p',q,q',r,r'.s,s',l,l'\le 1$ *such that* $2p=p+p'=q+q'+r+r'+l+l'$.
$(C3)$ *Any one of the mappings A or B is continuous.*
$(C4)$ *the pair of mappings* (G,J) *and* (I,H) *are weakly compatible. Then the above mappings have a unique common fixed point.*

Now we give the statement of important lemmas which plays vital role in our main result.

Lemma 2.9. *[5] If* $\phi_k\in\phi$ *and* $k\in\{0,1,2,3,4,5\}$ *where* ϕ *is upper semi continuous and contractive modulus such that* $max\{\phi_k(t)\}\le\phi(t)$ *for all* $t>0$ *and* $\phi(t)<t$ *for* $t>0$.

Lemma 2.10. *[1] Let* $\phi_j\in\phi$ *and* β_j *be a sequence of non-negative real numbers.If* $\beta_{j+1}\le\phi(\beta_j)$ *for* $j\in N$ *,then the sequence* β_j *converges to* 0.

3. Main Result

Now we prove the existence of fixed point theorem (2.8) in Banach Space under some modified conditions.

Theorem 3.1. *Suppose in a Banach Space* $(X,\|.\|)$ *,there are four mappings G,H,I and J holding the conditions*
$(C1)$ $G(X)\subseteq H(X)$ *and* $I(X)\subseteq J(X)$
$(C2)$ $\|(G\alpha-I\beta)\|^{2p}\le[a\phi_0(\|(J\alpha-H\beta)\|^{2p})$
$+(1-a)max\{\phi_1(\|J\alpha-H\beta\|^{2p}),$
$\phi_2(\|J\alpha-G\alpha\|^q\|H\beta-I\beta\|^{q'}),$
$\phi_3(\|J\alpha-I\beta\|^r\|H\beta-G\alpha\|^{r'}),$
$\phi_4(\frac{1}{2}\|J\alpha-G\alpha\|^s\|H\beta-I\beta\|^{s'}),$
$\phi_5(\frac{1}{2}\|J\alpha-I\beta\|^l\|H\beta-I\beta\|^{l'})\}]$
for all $\alpha,\beta\in X$ *where* $\phi_k\in\phi, k=0,1,2,3,4,5$, $0\le a\le 1$, $0<p,p',q,q',r,r'.s,s',l,l'\le 1$ *such that* $2p=p+p'=q+q'+r+r'+l+l'$.
$(C3)$ *The pairs* (G,J) *and* (I,H) *satisfy the E.A like property*
$(C4)$ *the pair of mappings* (G,J) *and* (I,H) *are weakly compatible.*
Then the above mappings have a unique common fixed point.

Proof. Begin with using the condition (C1), there is a point $\alpha_0\in X$ such that $G\alpha_0=H\alpha_1$.For this point $\alpha_1\in X$ there exists a point α_2 in X such that $I\alpha_1=J\alpha_2$ and so on.
Continuing this process it is possible to construct a sequence $\{\beta_j\}$ for $j=1,2,3......\in X$ such that $G\alpha_{2j}=H\alpha_{2j+1}=\beta_{2j}$ (say),
$I\alpha_{2j+1}=J\alpha_{2j+2}=\beta_{2j+1}$ (say) for $j\ge 0$.
We now prove $\{\beta_j\}$ is a cauchy sequence.
Putting $\alpha=\alpha_{2j}$ and β_{2j+1} in (C2), we get
$\|\beta_{2j}-\beta_{2j+1}\|^{2p}\le[a\phi_0(\|\beta_{2j-1}-\beta_{2j}\|^{2p})$
$+(1-a)max\{\phi_1(\|\beta_{2j-1}-\beta_{2j}\|^{2p}),$
$\phi_2(\|\beta_{2j-1}-\beta_{2j}\|^q\|\beta_{2j}-\beta_{2j+1}\|^{q'}),$
$\phi_3(\|\beta_{2j-1}-\beta_{2j+1}\|^r\|\beta_{2j}-\beta_{2j}\|^{r'}),\phi_4(\frac{1}{2}\|\beta_{2j-1}-\beta_{2j}\|^s\|\beta_{2j}-\beta_{2j}\|^{s'}),\phi_5(\frac{1}{2}\|\beta_{2j-1}-I\beta_{2j+1}\|^l\|\beta_{2j}-\beta_{2j+1}\|^{l'})\}]$.

Denote $\rho_j=\|\beta_j-\beta_{j+1}\|$
$(\rho_{2j})^{2p}\le[a\phi_0(\rho_{2j-1})^{2p})$
$+(1-a)\max\{\phi_1(\rho_{2j-1})^{2p},\phi_2((\rho_{2j-1})^q(\rho_{2j})^{q'}),$
$\phi_3(0),\phi_4(0),\phi_5(\frac{1}{2}[(\rho_{2j-1})^l+(\rho_{2j})^{l'})](\rho_{2j})^l)\}]$.

$(\rho_{2j})^{2p}\le[a\phi_0(\rho_{2j-1})^{2p})$
$+(1-a)\max\{\phi_1(\rho_{2j-1})^{2p},\phi_2((\rho_{2j-1})^q(\rho_{2j})^{q'}),\phi_3(0),\phi_4(0),$
$\phi_5(\frac{1}{2}[(\rho_{2j-1})^l(\rho_{2j})^l+(\rho_{2j}^{l'})(\rho_{2j})^l)])\}]$.

If $\rho_{2j}>\rho_{2j-1}$ then we have
$(\rho_{2j})^{2p}\le[a\phi_0(\rho_{2j})^{2p})$
$+(1-a)\max\{\phi_1(\rho_{2j})^{2p},\phi_2(\rho_{2j})^{q+q'},\phi_3(0),\phi_4(0),$
$\phi_5(\frac{1}{2}[(\rho_{2j})^{l+l'}(\rho_{2j})^{l+l'}+(\rho_{2j}^{l+l'})])\}(\rho_{2j})^{2p})]$.
$\le[a\phi_0(\rho_{2j})^{2p})+(1-a)\max\{\phi_1(\rho_{2j})^{2p},\phi_2(\rho_{2j})^{2p},$
$\phi_3(0),\phi_4(0),\phi_5(\rho_{2j})^{2p})]$.
Using Lemma(2.9)
$(\rho_{2j})^{2p}\le\phi(\rho_{2j})^{2p}<(\rho_{2j})^{2p}$ which is a contradiction. Thus

we must have $\rho_{2j} \leq \rho_{2j-1}$ then using this inequality the condition (C2) yields

$$\rho_{2j} \leq \phi(\rho_{2j-1}). \tag{3.1}$$

Similarly taking $\alpha = \alpha_{2j+2}$ and $\beta = \alpha_{2j+1}$ in (C2), we get

$$\|\beta_{2j+1} - \beta_{2j+2}\|^{2p} \leq [a\phi_0(\|\beta_{2j} - \beta_{2j+1}\|^{2p}) + (1-a)max\{\phi_1(\|\beta_{2j} - \beta_{2j+1}\|^{2p}), \phi_2(\|\beta_{2j+1} - \beta_{2j+2}\|^q \|\beta_{2j} - \beta_{2j+1}\|^{q'}), \phi_3(\|\beta_{2j+1} - \beta_{2j+1}\|^r \|\beta_{2j} - \beta_{2j+1}\|^{r'}), \phi_4(\tfrac{1}{2}\|\beta_{2j+11} - \beta_{2j+2}\|^s \|\beta_{2j} - \beta_{2j+2}\|^{s'}), \phi_5(\tfrac{1}{2}\|\beta_{2j+2} - I\beta_{2j+1}\|^l \|\beta_{2j} - \beta_{2j+1}\|^{l'})\}].$$

$$(\rho_{2j+1})^{2p} \leq [a\phi_0(\rho_{2j})^{2p}) + (1-a)\max\{\phi_1(\rho_{2j})^{2p}, \phi_2((\rho_{2j+1})^q(\rho_{2j})^{q'}), \phi_3(0), \phi_4(\tfrac{1}{2}[(\rho_{2j+1})^s(\rho_{2j})^{s'} + (\rho_{2j+1})^{s'}], \phi_5(0)\}].$$

$$(\rho_{2j+1})^{2p} \leq [a\phi_0(\rho_{2j})^{2p}) + (1-a)\max\{\phi_1(\rho_{2j})^{2p}, \phi_2((\rho_{2j+1})^q(\rho_{2j})^{q'}), \phi_3(0), \phi_4(\tfrac{1}{2}[(\rho_{2j+1})^s(\rho_{2j})^{s'} + (\rho_{2j+1})^{s'}(\rho_{2j+1})^{s'}]), \phi_5(0)\}].$$

If $\rho_{2j+1} > \rho_{2j}$, then we have

$$(\rho_{2j+1})^{2p} \leq [a\phi_0(\rho_{2j+1})^{2p}) + (1-a)\max\{\phi_1(\rho_{2j+1})^{2p}, \phi_2((\rho_{2j+1})^{q+q'}), \phi_3(0), \phi_4(\rho_{2j+1}), \phi_5(0)\}].$$

Using Lemma(2.9)

$(\rho_{2j+1})^{2p} \leq \phi(\rho_{2j+1})^{2p} < (\rho_{2j+1})^{2p}$ which is a contradiction.

Thus we must have $\rho_{2j+1} \leq \rho_{2j}$. Again applying (C2) to the above inequality, we obtain

$$(\rho_{2j+1}) \leq \phi(\rho_{2j}) \tag{3.2}$$

From (3.1) and (3.2), in general $\rho_{j+1} \leq \phi(\rho_j)$, for j=0,1,2,3.... by Lemma (2.10) we get $\rho_j \to 0$ as $j \to \infty$.

This shows that $\rho_j = \|\beta_j - \beta_{j+1}\| \to 0$ as $j \to \infty$.

Hence $\{\beta_j\}$ is a cauchy sequence.

Now, X being Banach space, there exists a point $\mu \in X$ such that $\beta_j \to \mu$ as $j \to \infty$.

Consequently, the subsequences $G\alpha_{2j}, J\alpha_{2j}, I\alpha_{2j+1}$ and $H\alpha_{2j+1}$ of $\{\beta_j\}$ also converges to the same point $\mu \in X$.

Now on using E.A like property of the pair (G,J) there exists a sequence $\{\alpha_j\}$ in X such that $G\alpha_j = J\alpha_j = \mu$ for some $\mu \in G(X) \cup J(X)$ as $j \to \infty$.

Since $G(X) \subseteq H(X)$ then there exists a sequence $\{\beta_j\}$ in X such that $G\alpha_j = H\beta_j$.

Therefore $G\alpha_j = J\alpha_j = H\beta_j = \mu$ as for some $\mu \in G(X) \cup J(X)$.

Now we prove that $I\beta_j = \mu$ as $j \to \infty$.

Now in (C2) on putting $\alpha = \alpha_j$, $\beta = \beta_j$,we get

$$\|(G\alpha_j - I\beta_j)\|^{2p} \leq [\phi_0(\|(J\alpha_j - H\beta_j)\|^{2p}) + (1-a)max\{\phi_1(\|J\alpha_j - H\beta_j\|^{2p}), \phi_2(\|J\alpha_j - G\alpha_j\|^q \|H\beta_j - I\beta_j\|^{q'}), \phi_3(\|J\alpha_j - I\beta_j\|^r \|H\beta_j - G\alpha_j\|^{r'}), \phi_4(\tfrac{1}{2}\|J\alpha_j - G\alpha_j\|^s \|H\beta_j - I\beta_j\|^{s'}), \phi_5(\tfrac{1}{2}\|J\alpha_j - I\beta_j\|^l \|H\beta_j - I\beta_j\|^{l'})\}].$$

Letting $j \to \infty$, which gives

$$\|\mu - I\beta_j\|^{2p} \leq [a\phi_0(\|\mu - \mu\|^{2p}) + (1-a)\max\{\phi_1(\|\mu - \mu\|^{2p}), \phi_2(\|\mu - \mu\|^q \|\mu - I\beta_j\|^{q'}), \phi_3(\|\mu - I\beta_j\|^r \|\mu - \mu\|^{r'}), \phi_4(\tfrac{1}{2}\|\mu - \mu\|^s \|\mu - \mu\|^{s'}), \phi_5(\tfrac{1}{2}\|\mu - I\beta_j\|^l \|\mu - \beta_j\|^{l'})\}]$$

.

$$\|\mu - I\beta_j\|^{2p} \leq [a\phi_0(0) + (1-a)max\{\phi_1(0), \phi_2(0), \phi_3(0), \phi_4(0), \phi_5(\tfrac{1}{2}\|\mu - I\beta_j\|^{2p})\}].$$

Since by Lemma(2.9)

$\|\mu - I\beta_j\|^{2p} \leq \phi(\|\mu - I\beta_j\|)^{2p} < \|\mu - I\beta_j\|^{2p}$

which is a contradiction.

Hence $I\beta_j = \mu$.

Therefore $G\alpha_j = J\alpha_j = H\beta_j = I\beta_j = \mu$ as $j \to \infty$ for some $\mu \in G(X) \cup J(X)$.

Let us assume that for $G\alpha_j = \mu$ as $j \to \infty$ for some $\mu \in J(X)$ then we can find $u \in X$ such that $Ju = \mu$.

Now in the inequality (C2) substitute $\alpha = u$ and $\beta = \beta_{2j+1}$,we get

$$\|Gu - I\beta_{2j+1}\|^{2p} \leq [a\phi_0(\|Ju - H\beta_{2j+1}\|^{2p} + (1-a)max\{\phi_1(\|Ju - H\beta_{2j+1}\|^{2p}), \phi_2(\|Ju - Gu\|^q \|H\beta_{2j+1} - I\beta_{2j+1}\|^{q'}), \phi_3(\|Ju - I\beta_{2j+1}\|^r \|H\beta_{2j+1} - Gu\|^{r'}), \phi_4(\tfrac{1}{2}\|Ju - Gu\|^s \|H\beta_{2j+1} - Gu\|^{s'}), \phi_5(\tfrac{1}{2}\|Ju - I\beta_{2j+1}\|^l \|H\beta_{2j+1} - I\beta_{2j+1}\|^{l'})\}].$$

Letting $j \to \infty$, which gives

$$\|Gu - \mu\|^{2p} \leq [a\phi_0(\|\mu - \mu\|^{2p}) + (1-a)\max\{\phi_1(\|\mu - \mu\|^{2p}), \phi_2(\|\mu - Gu\|^q \|\mu - \mu\|^{q'}), \phi_3(\|\mu - \mu\|^r \|\mu - Gu\|^{r'}), \phi_4(\tfrac{1}{2}\|\mu - Gu\|^s \|\mu - Gu\|^{s'}), \phi_5(\tfrac{1}{2}\|Ju - \mu\|^l \|\mu - \mu\|^{l'})\}]$$

$$\|Gu - \mu\|^{2p} \leq [a\phi_0(0) + (1-a)\max\{\phi_1(0), \phi_2(0), \phi_3(0), \phi_4(\tfrac{1}{2}\|\mu - Gu\|^{s+s'}), \phi_5(0)\}]$$

$$\|Gu - \mu\|^{2p} \leq [a\phi_0(0) + (1-a)\max\{\phi_1(0), \phi_2(0), \phi_3(0), \phi_4(\tfrac{1}{2}\|\mu - Gu\|^{2p}), \phi_5(0)\}].$$

Since by Lemma(2.9)

$\|Gu - \mu\|^{2p} \leq \phi(\|Gu - \mu\|)^{2p} < \|Gu - \mu\|^{2p}$

which is a contradiction,and hence $Gu = \mu$.

Therefore

$$Gu = Ju = \mu. \tag{3.3}$$

Since the pair (G,J) weakly compatible, G and J commute at a point of coincedence.

This gives $GJu = JGu$ and hence $G\mu = J\mu$

Now the pair (H,I) satisfies the E.A like property, there exists a sequence $\{\beta_j\}$ in X such that $H\beta_j = I\beta_j = \mu$ for some

$\mu \in H(X) \cup I(X)$ as $j \to \infty$.
Since $I\beta_j = \mu$ as $j \to \infty$ for some $\mu \in H(X)$ there exists a point $v \in X$ such that $Hv = \mu$

Now we prove that $Iv = \mu$.
Again in the inequality (C2) substitute $\alpha = \alpha_{2j}, \beta = v$
$\|G\alpha_{2j} - Iv\|^{2p} \leq [a\phi_0(\|J\alpha_{2j} - Hv\|^{2p})$
$+ (1-a)\max\{\phi_1(\|J\alpha_{2j} - Hv\|^{2p}),$
$\phi_2(\|J\alpha_{2j} - G\alpha_{2j}\|^q \|Hv - Iv\|^{q'}),$
$\phi_3(\|J\alpha_{2j} - Iv\|^r \|Hv - G\alpha_{2j}\|^{r'}),$
$\phi_4(\frac{1}{2}\|J\alpha_{2j} - G\alpha_{2j}\|^s \|Hv - G\alpha_{2j}\|^{s'}),$
$\phi_5(\frac{1}{2}\|J\alpha_{2j} - Iv\|^l \|Hv - Iv\|^{l'})\}].$
Letting $j \to \infty$, which gives

$\|\mu - Iv\|^{2p} \leq [a\phi_0(\|\mu - Hv\|^{2p})$
$+ (1-a)max\{\phi_1(\|\mu - Hv\|^{2p}),$
$\phi_2(\|\mu - \mu\|^q \|\mu - Iv\|^{q'}),$
$\phi_3(\|\mu - Iv\|^r \|\mu - \mu\|^{r'}),$
$\phi_4(\frac{1}{2}\|\mu - \mu\|^s \|\mu - \mu\|^{s'}),$
$\phi_5(\frac{1}{2}\|\mu - Iv\|^l \|\mu - Iv\|^{l'})\}].$

$\|\mu - Iv\|^{2p} \leq [a\phi_0(\|\mu - I\mu\|^{2p}) +$
$(1-a)\max\{\phi_1(\|0\|^{2p}), \phi_2(0),$
$\phi_3(0), \phi_4(0),$
$\phi_5(\frac{1}{2}\|\mu - Iv\|^{2p})\}].$
Since by Lemma(2.9)
$\|\mu - Iv\|^{2p} \leq \phi(\|\mu - Iv\|)^{2p} < \|\mu - Iv\|^{2p}$
which is a contradiction. Therefore $Iv = \mu$. This implies that

$$Iv = Hv = \mu. \tag{3.4}$$

Again the pair (I, H) is weakly compatible, I and H commute at point of coincidence.
This gives $IHv_j = HIv$ and this implies $I\mu = H\mu$.

Now we prove that $G\mu = \mu$. Again in the inequality (C2) substitute $\alpha = \mu, \beta = v$
$\|G\mu - Iv\|^{2p} \leq [a\phi_0(\|J\mu - Hv\|^{2p})$
$+ (1-a)\max\{\phi_1(\|J\mu - Hv\|^{2p}),$
$\phi_2(\|J\mu - G\mu\|^q \|Hv - Iv\|^{q'}),$
$\phi_3(\|J\mu - Iv\|^r \|Hv - G\mu\|^{r'}),$
$\phi_4(\frac{1}{2}\|J\mu - G\mu\|^s \|Hv - G\mu\|^{s'}),$
$\phi_5(\frac{1}{2}\|J\mu - Iv\|^l \|Hv - Iv\|^{l'})\}].$

$\|G\mu - \mu\|^{2p} \leq [a\phi_0(\|G\mu - \mu\|^{2p}) +$
$(1-a)\max\{\phi_1(\|G\mu - \mu\|^{2p}),$
$\phi_2(\|G\mu - G\mu\|^q \|\mu - \mu\|^{q'}),$
$\phi_3(\|\mu - \mu\|^r \|\mu - G\mu\|^{r'}),$
$\phi_4(\frac{1}{2}\|G\mu - G\mu\|^s \|\mu - G\mu\|^{s'}),$
$\phi_5(\frac{1}{2}\|G\mu - \mu\|^l \|\mu - \mu\|^{l'})\}].$

$\|G\mu - \mu\|^{2p} \leq [a\phi_0(\|G\mu - \mu\|^{2p}) +$
$(1-a)\max\{\phi_1(\|G\mu - \mu\|^{2p}),$
$\phi_2(0), \phi_3(\|\mu - G\mu\|^{2p}),$
$\phi_4(\frac{1}{2}\|G\mu - G\mu\|^{2p}), \phi_5(0)\}].$

Since by Lemma(2.9)
$\|G\mu - \mu\|^{2p} \leq \phi(\|G\mu - \mu\|)^{2p} < \|G\mu - \mu\|^{2p}$
which is a contradiction. Therefore $G\mu = \mu$.
Hence

$$G\mu = J\mu = \mu. \tag{3.5}$$

Now we prove that $I\mu = \mu$. For this substitute $\alpha = u$ and $\beta = \mu$ in the inequality (C2)
$\|Gu - I\mu\|^{2p} \leq [a\phi_0(\|Ju - H\mu\|^{2p})$
$+ (1-a)\max\{\phi_1(\|Ju - H\mu\|^{2p}),$
$\phi_2(\|Ju - Gu\|^q \|H\mu - I\mu\|^{q'}),$
$\phi_3(\|Ju - I\mu\|^r \|H\mu - Gu\|^{r'}),$
$\phi_4(\frac{1}{2}\|Ju - Gu\|^s \|H\mu - Gu\|^{s'}),$
$\phi_5(\frac{1}{2}\|Ju - I\mu\|^l \|H\mu - I\mu\|^{l'})\}]$

$\|\mu - I\mu\|^{2p} \leq [a\phi_0(\|\mu - I\mu\|^{2p}) +$
$(1-a)\max\{\phi_1(\|\mu - I\mu\|^{2p}),$
$\phi_2(\|\mu - \mu\|^q \|I\mu - I\mu\|^{q'}),$
$\phi_3(\|\mu - I\mu\|^r \|I\mu - \mu\|^{r'}),$
$\phi_4(\frac{1}{2}\|\mu - \mu\|^s \|I\mu - \mu\|^{s'}),$
$\phi_5(\frac{1}{2}\|\mu - I\mu\|^l \|I\mu - I\mu\|^{l'})\}]$
letting $j \to \infty$, which gives
$\|\mu - I\mu\|^{2p} \leq [a\phi_0(\|\mu - I\mu\|^{2p})$
$+ (1-a)\max\{\phi_1(\|\mu - I\mu\|^{2p}),$
$\phi_2(0),$
$\phi_3(\|\mu - I\mu\|^{2p}),$
$\phi_4(0), \phi_5(0)\}]$

$\|\mu - I\mu\|^{2p} \leq \phi(|\mu - I\mu\|^{2p}) < |\mu - I\mu\|^{2p}$ which is a contradiction. Therefore $I\mu = \mu$.

$$I\mu = H\mu = \mu. \tag{3.6}$$

Therefore from (3.5) and (3.6)
we get $G\mu = J\mu = I\mu = H\mu = \mu$.
Hence this shows μ is a common fixed point for the four mappings.

For uniqueness
Suppose μ and μ^* ($\mu \neq \mu^*$) are common fixed points of G,J,H and I and then substitute $\alpha = \mu$ and $\beta = \mu^*$ in the inequality (C2)
$\|G\mu - I\mu^*\|^{2p} \leq [a\phi_0(\|J\mu - H\mu^*\|^{2p}) +$
$(1-a)\max\{\phi_1(\|J\mu - H\mu^*\|^{2p}),$
$\phi_2(\|J\mu - G\mu\|^q \|H\mu^* - I\mu^*\|^{q'}),$
$\phi_3(\|J\mu - I\mu^*\|^r \|H\mu^* - G\mu\|^{r'}),$
$\phi_4(\frac{1}{2}\|J\mu - G\mu\|^s \|H\mu^* - G\mu\|^{s'}),$
$\phi_5(\frac{1}{2}\|J\mu - I\mu^*\|^l \|H\mu^* - I\mu^*\|^{l'})\}]$

$\|\mu - \mu^*\|^{2p} \leq [a\phi_0(\|\mu - \mu^*\|^{2p})$
$+ (1-a)\max\{\phi_1(\|\mu - \mu^*\|^{2p}), \phi_2(0),$
$\phi_3(\|\mu - \mu^*\|^r \|\mu^* - \mu\|^{r'}), \phi_4(0), \phi_5(0)\}]$
Since by Lemma(2.9)
$\|\mu - \mu^*\|^{2p} \leq \phi(\|\mu - \mu^*\|^{2p}) < \|\mu - \mu^*\|^{2p}$

which is a contraction. Therefore $\mu = \mu^*$, this proves the uniqueness.

Now we justify our main result with an example

□

4. Example

Suppose $X = [0,1]$ in Banach space with $\|\alpha - \beta\| = |\alpha - \beta|, \forall \alpha, \beta \in X$. We define self maps G,J,H and I as follows

$$G(\alpha) = I(\alpha) = \begin{cases} \frac{4-3\alpha}{5} & \text{if } 0 \le \alpha \le \frac{1}{2}; \\ \frac{4\alpha-1}{4} & \text{if } \frac{1}{2} < \alpha \le 1. \end{cases}$$

$$H(\alpha) = J(\alpha) = \begin{cases} \frac{2+\alpha}{5} & \text{if } 0 \le \alpha \le \frac{1}{2}; \\ \frac{3\alpha-1}{2} & \text{if } \frac{1}{2} < \alpha \le 1. \end{cases}$$

G(X)=I(X)=$[\frac{1}{2}, \frac{4}{5}] \cup (\frac{1}{4}, \frac{3}{4}]$ while H(X)=J(X)=$[\frac{2}{5}, \frac{1}{2}] \cup (\frac{1}{4}, 1]$
$G(X) \subseteq H(X), I(X) \subseteq J(X)$ so that the condition($\mathbf{C}-1$) is satiesfied.
Take a sequence $\alpha_k = \frac{1}{2} - \frac{1}{k}$ for $k > 0$.
Then $G(\alpha_k) = G(\frac{1}{2} - \frac{1}{k})$=$\frac{4+3(\frac{1}{2}-\frac{1}{k})}{5}$=$\frac{1}{2}$
and $J(\alpha_k) = J(\frac{1}{2} - \frac{1}{k})$=$\frac{2-(\frac{1}{2}-\frac{1}{k})}{5}$=$\frac{1}{2}$.
This gives $G\alpha_k$=$J\alpha_k$=$\frac{1}{2}$ as $k \to \infty$ and $\frac{1}{2} \in G(X) \cup J(X)$.
Similarly $H\alpha_k$=$I\alpha_k$=$\frac{1}{2}$ ask $\to \infty$and,$\frac{1}{2}$,$\in G(X) \cup J(X)$.
Hence the pairs (G,J) and (H,I) satisfy E.A like property.
Now $GJ(\alpha_k) = GJ(\frac{1}{2} - \frac{1}{k})$
=$G(\frac{(2+\frac{1}{2}-\frac{1}{k})}{5})$=G$(\frac{1}{2} - \frac{1}{5k})$
=$\frac{4-3(\frac{1}{2}-\frac{1}{5k})}{5}$=$\frac{1}{2}$
and $JG(\alpha_k) = JG(\frac{1}{2} - \frac{1}{k})$=J$(\frac{4-3(\frac{1}{2}-\frac{1}{k})}{5})$=$J\frac{(4-\frac{3}{2}+\frac{3}{k})}{5}$=$J(\frac{1}{2} + \frac{3}{5k})$
$= \frac{3(\frac{1}{2}+\frac{3}{5k}-1)}{2}$
$= \frac{1}{4} + \frac{9}{10k} = \frac{1}{4}$
$\lim_k \to \infty\|(GJ\alpha_k - JG\alpha_k)\| \neq 0$,
similarly $\lim_k \to \infty\|(HI\alpha_k - IH\alpha_k)\| \neq 0$ which implies the pairs $(G,J) and (H,I)$ are not compatible.
But $G(\frac{1}{2})$=$J(\frac{1}{2})$=$(\frac{1}{2})$.
This gives $GJ\frac{1}{2})$=$G(\frac{2+\frac{1}{2}}{5})$=$G(\frac{1}{2}) = \frac{4-3(\frac{1}{2})}{5} = \frac{1}{2}$
and $JG(\frac{1}{2})$=$J(\frac{4-\frac{1}{2}}{5})$=$J(\frac{1}{2}) = \frac{2+\frac{1}{2}}{5}$=$\frac{1}{2}$.
So that $GJ(\frac{1}{2}) = JG(\frac{1}{2})$.
Hence the pairs (G,J) and (H,I) are weakly compatiable.

Now we establish the condition **(C-2)**

Case I

If $\alpha, \beta \in [0, \frac{1}{2}]$,then we have $\|(G\alpha - I\beta)\| = |G\alpha - H\beta|$
put $\alpha = \frac{1}{6}, \beta = \frac{1}{8}$
Then the inequality**(C-2)** implies
$\|G(\frac{1}{6}) - I(\frac{1}{8})\|^{2p} \le [a\phi_0(\|J(\frac{1}{6}) - H(\frac{1}{8}))\|^{2p})$
$+(1-a)max\{\phi_1(\|J(\frac{1}{6}) - H(\frac{1}{8})\|^{2p}),$
$\phi_2(\|J(\frac{1}{6}) - G(\frac{1}{6})\|^q \|H(\frac{1}{8}) - I(\frac{1}{8})\|^{q'}),$
$\phi_3(\|J(\frac{1}{6}) - I(\frac{1}{8})\|^r \|H(\frac{1}{8}) - G(\frac{1}{6})\|^{r'}),$
$\phi_4(\frac{1}{2}\|J(\frac{1}{6}) - G(\frac{1}{6})\|^s \|H(\frac{1}{8}) - I(\frac{1}{8})\|^{s'}),$
$\phi_5(\frac{1}{2}\|J(\frac{1}{6}) - I(\frac{1}{8})\|^l \|H(\frac{1}{8}) - I(\frac{1}{8})\|^{l'})\}]$

for $a = \frac{1}{2}$ and $p = p' = q = q' = r = r' = s = s' = l = l' = \frac{1}{2}$
$\|0.025\| \le [a\phi_0(0.0025)\| + (1-a)max\{\phi_1(\|J(0.008)\|,$
$\phi_2(0.275)\|, \phi_3(0.28)\|,$
$\phi_4(0.133), \phi_5(0.14)\}]$
$|0.025| < |0.1425|$ Hence the condition (C2) is satisfied.

Case II

If $\alpha, \beta \in (\frac{1}{2}, 1]$,then we have $\|(G\alpha, I\beta)\| = |G\alpha - I\beta|$ put $\alpha = \frac{3}{4}, \beta = 1$
Then the inequality**(C-2)** implies

$\|G(\frac{3}{4}) - I(1)\|^{2p} \le [a\phi_0(\|J(\frac{3}{4}) - H(1))\|^{2p})$
$+(1-a)max\{\phi_1(\|J(\frac{3}{4}) - H(1)\|^{2p}),$
$\phi_2(\|J(\frac{3}{4}) - G(\frac{3}{4})\|^q \|H(1) - I(1)\|^{q'}),$
$\phi_3(\|J(\frac{3}{4}) - I(1)\|^r \|H(1) - G(\frac{3}{4})\|^{r'}),$
$\phi_4(\frac{1}{2}\|J(\frac{3}{4}) - G(\frac{3}{4})\|^s \|H(1) - I(1)\|^{s'}),$
$\phi_5(\frac{1}{2}\|J(\frac{3}{4}) - I(1)\|^l \|H(1) - I(1)\|^{l'})\}]$

for $a = \frac{1}{2}$ and $p = p' = q = q' = r = r' = s = s' = l = l' = \frac{1}{2}$
$\|0.25\| \le [\frac{1}{2}\phi_0(0.375)\| + (1-\frac{1}{2})max\{\phi_1(\|0.375\|,$
$\phi_2(0.175)\|, \phi_3(0.024)\|,$
$\phi_4(0.1237), \phi_5 0.0875)\}]$

$|0.25| \le [0.1875 + (1-\frac{1}{2})max\{0.375,$
$0.175, 0.024,$
$0.1237, 0.0875\}]$
$|0.25| < |0.375|$
Hence the inequality**(C-2)** holds.

Also we observe that $\frac{1}{2}$ is the
unique common fixed point for the four self mappings.

5. Corollary

As a particular case on letting $p = 1$, we get a corollary from Theorem 3.1.

Corollary 5.1. *Suppose in a Banach Space $(X, \|.\|)$,there are four mappings G,H,I and J holding the conditions*
$(C1)$ $G(X) \subseteq H(X)$ *and* $I(X) \subseteq J(X)$
$(C2)$ $\|(G\alpha - I\beta)\|^2 \le [a\phi_0(\|(J\alpha - H\beta)\|^2)$
$+(1-a)max\{\phi_1(\|J\alpha - H\beta\|^2),$
$\phi_2(\|J\alpha - G\alpha\|\|H\beta - I\beta\|),$
$\phi_3(\|J\alpha - I\beta\|\|H\beta - G\alpha\|),$
$\phi_4(\frac{1}{2}\|J\alpha - G\alpha\|\|H\beta - I\beta\|),$
$\phi_5(\frac{1}{2}\|J\alpha - I\beta\|\|H\beta - I\beta\|)\}$
for all $\alpha, \beta \in X$ *where* $\phi_k \in \phi, k = 0,1,2,3,4,5$, $0 \le a \le 1$, $0 < p, p', q, q', r, r'.s, s', l, l' \le 1$ *such that* $2p = p + p' = q + q' + r + r' + l + l'$.
$(C3)$ *The pairs* (G,J) *and* (I,H) *satisfy the E.A like property*

$(C4)$ *the pair of mappings* (G,J) *and* (I,H) *are weakly compatible.*
Then the above mappings have a unique common fixed point.

Similarly taking $p = \frac{1}{2}$, we get another corollary from Theorem 3.1.

Corollary 5.2. *Suppose in a Banach Space* $(X, \|.\|)$ *,there are four mappings G,H,I and J holding the conditions*
$(C1)$ $G(X) \subseteq H(X)$ *and* $I(X) \subseteq J(X)$
$(C2)\|(G\alpha - I\beta)\| \leq [a\phi_0(\|(J\alpha - H\beta)\|)$
$+(1-a)max\{\phi_1(\|J\alpha - H\beta\|),$
$\phi_2(\|J\alpha - G\alpha\|^{\frac{1}{2}}\|H\beta - I\beta\|^{\frac{1}{2}}),$
$\phi_3(\|J\alpha - I\beta\|^{\frac{1}{2}}\|H\beta - G\alpha\|^{\frac{1}{2}}),$
$\phi_4(\frac{1}{2}\|J\alpha - G\alpha\|^{\frac{1}{2}}\|H\beta - I\beta\|^{\frac{1}{2}}),$
$\phi_5(\frac{1}{2}\|J\alpha - I\beta\|^{\frac{1}{2}}\|H\beta - I\beta\|^{\frac{1}{2}})\}$
for all $\alpha,\beta \in X$ *where* $\phi_k \in \phi, k = 0,1,2,3,4,5$, $0 \leq a \leq 1$, $0 < p,p^{'},q,q^{'},r,r^{'}.s,s^{'},l,l^{'} \leq 1$ *such that* $2p = p + p^{'} = q + q' + r + r' + l + l'$.
$(C3)$ *The pairs* (G,J) *and* (I,H) *satisfy the E.A like property*

$(C4)$ *the pair of mappings* (G,J) *and* (I,H) *are weakly compatible.*
Then the above mappings have a unique common fixed point.

6. Conclusion

This paper aimed on a Banach space to establish a common fixed point theorem without using the continuity condition and also adopting E.A like property. Also two corollaries are obtained at the end of the paper from the main result.

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