Malaya Journal of Matematik, Vol. 8, No. 3, 961-965, 2020
https://doi.org/10.26637/MJM0803/0040

Automorphism group of L-fuzzy topologies when L is a diamond-type lattice

N. M. Madhavan Namboothiri [1]*, P. K. Santhosh [2] and P.N. Jayaprasad [3]

Abstract
We determined the group of all automorphisms of the lattice of all L-fuzzy topologies on a fixed set X, when L is a diamond-type lattice.

Keywords
Automorphism group, homomorphism, Lattice, Fuzzy Topology.

AMS Subject Classification
54A40, 54A05, 03E72, 06A05.

[1,3] *Department of Mathematics, Government College Kottayam, Nattakom , Kottayam, Kerala, India.*
[2] *Department of Applied Science, Government Engineering College Kozhikode, Kozhikode, Kerala, India.*
***Corresponding author**: [1] madhavangck@gmail.com; [2]santhoshgpm2@gmail.com, [3]jayaprasadpn@gmail.com
Article History: Received **12** March **2017**; Accepted **19** June **2020**

Contents

1. Introduction

The group of all automorphisms of lattices are investigated by many authors in different contexts [5–7, 10]. In 1965 Zadeh [16] introduced the fundamental concepts of fuzzy set. Subsequently Chang [2] introduced fuzzy theory into topology and that led to the discussion of various aspects of L-fuzzy topology by many authors. It is a fact that order relations play an important role in L-fuzzy topological spaces. Many authors made attempts to study the relation between fuzzy topological structures and corresponding order structures [3, 4, 6, 8, 12, 15]. In 1958, Hartmanis J. [7] determined the automorphisms of the lattice $LT(X)$ of all topologies on a fixed set X. If X is an infinite set and P is any topological property, then the set of all topologies in $LT(X)$ possessing the property P can be identified exclusively from the lattice structure of $LT(X)$ and hence the position of the topologies in $LT(X)$ determines the topological properties of elements of $LT(X)$ [10]. Here we determine the group of automorphisms of the lattice $LFT(X,\ L)$, in the cases when L is a diamond-type lattice.

The contents of this paper are arranged as follows. In section 2, some basic definitions and results in Lattices, Fuzzy set theory and Fuzzy topology. The main results of this paper are discussed in Section 3. The most of the fuzzy topological terminologies carried in this paper are that of Liu Ying Ming, Liu Mao Kang [11] and the order terminologies are that of Birkhoff G. [1].

2. Preliminaries

A partially ordered set L is called a lattice, if every finite subset of L has a join (or the least upper bound, or supremum) and meet (or greatest lowerbound or infimum) in L. An element α in a lattice L is called an atom in L, if it is a minimal element in $L-\{0\}$. A lattice L is called a complete lattice, if every subset of L has a meet and join and such a lattice is is called completely distributive if for any non empty index set I, collection of index sets $\{J_i : i \in I\}$ and for all $\{\{a_{ij} : j \in J_i\} : i \in I\} \subseteq (P(L)-\{\emptyset\})$,

(i) $\bigwedge_{i\in I}(\bigvee_{j\in J_i} a_{ij}) = \bigvee_{\varphi\in\prod_{i\in I} J_i}(\bigwedge_{i\in I} a_{i,\varphi(i)})$,

(ii) $\bigvee_{i\in I}(\bigwedge_{j\in J_i} a_{ij}) = \bigwedge_{\varphi\in\prod_{i\in I} J_i}(\bigvee_{i\in I} a_{i,\varphi(i)})$.

Let $L = \{\ 0,\ a,\ b,\ 1\}$. Define a partial order $\leq$ on L as $0 \leq a \leq 1$, $0 \leq b \leq 1$ but a and b are not comparable. Then L is a lattice, called the diamond-type lattice or simply diamond lattice. In a lattice L a mapping $' : L \longrightarrow L$ is called order reversing, if $a \leq b$ implies $a' \geq b'$ for all $a, b \in L$ and a mapping

$' : L \longrightarrow L$ is called an involution on L, if $'' = id_L$. A completely distributive lattice L is called an F-lattice, if there is an order reversing involution from L to L. A homomorphism from a lattice L into a lattice M is a function $f : L \longrightarrow M$ such that $a \leq b$ implies $f(a) \leq f(b)$ for all $a, b \in L$. A bijective homomorphism $f : L \longrightarrow M$ is called an isomorphism if its inverse is also a homomorphism. An isomorphism from a lattice onto itself is called an automorphism. Observe that an automorphism of a lattice maps atoms to atoms and dual atoms to dual atoms. For an L fuzzy space L^X corresponding to an F-lattice L, L^X is an F-lattice, with respect to the ordering, for $C, D \in L^X$, $C \leq D \Longleftrightarrow C(t) \leq D(t)$ for all $t \in X$ and the order-reversing involution as the pseudo complementary operation on L^X. Also it can be noted that if L is a complemented F-lattice then L^X is also a complemented F-lattice and hence it is a Boolean lattice.

Let X be any nonempty set, L an F-Lattice, and $\delta \subseteq L^X$ then δ is called an L-fuzzy topology on X or (L^X, δ) is called an L-fuzzy topological space if (i) $\underline{0}, \underline{1} \in \delta$ (ii) for all $\mathscr{A} \subseteq \delta$, $\vee \mathscr{A} \in \delta$ and (iii) for all $A, B \in \delta$, $A \wedge B \in \delta$. An L-fuzzy point in a nonempty set X is an L-fuzzy subset $x_a \in L^X$ defined by $x_a(y) = a$ for $y = x$ and $x_a(y) = 0$ for $y \neq x$. The set of all L-fuzzy points in X is denoted by $Pt(L^X)$. It can be noted that the set of all L-fuzzy topologies on L^X is a complete lattice under the usual inclusion relation and is denoted by $LFT(X, L)$.

3. Main Results

Let us consider the diamond-type lattice $L = \{0, a, b, 1\}$. Define an order reversing involution $'$ on L by $0' = 1$, $1' = 0$, $a' = b$, $b' = a$. Then L is an F-lattice. Note that L is complemented F-lattice and hence it is a Boolean. Here onwards L represents the diamond-type lattice, unless otherwise specified. Here the atoms of L^X are precisely $\{x_l \in L^X : x \in X, l \in \{a, b\}\}$. Also the lattice L^X is atomistic.

Let X be a nonempty set. For any permutation $p = (p_1, p_2)$ with $p_1 \in S(X)$ and $p_2 \in S(\{a, b\})$, define p^* on $Pt(L^X)$ by $p^*(x_l) = y_m$ if and only if $p(x, l) = (y, m)$ for all $x_l \in Pt(L^X)$, $l \neq 1$ and $p^*(x_1) = p^*(x_a) \vee p^*(x_b)$ for all $x \in X$. Clearly p^* is a bijection and for $C, D \in L^X$ with $C(x) \neq 0$ and $D(x) \neq 0$, $p^*(x_{(C \vee D)(x)}) = p^*(x_{C(x)}) \vee p^*(x_{D(x)})$.

Theorem 3.1. *Let X be a nonempty set. Then the automorphisms of L^X are precisely $\{E_p : p \in (S(X) \times S(\{a, b\}))\}$, where $E_p(C) = \bigvee_{x \in X,\ C(x) \neq 0} p^*(x_{C(x)})$ for any $C \in L^X$.*

Proof. For $C, D \in L^X$ with $C(x) \neq 0$ and $D(x) \neq 0$ $p^*(x_{(C \vee D)(x)}) = p^*(x_{C(x)}) \vee p^*(x_{D(x)})$. Also note that L^X is completely distributive. Therefore, for any $C, D \in L^X$,
$E_p(C \vee D) = \bigvee_{x \in X,\ (C \vee D)(x) \neq 0} p^*(x_{(C \vee D)(x)})$
$= (\bigvee_{x \in X,\ C(x) \neq 0} p^*(x_{C(x)})) \vee (\bigvee_{x \in X,\ D(x) \neq 0} p^*(x_{D(x)})) = E_p(C) \vee E_p(D)$.
By definition of E_p, $E_p(C) = \underline{0}$ if and only if $C = \underline{0}$ and $E_p(C) = \underline{1}$ if and only if $C = \underline{1}$.

Now consider any $C, D \in L^X$ with $C \neq 0$ and $D \neq 0$. Suppose that $E_p(C) = E_p(D)$, then we want to prove that $C = D$. Note that $E_p(C) = \bigvee_{x \in X,\ C(x) \neq 0} p^*(x_{C(x)})$ and
$E_p(D) = \bigvee_{x \in X,\ D(x) \neq 0} p^*(x_{D(x)})$.

Since p^* is a bijection on $Pt((L - \{0\})^X)$ and $E_p(C) = E_p(D)$ it is clear that for any $x \in X$, $C(x) \neq 0$ there exists $y \in X, D(y) \neq 0$ such that $p^*(x_{C(x)}) = p^*(y_{D(y)})$. Now p^* is a bijection implies that $x_{C(x)} = y_{D(y)}$ and hence $x = y$ and $C(x) = D(y)$. Thus for any $x \in X, C(x) \neq 0$, we have $C(x) = D(x)$ and similarly for any $x \in X$, $D(x) \neq 0$, we have $C(x) = D(x)$. Thus $C(x) = D(x)$ for all $x \in X$. Therefore $C = D$. Therefore, E_p is an injection on L^X. Now for $C \in L^X$ with $C \neq 0$, define $D = \bigvee_{x \in X,\ C(x) \neq 0} (p^*)^{-1}(x_{C(x)})$ then $D \in L^X$ and
$E_p(D) = \bigvee_{x \in X,\ C(x) \neq 0} x_{C(x)} = C$. Hence E_p is onto. Thus E_p is a bijection on L^X. Thus E_p is a bijection which preserves order and hence an automorphism of L^X.

Conversely suppose that A is an automorphism of L^X. Then A maps atoms of L^X onto the atoms of L^X. That is A is a one to one, order preserving and maps $Pt(L^X)$ onto $Pt(L^X)$. Also $A(x_1) = A(x_a) \vee A(x_b)$. Thus $A = p^*$ on $Pt(L^X)$ for some $p \in (S(X) \times S(\{a, b\}))$.
Hence $A(C) = \bigvee_{x \in X,\ C(x) \neq 0} A(x_{C(x)}) = \bigvee_{x \in X,\ C(x) \neq 0} p^*(x_{C(x)}) = E_p(C)$.
That is $A = E_p$ for some $p \in (S(X) \times S(\{a, b\}))$. □

Definition 3.2. *Let X be a nonempty set. Then for each $p \in (S(X) \times S(\{a, b\}))$, define a mapping E_p^* on $LFT(X, L)$ by $E_p^*(\delta) = \{E_p(C) : C \in \delta\}$, for all $\delta \in LFT(X, L)$, where E_p as in Theorem 3.1.*

Remark 3.3. *Let X, L, p and E_p^* are as in Definition 3.2. Then E_p^* is an automorphism on $LFT(X, L)$.*

Definition 3.4. *Let X be any finite nonempty set. Then for each $p \in (S(X) \times S(\{a, b\}))$, define a mapping F_p on L^X by $F_p(C) = comp(E_p(C))$for all $C \in L^X$, where $comp(C)$ denotes the pseudo complement of C in L^X and E_p as in Theorem 3.1. Also define F_p^* on $LFT(X, L)$ for each $p \in (S(X) \times S(\{a, b\}))$ by $F_p^*(\delta) = \{F_p(C) : C \in \delta\}$ for all $\delta \in LFT(X, L)$.*

Remark 3.5. *Let X, L, p and F_p^* are as in Definition 3.4. Then F_p^* is an automorphism on $LFT(X, L)$.*

Definition 3.6. *Let us consider any atom $\{\underline{0}, C, \underline{1}\}$ in $LFT(X, L)$. Denote this atom by I_C . Now define the sets η, η', ζ and ζ' as follows.*
$\eta = \{I_{x_a} : x \in X\}$, $\eta' = \{I_{x^b} : x \in X\}$
$\zeta = \{I_{x_b} : x \in X\}$ *and* $\zeta' = \{I_{x^a} : x \in X\}$

where $x_l(t) = \begin{cases} l & \text{at } t{=}x \\ 0 & \text{otherwise} \end{cases}$ *for* $l \in \{a, b, 1\}$

and $x^l(t) = \begin{cases} l & \text{at } t{=}x \\ 1 & \text{otherwise} \end{cases}$ *for* $l \in \{a, b, 0\}$.

Now we will obtain the following results. The proof of these results needs only usual arguments hence we omit those proofs.

Lemma 3.7. *The join any atom from $G = (\eta \cup \eta') \cup (\zeta \cup \zeta')$ with any atom of $LFT(X,L)$ consists of at most 5 open sets.*

Lemma 3.8. *Let X be a nonempty set with more than one element,and I_C be an atom in $LFT(X,\ L)$ such that $I_C \notin G = (\eta \cup \eta') \cup (\zeta \cup \zeta')$. Then there exists $x,\ y,\ z_1,\ z_2 \in X$ with $x \neq y,\ z_1 \neq z_2$ so that it satisfies one of the following conditions.*

(i) $x_a < C < y^b$ and $(x_a(z_1) < C(z_1)$ and $C(z_2) < y^b(z_2))$

(ii) $x_a < C < y^a$ and $(x_a(z_1) < C(z_1)$ and $C(z_2) < y^a(z_2))$

(iii) $x_b < C < y^a$ and $(x_b(z_1) < C(z_1)$ and $C(z_2) < y^a(z_2))$

(iv) $x_b < C < y^b$ and $(x_b(z_1) < C(z_1)$ and $C(z_2) < y^b(z_2))$

Lemma 3.9. *Let X be a nonempty set with more than one element and p be an atom in $LFT(X,\ L)$ such that $u \notin G$. Then there is an atom v of $LFT(X,L)$ such that $u \vee v$ consists of 6 open sets.*

Lemma 3.10. *Let $\eta, \eta', \zeta,$ and ζ' are as in Definition 3.6. Then,*

(i) The join of any two distinct atoms in η consists of five open sets.

(ii) The join of any two distinct atoms in η' consists of five open sets.

(iii) The join of any two distinct atoms in ζ consists of five open sets.

(iv) The join of any two distinct atoms in ζ' consists of five open sets.

(v) The join of any atom in η with an atom in ζ consists of five open sets.

(vi) The join of any atom in η' with an atom in ζ' consists of five open sets.

(vii) The join of any atom in η with an atom in η' consists of four open sets.

(viii) The join of any atom in ζ with an atom in ζ' consists of four open sets.

(ix) The join of any atom in η with an atom in ζ' consists of four open sets.

(x) The join of any atom in ζ with an atom in η' consists of four open sets.

Remark 3.11. *From Lemma 3.10, we have,*

(1) The join of any two distinct atoms in $(\eta \cup \zeta)$ consists of five open sets.

(2) The join of any two distinct atoms in $(\eta\,' \cup \zeta\,')$ consists of five open sets.

(3) The join of any atom in $(\eta \cup \zeta)$ with an atom in $(\eta\,' \cup \zeta\,')$ consists of four open sets.

Lemma 3.12. *Let X be a nonempty set with more than one element. Then an automorphism of the lattice $LFT(X,\ L)$ maps $G = (\eta \cup \eta\,') \cup (\zeta \cup \zeta\,')$ onto itself.*

Lemma 3.13. *Let X be a nonempty set with more than one element. Then an automorphism of the lattice $LFT(X,\ L)$ maps either $[(\eta \cup \zeta)$ onto $(\eta \cup \zeta)$ and $(\eta\,' \cup \zeta\,')$ onto $(\eta\,' \cup \zeta\,')]$ or $[(\eta \cup \zeta)$ onto $(\eta\,' \cup \zeta\,')$ and $(\eta\,' \cup \zeta\,')$ onto $(\eta \cup \zeta)]$.*

Theorem 3.14. *For a finite nonempty set X, the set of all automorphisms of the lattice $LFT(X,L)$ is precisely given by $\{E_p^* : p \in (S(X) \times S(\{a,b\}))\} \cup \{F_p^* : p \in (S(X) \times S(\{a,b\}))\}$, where E_p^* and F_p^* are as in Definition 3.2 and 3.4.*

Proof. From remarks 3.3 and 3.5 E_p^* and F_p^* are automorphisms of $LFT(X,L)$ for each $p \in S(X \times \{a,b\})$. Thus we have to prove that any automorphism of $LFT(X,L)$ is of the form E_p^* or F_p^* for some $p \in (S(X) \times S(\{a,b\}))$.

If $X = \{x\}$, there is only two atoms I_{x_a} and I_{x_b} in $LFT(X,L)$. Also note that $comp(x_a) = x_b$. Let A be an automorphism of the lattice $LFT(X,L)$. If A maps I_{x_a} to I_{x_b} then it maps I_{x_b} and I_{x_a} and hence $A = E_p^*$ where $p \in (S(X) \times S(\{a,b\}))$ by $p(x,a) = (x,b)$ and $p(x,b) = (x,a)$. If A maps I_{x_a} to I_{x_a} then it maps I_{x_b} and I_{x_b} and hence $A = E_p^*$ where p is the identity permutation on $X \times \{a,b\}$. Since there is only two permutations on $X \times \{a,b\}$, the result is clear.

Now suppose that X contains more than one element. Let A be an automorphism of the lattice $LFT(X,L)$. Then by Lemma 3.13 A maps either $[(\eta \cup \zeta)$ onto $(\eta \cup \zeta)$ and $(\eta\,' \cup \zeta\,')$ onto $(\eta\,' \cup \zeta\,')]$ or $[(\eta \cup \zeta)$ onto $(\eta\,' \cup \zeta\,')$ and $(\eta\,' \cup \zeta\,')$ onto $(\eta \cup \zeta)]$. Suppose that A maps $(\eta \cup \zeta)$ onto $(\eta \cup \zeta)$ and $(\eta\,' \cup \zeta\,')$ onto $(\eta\,' \cup \zeta\,')$.

Consider the atom $I_{(x_l \vee z_p)}$. Note that join of $I_{(x_l \vee z_p)}$ with I_{x_l} and I_{z_p} gives four open sets, but the join of $I_{(x_l \vee z_p)}$ with any other atom in $(\eta \cup \zeta)$ gives five open sets. This property is preserved by any automorphism A of $LFT(X,\ L)$. Thus the join of $A(I_{(x_l \vee z_p)})$ with $A(I_{x_l}) = I_{y_m}$ and $A(I_{z_p}) = I_{w_q}$ gives four open sets, but the join of $A(I_{(x_l \vee z_p)})$ with any other atom in $(\eta \cup \zeta)$ gives five open sets. Let $A(I_{(x_l \vee z_p)}) = I_D$. Then D is comparable or complementary with y_m and w_q. But $I_D \notin (\eta\,' \cup \zeta\,')$. Thus D is not complementary with y_m or w_q. Thus D is comparable with y_m and w_q. By previous argument it can also note that D is not comparable with any $C \in L^X$ with $C \neq y_m$, $C \neq w_q$ and $I_C \in (\eta \cup \zeta)$.

Therefore $D(t) = 0$ for all $t \neq y$ and $t \neq w$. Also $D \nleq y^{m'}$ and $D \nleq w^{q'}$. Hence we have $D = y^m \vee w^q$. That is $A(I_{(x_l \vee z_p)}) = I_{(y_m \vee w_q)}$. Thus If $A(I_{x_l}) = I_{y_m}$ and $A(I_{z_p}) = I_{w_q}$, where $x,y,z,w \in X$ and $l,m,p,q \in \{a,b\}$, then $A(I_{(x_l \vee z_p)}) = I_{(y_m \vee w_q)}$. Similarly we have If $A(I_{x^l}) = I_{y^m}$ and $A(I_{z^p}) = I_{w^q}$, where $x,y,z,w \in X$ and $l,m,p,q \in \{a,b\}$, then $A(I_{(x^l \wedge z^p)}) = I_{(y^m \wedge w^q)}$.

Also note that $A(I_{x_l}) = I_{y_m}$ and $A(I_{x_{l'}}) = I_{w_p}$, where $x,y,w \in X$ and $l,m,p \in \{a,b\}$, if and only if $A(I_{x^{l'}}) = I_{y^{m'}}$ and $A(I_{x^l}) = I_{w^{p'}}$.

Now let $C \in L^X$, $I_c \notin G = (\eta \cup \zeta) \cup (\eta' \cup \zeta')$, $C \neq \underline{0}$ and $C \neq \underline{1}$, $B_1 = \bigvee_{x\in X,\ C(x)\neq 0, C(x)\neq 1} B(x_{C(x)})$, $B_a = \bigvee_{x\in X,\ C(x)=1} B(x_a)$ and $B_b = \bigvee_{x\in X,\ C(x)=1} B(x_b)$. Note that the join of I_c with the atoms $\{I_{x_{C(x)}} : C(x) \neq 0, C(x) \neq 1, x \in X\} \bigcup \{I_{x_a} : C(x) = 1,\ x \in X\} \bigcup \{I_{x_b} : C(x) = 1,\ x \in X\}$ gives four open sets and all other atoms in $(\eta \cup \zeta)$ gives five open sets. This property is preserved by automorphisms. Hence the join of $A(I_c)$ with the atoms $\{A(I_{x_{C(x)}}) : C(x) \neq 0, C(x) \neq 1, x \in X\} \bigcup \{A(I_{x_a}) : C(x) = 1, x \in X\} \bigcup \{A(I_{x_b}) : C(x) = 1, x \in X\}$ gives four open sets and all other atoms in $(\eta \cup \zeta)$ gives five open sets. Since $I_C \notin G$, we have $A(I_C) \notin G$. Let $A(I_C) = I_{C'}$. Then C' is not complementary with any $D \in L^X$, such that $I_D \in (\eta \cup \zeta)$. Thus the only possibility to get, join of I_c with the atoms of $(\eta \cup \zeta)$ gives four open sets is comparability. Thus $B(x_{C(x)}) \leq C'$ if $C(x) \neq 0, C(x) \neq 1, x \in X$, $B(x_a) \leq C'$ if $C(x) = 1,\ x \in X$ and $B(x_b) \leq C'$ if $C(x) = 1,\ x \in X$. Therefore $B_1 \vee B_a \vee B_b \leq C'$ and C' is not comparable with any $D \in L^X$ such that $I_D \in (\eta \cup \zeta)$ and $D \neq B(x_{C(x)})$ for any $x \in X,\ C(x) \neq 0$. Therefore, $C' = B_1 \vee B_a \vee B_b$. Hence $A(I_C) = I_{B_1 \vee B_a \vee B_b}$ where $A(I_{x_{C(x)}}) = I_{y_m}$ if and only if $B(x_{C(x)}) = y_m$.

Since A maps $(\eta \cup \zeta)$ onto $(\eta \cup \zeta)$, it induces a permutation $p \in (S(X) \times S(\{a,b\}))$, given by $p(x,\ l) = (y, m)$ if $A(I_{x_l}) = I_{y_m}$ where $l, m \in \{a,\ b\}$. Note that, for any $x,\ y \in X,\ l,\ m \in \{a,\ b\}$, $A(I_{x_l}) = I_{y_m}$ implies $p(x,\ l) = (y,\ m)$ and hence $p^*(x_l) = y_m$. Therefore, $A(I_{x_l}) = I_{y_m} = \{\underline{0},\ y_m,\ \underline{1}\} = \{E_p(\underline{0}), E_p(x_l), E_p(\underline{1})\} = E_p^*(I_{x_l})$. Hence $A = E_p^*$ on $(\eta \cup \zeta)$.

Also note that A maps $(\eta' \cup \zeta')$ onto $(\eta' \cup \zeta')$. Then for any $x, y \in X, l, m \in \{a, b\}$, $A(I_{x^l}) = I_{y^m}$ implies $A(I_{x_{l'}}) = I_{y_{m'}}$, and hence $A(I_{x^l}) = I_{y^m}$ implies $p^*(x_{l'}) = y_{m'}$.

Now, $x^l = \bigvee_{y\in X,\ y\neq x} (y_a \vee y_b) \vee x_l$.

Therefore,$E_P(x^l) = \bigvee_{y\in X,\ y\neq x} (p^*(y_a) \vee p^*(y_b)) \vee p^*(x_l)$. Since p^* is a bijection on $Pt(L^X)$, it is clear that $E_P(x^l) \geq z_m$ for all $z \in X,\ z \neq y,\ m \in \{a,\ b\}$ (since $p^*(x_{l'}) = y_{m'}$) and $E_P(x^l) \geq y_m$ but $E_P(x^l) \not\geq y_{m'}$. Thus $E_P(x^l) = y^m$. Therefore, $A(I_{x^l}) = I_{y^m} = \{\underline{0},\ y^m,\ \underline{1}\} = \{E_p(\underline{0}),\ E_p(x^l),\ E_p(\underline{1})\} = E_p^*(I_{x^l})$. Hence $A = E_p^*$ on $(\eta' \cup \zeta')$.

Now consider $C \in L^X$, $I_c \notin G$, $C \neq \underline{0}$ and $C \neq \underline{1}$. Then, $A(I_C) = I_{(B_1 \vee B_a \vee B_b)}$. Now for $C(x) \in \{a,b\}$, $A(I_{x_{C(x)}}) = I_{y_m}$ if and only if $p^*(x_{C(x)}) = y_m$. Therefore, $B(x_{C(x)}) = y_m$ if and only if $p^*(x_{C(x)}) = y_m$. Therefore,

$$A(I_C) = I_{(\bigvee_{x\in X,\ C(x)\neq 0,\ C(x)\neq 1} B(x_{C(x)})) \vee (\bigvee_{x\in X,\ C(x)=1} B(x_a)) \vee (\bigvee_{x\in X,\ C(x)=1} B(x_b))}$$
$$= I_{(\bigvee_{x\in X,\ C(x)\neq 0,\ C(x)\neq 1} p^*(x_{C(x)})) \vee (\bigvee_{x\in X,\ C(x)=1} p^*(x_a)) \vee (\bigvee_{x\in X,\ C(x)=1} p^*(x_b))}$$
$$= I_{\bigvee_{x\in X,\ C(x)\neq 0} p^*(x_{C(x)})}$$
$$= I_{E_p(C)} = \{\underline{0},\ E_p(C),\ \underline{1}\} = \{E_p(\underline{0}),\ E_p(C),\ E_p(\underline{1})\} = E_p^*(I_C).$$

Hence $A = E_p^*$ on every atom not in G.

Thus $A = E_p^*$ on every atoms of $LFT(X,L)$ and since $LFT(X,L)$ is atomistic it is clear that $A = E_p^*$ on $LFT(X,L)$.

Now suppose that A maps $[(\eta \cup \zeta)$ onto $(\eta' \cup \zeta')$ and $(\eta' \cup \zeta')$ onto $(\eta \cup \zeta)]$.

Consider the atom I_{x_1}. Note that join of I_{x_1} with I_{x_a} and I_{x_b} gives four open sets, but the join of I_{x_1} with any other atom in $(\eta \cup \zeta)$ gives five open sets. This property is preserved by any automorphism A of $LFT(X,\ L)$. Also note that A maps $(\eta \cup \zeta)$ onto $(\eta' \cup \zeta')$. Thus the join of $A(I_{x_1})$ with $A(I_{x_a}) = I_{y^l}$ and $A(I_{x_b}) = I_{w^p}$ gives four open sets, but the join of $A(I_{x_l})$ with any other atom in $(\eta' \cup \zeta')$ gives five open sets.

Let $A(I_{x_l}) = I_D$. Then D is comparable or complementary with y^l and w^p. But $I_D \notin (\eta \cup \zeta)$ since $I_{x_1} \notin (\eta' \cup \zeta')$. Thus D is not complementary with y^l or w^p. Thus D is comparable with y^l and w^p. Thus since $I_D \notin (\eta' \cup \zeta')$ and $D \neq \underline{1}$, we have $D \leq y^l$ and $D \leq w^p$.

By previous argument it can also note that D is not comparable with any $C \in L^X$ with $C \neq y^l$, $C \neq w^p$ and $I_C \in (\eta' \cup \zeta')$. Therefore $D(t) = 1$ for all $t \neq y$ and $t \neq w$. Also $D \nleq y^{l'}$ and $D \nleq w^{p'}$. Hence we have $D = y^l \wedge w^p$. That is $A(I_{x_1}) = I_{(y^l \wedge w^p)}$. Thus $A(I_{x^a}) = I_{y_l}$ and $A(I_{x^b}) = I_{w_p}$, where $x, y, w \in X$ and $l, p \in \{a, b\}$, then $A(I_{x^0}) = I_{(y_l \vee w_p)}$. Similarly if $A(I_{x^a}) = I_{y_l}$ and $A(I_{x^b}) = I_{w_p}$, where $x, y, w \in X$ and $l, p \in \{a, b\}$, then $A(I_{x^0}) = I_{(y_l \vee w_p)}$.

Now Suppose that $A(I_{x^l}) = I_{y_m}$ and $A(I_{x^{l'}}) = I_{w_p}$. Consider the atom I_{x_l}. If $A(I_{x_l}) = I_D$, then $I_D \in (\eta' \cup \zeta')$. Also note that $I_{x_l} \vee I_{x^0}$ consists of five open sets and $I_{x^l} \subseteq I_{x_l} \vee I_{x^0}$. Therefore we have, $A(I_{x_l}) \vee A(I_{x^0})$ consists of five open sets and $A(I_{x^l}) \subseteq A(I_{x_l}) \vee A(I_{x^0})$. Also$A(I_{x^0}) = A(I_{(x^l \wedge x^{l'})}) = I_{(y_m \vee w_p)}$. Hence $I_D \vee I_{(y_m \vee w_p)}$ consists of five open sets and $I_{y_m} \subseteq I_D \vee I_{(y_m \vee w_p)}$. Now $I_D \in (\eta' \cup \zeta')$ and by above arguments it is clear that $A(I_{x_l}) = I_D = I_{w^{p'}}$. Similarly we have $A(I_{x_{l'}}) = I_{y^{m'}}$. Thus if $A(I_{x^l}) = I_{y_m}$ and $A(I_{x^{l'}}) = I_{w_p}$, where $x, y, w \in X$ and $l, m, p \in \{a, b\}$, then, $A(I_{x_{l'}}) = I_{y^{m'}}$ and $A(I_{x_l}) = I_{w^{p'}}$.

The join of I_C with the atoms $\{I_{x_{C(x)}} : C(x) \neq 0,\ C(x) \neq 1,\ x \in X\} \bigcup \{I_{x_a} : C(x) = 1,\ x \in X\} \bigcup \{I_{x_b} : C(x) = 1,\ x \in X\}$ gives four open sets and all other atoms in $(\eta \cup \zeta)$ gives five open sets. This property is preserved by automorphisms. Hence the join of $A(I_C)$ with the atoms $\{A(I_{x_{C(x)}}) : C(x) \neq 0,\ C(x) \neq 1,\ x \in X\} \bigcup \{A(I_{x_a}) : C(x) = 1,\ x \in X\} \bigcup \{A(I_{x_b}) : C(x) = 1,\ x \in X\}$ gives four open sets and all other atoms in $(\eta' \cup \zeta')$ gives five open sets.

Since $I_C \notin G$, we have $A(I_C) \notin G$. Let $A(I_C) = I_{C'}$. Then C' is not complementary with any $D \in L^X$, such that $I_D \in (\eta' \cup \zeta')$. Thus the only possibility to get, join of I_C with the atoms of $(\eta' \cup \zeta')$ gives four open sets is comparability. Thus $C' \leq B(x_{C(x)})$ if $C(x) \neq 0,\ C(x) \neq 1,\ x \in X$, $C' \leq B(x_a)$ if $C(x) = 1,\ x \in X$ and $C' \leq B(x_b)$ if $C(x) = 1,\ x \in X$. Therefore $C' \leq (\bigwedge_{x\in X,\ C(x)\neq 0,\ C(x)\neq 1} B(x_{C(x)})) \wedge (\bigwedge_{x\in X,\ C(x)=1} B(x_a)) \wedge (\bigwedge_{x\in X,\ C(x)=1} B(x_b))$ and C' is not comparable with any $D \in L^X$ such that $I_D \in (\eta' \cup \zeta')$ and $D \neq B(x_{C(x)})$ for any $x \in X,\ C(x) \neq 0$. Therefore, $C' = (\bigwedge_{x\in X,\ C(x)\neq 0,\ C(x)\neq 1} B(x_{C(x)})) \wedge (\bigwedge_{x\in X,\ C(x)=1} B(x_a)) \wedge (\bigwedge_{x\in X,\ C(x)=1} B(x_b))$.

Since A maps $(\eta \cup \zeta)$ onto $(\eta' \cup \zeta')$ it induces a permutation p on $X \times \{a,\ b\}$, given by $p(x,\ l) = (y, m')$ if and only if $A(I_{x_l}) = I_{y^m}$ where $x,\ y \in X, l,\ m \in \{a,b\}$. Now we want to prove that $A = F_p^*$.

First note that, for any $x,\ y \in X,\ l,\ m \in \{a,\ b\}$,

$A(I_{x_l}) = I_{y^m}$ implies $p(x,\ l) = (y,\ m')$ and hence $p^*(x_l) = y_{m'}$. Thus $E_p(x_l) = y_{m'}$. Therefore, $A(I_{x_l}) = I_{y^m} = \{\underline{0},\ y^m,\ \underline{1}\}$
$= \{E_p(\underline{0}),\ comp(E_p(x_l)), E_p(\underline{1})\}$
$= \{comp(E_p(\underline{1})),\ comp(E_p(x_l)),\ comp(E_p(\underline{1}))\}$
$= \{F_p(\underline{1}),\ F_p(x_l),\ F_p(\underline{0})\} = F_p^*(I_{x_l})$.

Hence $A = F_p^*$ on $(\eta \cup \zeta)$.

Now note that A maps $(\eta' \cup \zeta')$ onto $(\eta \cup \zeta)$. Then for any $x,\ y \in X,\ l,\ m \in \{a,\ b\}$, $A(I_{x^l}) = I_{y_m}$ implies $A(I_{x_{l'}}) = I_{y^{m'}}$. Thus, if $A(I_{x^l}) = I_{y_m}$ then $p^*(x_{l'}) = y_m$.

Also we have, $E_P(x^l) = \bigvee_{y \in X,\ y \neq x} (p^*(y_a) \vee p^*(y_b)) \vee p^*(x_l)$, since $x^l = \bigvee_{y \in X,\ y \neq x} (y_a \vee y_b) \vee x_l$.

Since p^* is a bijection on $Pt(L^X)$, it is clear that $E_P(x^l) \geq z_q$ for all $z \in X,\ z \neq y,\ q \in \{a,\ b\}$ (since $p^*(x_{l'}) = y_m$) and $E_P(x^l) \geq y_{m'}$ but $E_P(x^l) \ngeq y_m$. Thus $E_P(x^l) = y^{m'}$. Therefore,
$A(I_{x^l}) = I_{y_m} = \{\underline{0},\ y_m,\ \underline{1}\} = \{\underline{0},\ comp(y^{m'}), \underline{1}\}$
$= \{comp(\underline{0}),\ comp(y^{m'}),\ comp(\underline{1})\}$
$= \{comp(E_p(\underline{1})),\ comp(E_P(x^l)),\ comp(E_p(\underline{0}))\}$
$= \{F_p^*(\underline{1}),\ F_p^*(x^l),\ F_p^*(\underline{0})\} = F_p^*(I_{x^l})$.
Hence $A = F_p^*$ on $(\eta' \cup \zeta')$.

Now consider $C \in L^X$, $I_c \notin G$, $C \neq \underline{0}$ and $C \neq \underline{1}$. Then $A(I_C) = I_{(\bigwedge_{x \in X,\ C(x) \neq 0,\ C(x) \neq 1} B(x_{C(x)})) \wedge (\bigwedge_{x \in X,\ C(x)=1} B(x_a)) \wedge (\bigwedge_{x \in X,\ C(x)=1} B(x_b))}$.
Now for $C(x) \in \{a,b\}$, $A(I_{x_{C(x)}}) = I_{y^m}$ if and only if $B(x_{C(x)}) = y^m = comp(p^*(x_{C(x)}))$.
Therefore, $I_{C'} = A(I_C)$
$= I_{(\bigwedge_{x \in X,\ C(x) \neq 0,\ C(x) \neq 1} comp(p^*(x_{C(x)}))) \wedge (\bigwedge_{x \in X,\ C(x)=1} comp(p^*(x_a))) \wedge (\bigwedge_{x \in X,\ C(x)=1} comp(p^*(x_b)))}$.
Also we have,
$E_p(C) = (\bigvee_{x \in X,\ C(x) \neq 0,\ C(x) \neq 1} p^*(x_{C(x)}))\ \vee\ (\bigvee_{x \in X,\ C(x)=1} p^*(x_a)) \vee (\bigvee_{x \in X,\ C(x)=1} p^*(x_b))$.
Therefore, $F_p(C) = comp(E_p(C)) = C'$.
Hence,
$A(I_c) = I_{C'} = \{\underline{0},\ C',\ \underline{1}\} = \{F_p(\underline{1}),\ F_p(C),\ F_p(\underline{0})\} = F_p^*(I_C)$
Hence $A = F_p^*$ on every atoms not in G. Thus $A = F_p^*$ on every atoms of $LFT(X,\ L)$. Since $LFT(X,\ L)$ is atomistic it is clear that $A = F_p^*$ on $LFT(X,L)$. □

Remark 3.15. *If X is infinite set, then there is no automorphism of $LFT(X,L)$ which maps $(\eta \cup \zeta)$ onto $(\eta' \cup \zeta')$ and $(\eta' \cup \zeta')$ onto $(\eta \cup \zeta)$.*

From the proof of the Theorem 3.14 and from the Remark 3.15, we have the following result.

Theorem 3.16. *If X is an infinite set, then the set of all automorphisms of the lattice $LFT(X,\ L)$ is precisely $\{E_p^* : p \in (S(X) \times S(\{a,b\}))\}$.*

References

[1] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloq. Pub., Third Edn., 1967.

[2] C. L. Chang, Fuzyy Topological Spaces, *J. Math. Anal. Appl.*, 24(1968), 182–190.

[3] Dexue Zhang and Hongliang Lai, Fuzzy Preorders and Fuzzy topology, *Fuzzy Sets and Systems*, 157(2006), 1865–1885.

[4] G. Elsamony, Connecting Fuzzifying Topologies And Generalized Ideals by means of Fuzzy Preorders, *Int. J. of Mathematics and Mathematical Sciences*, Volume 2009, Article ID 567482, 16 page.

[5] J. D. Farely, The Automorphism group of a function lattice, *Algebra Universalis*, 36 (1996), 8–45.

[6] Gerald Schrag, Every finite group is the Automorphism group of some finite orthomodular lattice, *Proceedings of the American Mathematical Society*, 55 (1976), 243–249.

[7] J. Hartmanis, On The Lattice of Topologies, *Cand. J. Math.*, 10 (1958), 547–553.

[8] Jinming Fang and Piwei Chen, One to one correspondencebetween fuzzifying topologies and fuzzy preorders, *Fuzzy Sets and Systems*, 158 (2007), 1814–1822.

[9] Jiri Adamek Horst Herrlich and George E. Strecker, Abstract and Concrete Categories The Joy of Cats, *Free Online Edn., http:// katmat.math,uni.bremen.de/acc*, (2004).

[10] R. E. Larson and S. J. Andima, The Lattice of Topologies a Survey, *Rocky Mountain J. Math.*, 5 (1975), 177–198.

[11] Liu Ying Ming and Liu Mao Kang, *Fuzzy Topology*, World Scientific Pub. Singapore, 1997.

[12] N. M. Madhavan Namboothiri, Order Induced L-Fuzzy Topological Properties, *Advances in Fuzzy Sets And Systems*, 9 (2011), 167–197.

[13] S. E. Rodabaugh, A Categorical Accomadation to Various Notions of Fuzzy Topology, *Fuzzy Sets and Systems*, 9 (1983), 241–265.

[14] Susan J. Andima and W. J. Thron, Order-Induced Topological Properties, *Pacific J. Math.*, 75(2)(1978), 297–318.

[15] I. Uljane, On the Order Type L-valued Relations on L-powersets, *Mathware and Soft Computing*, 14 (2007), 183–199.

[16] L. A. Zadeh, Fuzzy Sets, *Information and Control*, 8 (1965), 338–353.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆