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Weyl-semi symmetric special Para-Sasakian manifold

Indiwar Singh Chauhan[1]* and T.S. Chauhan[2]

Abstract
In this paper, we investigate the theory of Weyl-semi symmetric special Para-Sasakian. In section 1, we have defined special Para-Sasakian manifold and established a few properties thereof. Section 2 is devoted to the study of Weyl-pseudo symmetric and Weyl-semi symmetric special Para-Sasakian manifold. The results of this paper are believed to be new and unified in nature.

Keywords
Weyl-semi symmetric, Weyl-pseudo symmetric, Special Para-Sasakian manifold, Levi-Civita connection, Riemannian manifold.

AMS Subject Classification
53C25, 53Cxx, 53-XX.

[1,2] *Department of Mathematics, Bareilly College, Bareilly-243001, Uttar Pradesh, India.*
***Corresponding author**: [1] indiwarsingh.chauhan@gmail.com; [2]tarkeshwarsinghchauhan@gmail.com
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Contents

1. Introduction

Let M be a connected n-dimensional Riemannian manifold of class C^∞ with a positive definite metric g which admits a unit 1-from η satisfying

$$\nabla_\beta \eta_\alpha - \nabla_\alpha \eta_\beta = 0 \tag{1.1}$$

and

$$\nabla_\gamma \nabla_\beta \eta_\alpha = -(g_{\gamma\beta}\eta_\alpha + g_{\gamma\alpha}\eta_\beta) + 2\eta_\gamma\eta_\beta\eta_\alpha \tag{1.2}$$

wherein ∇ denotes the covariant differentiation with regard to Levi-Civita connection.

If we take

$$\xi^\alpha = g^{\alpha\beta}\eta_\beta \tag{1.3}$$

$$\eta_\alpha = g_{\alpha\beta}\xi^\beta \tag{1.4}$$

$$\phi^\alpha_\beta = \nabla_\beta \xi^\alpha \tag{1.5}$$

$$\phi_{\alpha\beta} = g_{\alpha\gamma}\phi^\gamma_\beta \tag{1.6}$$

Consequently, we obtain

$$\eta_\alpha \xi^\alpha = 1 \tag{1.7}$$

$$\phi_{\alpha\beta} = \phi_{\beta\alpha} \tag{1.8}$$

$$\phi^\alpha_\beta \xi^\beta = 0 \tag{1.9}$$

$$\phi^\alpha_\beta \eta_\alpha = 0 \tag{1.10}$$

$$\phi^\gamma_\beta \phi^\alpha_\gamma = \delta^\alpha_\beta - \eta_\beta \xi^\alpha \tag{1.11}$$

$$g_{\gamma\varepsilon}\phi^\gamma_\alpha \phi^\varepsilon_\beta = g_{\alpha\beta} - \eta_\alpha \xi^\beta \tag{1.12}$$

and

$$rank(\phi^{\alpha}_{\beta}) = (n-1) \quad (1.13)$$

These relations shows that the manifold M is a special para contact Riemannian manifold with a structure (ϕ,ξ,η,g). Such a manifold is called a Para-Sasakian manifold [1,5].

If in a Para-Sasakian manifold M the unit 1-form η-satisfying the relation

$$\bigtriangledown_{\beta}\eta_{\alpha} = \varepsilon(-g_{\beta\alpha}+\eta_{\beta}\eta_{\alpha}), \quad (1.14)$$

wherein $\varepsilon = \pm 1$, then the manifold M is termed as special Para-Sasakian manifold or briefly SP-Sasakian manifold [4].

From [2], we have

$$S_{\alpha\beta}\xi_{\beta} = -(n-1)\eta_{\alpha}. \quad (1.15)$$

$$\eta_{\lambda}R^{\lambda}_{\alpha\beta\gamma} = g_{\alpha\beta}\eta_{\gamma} - g_{\beta\gamma}\eta_{\alpha}. \quad (1.16)$$

$$g^{\alpha\beta}S_{\alpha\beta} = \tau. \quad (1.17)$$

2. Weyl-Semi Symmetric Special Para-Sasakian Manifold

Let M be an n-dimensional (n⩾3) differentiable manifold of class C^{∞} and $\bigtriangledown$ denotes its Levi-Civita connection. Also let S is the Ricci tensor of n-dimensional differentiable manifold M.

The Ricci operator S is defined as

$$S_{\alpha\beta}S^{\beta\gamma} = S^{\gamma}_{\alpha} \quad (2.1)$$

and the covariant tensor of rank two (S^2) is defined as

$$(S^2)_{\alpha\beta} = (S.S)_{\alpha\beta} = S_{\alpha a}S^{a}_{\beta}. \quad (2.2)$$

The Weyl conformal curvature operator is defined as

$$C^{\alpha}_{\beta} = R^{\alpha}_{\beta} - \frac{1}{(n-2)}[\delta^{\alpha}_{a}S^{a}_{\beta} + S^{\alpha}_{a}\delta^{a}_{\beta} - \frac{k}{(n-1)}\delta^{\alpha}_{a}S^{a}_{\beta}] \quad (2.3)$$

and the Weyl conformal curvature tensor is defined as

$$C_{\alpha\beta\gamma\varepsilon} = g_{\gamma\varepsilon}C_{\alpha\beta}, \quad (2.4)$$

wherein k is the scalar curvature of n-dimensional differentiable manifold M.

Definition 2.1. *If the tensor $R.C$ and $Q(g,C)$ are linearly dependent then the manifold M is termed as Weyl-Pseudo symmetric special Para-Sasakian manifold [2,3].*

This is equivalent to

$$R.C = L_{C}Q(g,C). \quad (2.5)$$

Definition 2.2. *A special Para-Sasakian manifold M with the properties*

$$C.S = 0 \quad (2.6)$$

is termed as Weyl semi-symmetric special Para-Sasakian manifold.

Remark 2.3. *It is noteworthy that a conformally symmetric special Para-Sasakian manifold is Weyl semi-symmetric.*

Next, we define the tensor C.S on (M,g) as follows

$$C^{\alpha}_{\beta}S^{\gamma}_{\varepsilon} = -(S_{\beta\varepsilon}C^{\alpha\gamma} + S^{\alpha\gamma}C_{\beta\varepsilon}). \quad (2.7)$$

Equation (2.7) can be written as

$$S_{\alpha\gamma}C^{\gamma}_{\beta} + S_{\alpha\varepsilon}C^{\varepsilon}_{\beta} = 0. \quad (2.8)$$

Contracting equation (2.8) by ξ^{α} and using equation (1.15) yields

$$\eta_{\gamma}C^{\gamma}_{\beta} + \eta_{\varepsilon}C^{\varepsilon}_{\beta} = 0. \quad (2.9)$$

By virtue of equations (1.15), (1.16), (2.2) and (2.3), we obtain

$$\begin{aligned}&\eta_{\beta}S_{\alpha\gamma} + \eta_{\gamma}S_{\alpha\beta} - (1-n)(\eta_{\gamma}g_{\alpha\beta} + \eta_{\beta}g_{\alpha\gamma})\\ &\quad + \frac{1}{(n-2)}[(S.S)_{\alpha\gamma} + \eta_{\gamma}(S.S)_{\alpha\beta} - (1-n)^2(\eta_{\beta}g_{\alpha\gamma}\\ &\quad + \eta_{\gamma}g_{\alpha\beta}) + \frac{k}{(n-1)(n-2)}\{(1-n)(\eta_{\beta}g_{\alpha\gamma}\\ &\quad + \eta_{\gamma}g_{\alpha\beta}) - \eta_{\beta}S_{\alpha\gamma} - \eta_{\gamma}S_{\alpha\beta}\}] = 0. \quad (2.10)\end{aligned}$$

Contracting equation (2.10) by ξ^{γ} and using equations (1.15), (2.2), we get

$$(S.S)_{\alpha\beta} = \frac{k-(n-1)(n-2)}{(n-1)}S_{\alpha\beta} + (k+n-1)g_{\alpha\beta}. \quad (2.11)$$

In view of above discussion, we observe the following theorem:

Theorem 2.4. *If n-dimensional special Para-Sasakian manifold is Weyl-semi symmetric then the following condition (2.11) holds good.*

Let us consider an η-Einstein special Para-Sasakian manifold, then we can write [2]:

$$S_{\alpha\beta} = ag_{\alpha\beta} + b\eta_{\alpha}\eta_{\beta}, \quad (2.12)$$

wherein a and b are smooth functions on M.

Contracting equation (2.12) with $g^{\alpha\beta}$ and using equation (1.17), we get

$$na + b = \tau. \quad (2.13)$$

Further, contracting equation (2.12) with ξ^{β} and using equations (1.7), (1.15) yields

$$a + b = (1-n). \quad (2.14)$$

Subtracting equation (2.14) from equation (2.13), we get

$$a = 1 - \frac{\tau}{(1-n)}. \tag{2.15}$$

Inserting this value of a in equation (2.14), we obtain

$$b = \frac{\tau}{(1-n)} - n. \tag{2.16}$$

Consequently, we have a theorem:

Theorem 2.5. *If η-Einstein special Para-Sasakian manifold is Weyl-semi symmetric admits a vector field ξ^α characterised by the relation (2.12) then the smooth functions are connected by the relations (2.15) and (2.16).*

Substituting the values of a and b in equation (2.12), we get

$$S_{\alpha\beta} = (1 - \frac{\tau}{(1-n)})g_{\alpha\beta} + (\frac{\tau}{(1-n)} - n)\eta_\alpha\eta_\beta. \tag{2.17}$$

Consequently, we have a theorem:

Theorem 2.6. *If a special Para-Sasakian manifold is an η-Einstein admits a condition C.S = 0, and a vector field ξ^α characterised by the relation (2.12) then the Ricci tensor holds the relation (2.17).*

In this regard, we have a theorem:

Theorem 2.7. *For an η-Einstein special Para-Sasakian manifold with the condition C.S = 0, the following relation $S_{\alpha\beta}\phi^\beta_\gamma = (1 - \frac{\tau}{(1-n)})\phi_{\alpha\gamma}$ holds good.*

Proof. Contracting equation (2.17) with ϕ^β_γ and using equations (1.6), (1.10) yields

$$S_{\alpha\beta}\phi^\beta_\gamma = (1 - \frac{\tau}{(1-n)})\phi_{\alpha\gamma}. \tag{2.18}$$

□

From equations (1.12) and (2.17), we get

$$S_{\alpha\beta} = (1-n)g_\alpha\beta - (\frac{\tau}{(1-n)} - n)g_{\gamma\varepsilon}\phi^\gamma_\alpha\phi^\varepsilon_\beta. \tag{2.19}$$

As a consequence of equations (1.6) and (2.19), we obtain

$$S_{\alpha\beta} = (1-n)g_\alpha\beta - (\frac{\tau}{(1-n)} - n)\phi_{\varepsilon\alpha}\phi^\varepsilon_\beta. \tag{2.20}$$

By virtue of equations (1.5) and (2.20), we observe that

$$S_{\alpha\beta} = (1-n)g_\alpha\beta - (\frac{\tau}{(1-n)} - n)(\nabla_\varepsilon\eta_\alpha)(\nabla_\beta\xi^\varepsilon). \tag{2.21}$$

Contracting equation (2.20) with ξ^β and using equation (1.9) yields

$$S_{\alpha\beta}\xi^\beta = -(n-1)\eta_\alpha. \tag{2.22}$$

This expression obtained above is similar to the expression (1.15) given by Mileva Prvanovic [2].

In view of above, we have the following theorems:

Theorem 2.8. *For η-Einstein special Para-Sasakian manifold, the relation τ=-(n-1) holds good.*

Proof. Contracting equation (2.22) with η_β and using equation (1.7), we obtain

$$S_{\alpha\beta} = -(n-1)\eta_\alpha\eta_\beta. \tag{2.23}$$

Again contracting equation (2.23) with $g^{\alpha\beta}$ and using equations (1.3), (1.7) yields

$$g^{\alpha\beta}S_{\alpha\beta} = -(n-1). \tag{2.24}$$

From equations (1.17) and (2.24), we get

$$\tau = -(n-1) \tag{2.25}$$

□

Theorem 2.9. *If η-Einstein special Para-Sasakian manifold admits C.S. = 0, then the following relation $(S.S)_{\alpha\beta}\phi^\beta_\gamma = (k+n-1)\phi_{\alpha\gamma}$ holds good.*

Proof. Contracting equation (2.23) with ϕ^β_γ and using the equation (1.10) yields

$$S_{\alpha\beta}\phi^\beta_\gamma = 0. \tag{2.26}$$

Contracting equation (2.11) with ϕ^β_γ and using equations (2.26), we get

$$(S.S)_{\alpha\beta}\phi^\beta_\gamma = (k+n-1)\phi_{\alpha\gamma}. \tag{2.27}$$

□

References

[1] Indiwar Singh Chauhan, T.S. Chauhan, Rajeev Kumar Singh and Mohd. Rizwan, A note on D-Conformal para killing vector field in a P-Sasakian manifold, *Journal of Xidian University* 14(2020), 326–329.

[2] Mileva Prvanovic, On a class of SP-Sasakian manifold, *Note di Mathematica,* X(1990),325–334.

[3] R. Deszcz and M. Hotlos, On geodesic mappings in pseudo symmetric manifolds, *Bull. Inst. Math. Acad. Sinica,* 16 (1988), 251–262.

[4] T.S. Chauhan, A note on SP-Sasakian manifold, *Acta Ciencia Indica,* XXVI(2000), 301–304.

[5] T.S. Chauhan, A note on recurrent P-Sasakian manifolds, *Tensor, N. S.,* 62(2000), 215–218.

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