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$\mathscr{AC}$ and $\mathscr{AC}_2$-Paracompact spaces

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Abstract
The reason for this paper is to present the two new ideas of $\mathscr{AC}$-Paracompact spaces and $\mathscr{AC}_2$-Paracompact spaces. Additionally we have demonstrated that each $\mathscr{AC}$-Paracompactness and $\mathscr{AC}_2$-Paracompactness has a topological property. We have likewise presented the $\mathscr{AC}$-normal and its properties.

Keywords
Angelic spaces, $\mathscr{C}$-Paracompact, $\mathscr{C}_2$–Paracompact, $\mathscr{C}$-Normal, $\mathscr{AC}$-Paracompact, $\mathscr{AC}_2$–Paracompact, $\mathscr{AC}$-normal.

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1. Introduction

In 1944, Dieudonne. J [7] presented the paracompact space. The idea of paracompactness [7] is one of the most helpful speculation of compactness. Some notable mathematicians of different occasions have contemplated certain stronger just as more weaker types of paracompactness. $\mathscr{C}$-Paracompact and $\mathscr{C}_2$-Paracompact were characterized by Arhangelskii. $\mathscr{C}$ -Paracompact and $\mathscr{C}_2$-Paracompact were concentrated in [17]. Alzahrani S [3] research $\mathscr{C}$ -normal topological property. Fermlin's idea of angelic space [6] and a portion of its emanation carry us with he required tools for introducing those outcomes in a mordant idea.

2. Preliminaries

Definition 2.1. *[6] A topological space T is termed as an angelic, if for each relatively countably compact subset $\mathscr{S}$ of T the ensuing hold: (a) $\mathscr{S}$ is relatively compact (b) If $s \in \overline{\mathscr{S}}$, then there is a sequence in $\mathscr{S}$ that converges to s.*

Definition 2.2. *[14] A topological space X is paracompact, if each open cover has a locally finite open refinement.*

Definition 2.3. *[17]*
A topological space M is termed as $\mathscr{C}$-paracompact if $\exists$ a paracompact space N and a bijective mapping $p : M \to N \ni$ the restriction $p|_S : S \to p(S)$ is a homeomorphism for every compact subspace $S \subseteq M$.

Definition 2.4. *[17] A topological space M is termed as $\mathscr{C}_2$-paracompact if $\exists$ a Hausdorff paracompact space N and a bijective mapping $p : M \to N \ni$ the restriction $p|_S : S \to p(S)$ is a homeomorphism for every compact subspace $S \subseteq M$.*

Definition 2.5. *[3] A space M is termed as $\mathscr{C}$-normal if $\exists$ a normal space N and function $p : M \to N \ni$ the restriction $p|_S : S \to p(S)$ is homeomorphism for each compact subspace $S \subseteq M$.*

Definition 2.6. *[10] A space (M,τ) is termed as submetrizable if $\exists$ a metric d on $M \ni$ the topology τ_d on M caused by d is coarser than τ, i.e.$\tau_d \subseteq \tau$.*

Definition 2.7. *[2] A space (M,τ) is supposed to be epinormal if $\exists$ a coarser topology τ' on $X \ni (M,\tau')$ is normal.*

Definition 2.8. *[4] A topology τ on a nonempty set M is supposed to be minimal Hausdorff if (M,τ) is Hausdorff and has no Hausdorff topology on M strictly coarser than τ.*

Definition 2.9. *[20] A space M is termed as mildly normal, k-normal, if any two disjoint closed domains U and V of M $\exists$ disjoint open sets $\mathscr{U}$, $\mathscr{V}$ of $M \ni U \subseteq \mathscr{U}$ and $V \subseteq \mathscr{V}$.*

Definition 2.10. *[17] A topological space (M,τ) is termed as lower compact if $\exists$ a coarser topology τ' on $M \ni (M,\tau')$ is T_2–compact.*

Definition 2.11. *[8] Let X be topological space. If $\exists\ X' = X \times \{1\}$ and $X \cap X' = \emptyset \ni A(X) = X \cup X'$, then $A(X)$ with the unique topology τ is termed as Alexandroff duplicate of X.*

3. $\mathscr{AC}$ and $\mathscr{AC}_2$-Paracompact Spaces

Definition 3.1. *Let M be an angelic space and S be an angelic compact subspace of M. If there is a bijection mapping $p: M \to N$, N is an angelic paracompact space and the restriction $p|_S : S \to p(S)$ is a homeomorphism, then M is said to be an $\mathscr{AC}$-paracompact space.*

Definition 3.2. *Let M be an angelic space and S be an angelic compact subspace of M. If there is a bijection function $p: M \to N$, N is a Hausdorff angelic paracompact space and the restriction $p|_S : S \to p(S)$ is a homeomorphism, then M is said to be an $\mathscr{AC}_2$-paracompact space.*

Theorem 3.3. *Every $\mathscr{AC}$-paracompact space ($\mathscr{AC}_2$-paracompact space) is a topological property.*

Proof. Suppose M is an $\mathscr{AC}$-paracompact ($\mathscr{AC}_2$-paracompact) space and $M \cong O$. Let N be an angelic paracompact (Hausdorff angelic paracompact) space and $p: M \to N$ be a bijective mapping $\ni$ the restriction $p|_S : S \to p(S)$ is homeomorphism for every angelic compact subspace $S \subseteq M$. Let $q: O \to M$ be a homeomorphism. Hence, N and $p \circ q : O \to N$ has the topological properties. □

Theorem 3.4. *Every $\mathscr{AC}$-paracompact space ($\mathscr{AC}_2$-paracompact space) has an additive property.*

Proof. Suppose M_α is an $\mathscr{AC}$-paracompact($\mathscr{AC}_2$-paracompact) space for each $\alpha \in A$. To prove that their sum $\oplus_{\alpha\in A} M_\alpha$ is an $\mathscr{AC}$-paracompact ($\mathscr{AC}_2$-paracompact). For each $\alpha \in \Lambda$, choose an angelic paracompact (Hausdorff angelic paracompact) space N_α and bijective mapping $p_\alpha : M_\alpha \to N_\alpha$ $\ni p_{\alpha\S_\alpha} : S_\alpha \to p_\alpha(S_\alpha)$ is a homeomorphism for every angelic compact subspace S_α of M_α. By reason of N_α is an angelic paracompact (Hausdorff angelic paracompact) for each $\alpha \in \Lambda$, then the sum $\oplus_{\alpha\in A} N_\alpha$ is an angelic paracompact (Hausdorff angelic paracompact). Consider the function sum $\oplus_{\alpha\in\Lambda} p_\alpha$:$\oplus_{\alpha\in\Lambda} M_\alpha \to \oplus_{alpha\in\Lambda} N_\alpha$ described by $\oplus_{\alpha\in\Lambda} p_\alpha(m) = p(m)$ if $m \in M_\beta$, $\beta \in \Lambda$. Currently, a subspace $S \subseteq \oplus_{\alpha\in\Lambda} M_\alpha$ is an angelic compact iff the set $\Lambda_0 = \{\alpha \in \Lambda : S \cap M_\alpha \notin \emptyset\}$ is finite, $S \cap M_\alpha$ is an angelic compact in M_α for every $\alpha \in \Lambda_0$. If $S \subseteq \oplus_{\alpha\in\Lambda} M_\alpha$ is an angelic compact, subsequently $(\oplus_{\alpha\in\Lambda} M_\alpha)|_C$ is a homeomorphism as $p_{\alpha/C\cap M_\alpha}$ is a homeomorphism for every $\alpha \in \Lambda_0$. □

Theorem 3.5. *If M is an $\mathscr{AC}$-paracompact ($\mathscr{AC}_2$-paracompact), then its Alexandroff duplicate $A(M)$ is also an $\mathscr{AC}$-paracompact ($\mathscr{AC}_2$-paracompact).*

Proof. Let M be any $\mathscr{AC}$-paracompact space. Choose an angelic paracompact space N and a bijective function $p: M \to N$ $\ni p|_S : S \to p(S)$ is a homeomorphism for every angelic compact subspace $S \subseteq M$. Suppose the Alexandroff duplicate spaces $A(M)$ and $A(N)$ of M and N commonly. By reason of N is an angelic paracompact, next $A(N)$ is also an angelic paracompact. Characterize $q: A(M) \to A(N)$ by $q(a) = A(a)$ if $a \in M$. If $a \in M_0$, consider the unique element b in $M \ni b_0 = a$, then characterize $q(a) = (p(b))'$. Next q is a bijective mapping. Currently, a subspace $S \subseteq A(M)$ is an angelic compact iff $S \cap M$ is an angelic compact in M and for every open set U in M with $S \cap M \subseteq A(M)$, we state $S \cap M'/U'$ is finite. take $S \subseteq A(M)$ is any angelic compact subspace. To prove $q|_S : S \to q(S)$ is a homeomorphism. Take $a \in S$ is arbitrary. If $a \in S \cap M'$, let $b \in X$ be the unique element $\ni b' = a$. For the smallest basic open neighborhood $\{(f(b))'\}$ of the point $g(a)$ we state that $\{a\}$ is open in $\mathscr{C}$ and $g(\{a\}) \subseteq \{(f(b))'\}$. If $a \in S \cap M$. Let W be any open set in $N \ni g(a) = f(a) \in W$. Consider $H = (W \cup (W'/\{f(a)'\})) \cap g(C)$ which is a basic open neighborhood of $p(a)$ in $q(S)$. By reason of $p|_(S \cap M) : S \cap M \to p(S \cap M)$ is a homeomorphism, then $\exists$ an open set U in M with $a \in U$ and $p|_{S\cap M(U\cap S)} \subseteq W$. Currently, $(U \cup U'/\{a'\})) \cap q(C)$ is open in $\mathscr{C} \ni a \in G$ and $q_{c(G)} \subseteq H$. Thus, $q|_S$ is continuous. Currently, to prove that $q|_S$ is open. Take $K \cap (K'/\{k'\})$, here $k \in K$ and K is open in M,be any basic open set in $A(M)$,then $(K \cap S) \cup ((K' \cap S)/\{k'\})$ is a basic open set in $\mathscr{C}$. By reason of $M \cap S$ is an angelic compact in M.then $q|_{S(K\cap(M\cap C))} = q|_{M\cap S(K\cap(M\cap S))})$ is open in $N \cap p(S \cap M)$ as $p|_{M\cap S}$ is a homeomorphism. Hence $K \cap S$ is open in $N \cap p(N \cap M)$. Also, $p(K' \cap S)/\{k'\})$ is open in $N' \cap q(S)$ be a set of isolated points. Hence $q|_S$ is an open function. Hence, $p|_S$ is a homeomorphism. □

Theorem 3.6. *If (M,τ') is a submetrizable space of (M,τ) with $\tau' \subseteq \tau$, then (M,τ') is an $\mathscr{AC}_2$-paracompact.*

Proof. Since τ' is a metrizable topology on $M \ni \tau' \subseteq \tau$. Next (M,τ') is $\mathscr{AC}_2$-paracompact and the identity function $id_M : (M,\tau) \to (M,\tau')$ is a continuous function. If S is some angelic compact subspace of (M,τ), then the restriction of the identity mapping on S onto $id_M(S)$ is a homemorphism as S is an angelic compact, $id_M(S)$ is Hausdorff be a subspace of the metrizable space (M,τ'), and every continuous one-to-one mapping of an angelic compact space onto a Hausdorff space is a homeomorphism. Hence, (M,τ') submetrizable space is an $\mathscr{AC}_2$-paracompact. □

Theorem 3.7. *If M is an $\mathscr{AC}$-paracompact ($\mathscr{AC}_2$- paracompact) Frechet space and $p: M \to N$ is a witness of the $\mathscr{AC}$-paracompactness ($\mathscr{AC}_2$-paracompactness) of M, then p is continuous.*

Proof. Suppose S is any nonempty subset of M. Take $n \in p(\overline{S})$ is arbitrary and $m \in M$ be the unique element $\ni p(m) = n$. Then $m \in S$. Choose a sequence $(s_n) \subseteq S \ni s_n \to s$ Let then $T = \{s, s_n : n \in N\}$ is an angelic compact subspace of M, being a convergent sequence with its limit, thus $p|_T : T \to p(T)$ is a homeomorphism. Currently, take $V \subseteq U$ is some open neighbourhood of y; next $V \cap p(T)$ is open in the subspace $p(T)$ including n. Thus, $p^{-1}(V) \cap T$ is open in the subspace T containing m. Thus, $p^{-1}(V) \cap B) \cap \{m_n : n \in N\} \neq \emptyset$, so $((p^{-1}(V) \cap T) \cap A \neq \emptyset$. Hence, $\emptyset \neq p((p^{-1}(V) \cap T) \cap S) \subseteq p((p^{-1}(V) \cap S) = V \cap p(S)$. hence, $n \in (p(\overline{S}))$ and $p(\overline{S}) \subseteq (p(\overline{S}))$ Thus, p is continuous. □

Corollary 3.8. *If M is an $\mathscr{AC}$-paracompact ($\mathscr{AC}_2$-paracompact) first countable space and $p : M \to Y$ is a witness of the $\mathscr{AC}$-paracompact ($\mathscr{AC}_2$-paracompact) of M, then p is continuous.*

Corollary 3.9. *If M is an $\mathscr{AC}_2$-paracompact Frechet space, then X is Hausdorff.*

Proof. Since N is a T_2 angelic paracompact space and $p : M \to N$ be a bijective mapping $\ni$ the restriction $p|_S : S \to p(S)$ is homeomorphism for every angelic compact subspace $S \subseteq M$. Through Theorem 3.5, p is continuous. Take A, B are some disjoint angelic compact space; then $f(A)$, $f(B)$ are disjoint angelic compact subspaces of N. By reason of N is T_2, then $f(A)$ and $f(B)$ are disjoint closed subspaces of N. By reason of N is T_2 angelic paracompact, N is normal and thus $\exists$ open subsets G and H of $Y \ni f(A) \subseteq G$, $f(B) \subseteq H$, and $G \cap H = \emptyset$. By the continuity of p, $U = p^{-1}(G)$ and $V = p^{-1}(H)$. Thus, for every disjoint angelic compact subspaces A and B, $\exists$ open sets U and $V \ni S \subseteq U$, $T \subseteq V$ and $U \cap V = \emptyset$. □

Theorem 3.10. *If M is a T_1 space $\ni$ the only angelic compact subsets are the finite subsets, then M is $\mathscr{AC}_2$ paracompact space.*

Proof. Suppose M is a T_1 space $\exists$ the only angelic compact subspaces of M are the finite subsets of M. Since T_1, finally some angelic compact subspace of M is discrete. Then, take $N = M$ and let Y with the discrete topology. Thus the identity mapping from M onto N. Hence, $\exists$ a bijection mapping $p : M \to N$, N is an angelic Hausdorff paracompact space and the restriction $p|_S : S \to p(S)$ is a homeomorphism, then M is said to be an $\mathscr{AC}_2$-paracompact. □

Theorem 3.11. *Let M be the Hausdorff locally angelic compact space. Then M is an $\mathscr{AC}_2$-Paracompact space.*

Proof. Since M is any Hausdorff locally angelic compact topological space. Then $\exists$ a T_2 angelic compact space N and hence N is T_2 angelic paracompact, and a bijective function $p : M \to N \ni p$ is continuous. By reason of p is continuous, Next for some angelic compact subspace $S \subseteq M$, we have $p|_S : S \to p(S)$ is a homeomorphism because 1 to 1, onto, and continuity are acquired from p, and $p|_S$ is closed as S is an angelic compact and $p(S)$ is Hausdorff. □

Example 3.12. *A Tychonoff $\mathscr{AC}_2$-paracompact space is not locally compact.*

Proof. Consider the quotient space $\mathbb{R}/\mathbb{N}$. We can describe it as follows: Let $i = \sqrt{-1}$. Let $N = \mathbb{R}/\mathbb{N} \cup i$. Define $p : \mathbb{R} \to Y$ as follows:

$$p(x) = \begin{cases} x; \text{ if } x \in \mathbb{R}/\mathbb{N} \\ i; \text{ if } x \in \mathbb{N} \end{cases}$$

Now consider on $\mathbb{R}$ the usual topology U. Define on N the topology $\tau = \{W \subseteq Y : p^{-1}(W) \in U\}$. Then $p : (R, U) \to (N, \tau)$ is a closed quotient mapping. We can describe the open neighborhoods of each element in as follows: The open neighborhoods of $i \in N$ are $(U/\mathbb{N}) \cup \{i\}$, here U is an open set in $(\mathbb{N}, U) \ni \mathbb{N} \subseteq U$. The open neighborhoods of any $y \in \mathbb{R}/\mathbb{N}$ are $(y - \varepsilon, y + \varepsilon)\mathbb{N}$ where ε is a positive real number. It is well known that (N, τ) is T_3, which is neither locally angelic compact nor first countable. Now, by reason of (N, τ) is Lindelöf, being a continuous image of $\mathbb{R}$ with its usual topology, and T_3, then (N, τ) is angelic paracompact and T_4. Hence, it is an $\mathscr{AC}_2$-paracompact. □

Definition 3.13. *A topological space (M, τ) is termed as lower angelic compact if $\exists$ a coarser topology τ' on $M \ni (X, \tau')$ is T_2 angelic compact.*

Theorem 3.14. *If (M, τ) lower angelic compact space, Then M is an $\mathscr{AC}_2$-paracompact.*

Proof. Suppose τ' is a T_2 angelic compact topology on $M \ni \tau' \subseteq \tau$. Next (M, τ') is T_2 angelic paracompact and the identity mapping $id_M : (M, \tau) \to (M, \tau')$ is a continuous function. If S is some angelic compact subspace of (M, τ), next the restriction of the identity mapping on S onto $id_M(S)$ is a homeomorphism as S is an angelic compact, $id_M(S)$ is Hausdorff being a subspace of the T_2 space (M, τ') and every continuous 1-1 function of an angelic compact space onto a Hausdorff space is a homeomorphism. Hence, M is an $\mathscr{AC}_2$-paracompact. □

Theorem 3.15. *If (M, τ) is an $\mathscr{AC}_2$-paracompact countably angelic compact Frechet, then (M, τ) is lower angelic compact.*

Proof. Consider a T_2 angelic paracompact space $(N, \tau*)$ and a bijection function $p : (M, \tau) \to (N, \tau*) \ni$ the restriction $p|_S : S \to p(S)$ is homeomorphism for every angelic compact subspace $S \subseteq M$. By reason of M is Frechet, then p is continuous. Hence, $(M, \tau*)$ is countably angelic compact. By reason of $(M, \tau*)$ is also an angelic paracompact, then $(M, \tau*)$ is T_2 angelic compact. Characterize a topology τ' on M as follows: $\tau' = \{p^{-1}(U) : U \in \tau*\}$. Then τ' is coarser than τ and $p : (M, \tau') \to (N, \tau*)$ is a bijection continuous function. Let $W \in \tau'$ be arbitrary; then W is $p^{-1}(U)$ for some $U \in \tau*$. Thus, $p(W) = p(p^{-1}(U)) = U$. Hence, p is open and p is a homeomorphism. Thus, (M, τ') is T_2 angelic compact. Therefore, (M, τ) is lower angelic compact. □

4. $\mathscr{AC}$-Normal and its Properties

Definition 4.1. *A space M is termed as an $\mathscr{AC}$-normal if $\exists$ a normal space N and a bijective $p : M \to N \ni$ the restriction $p|_S : S \to p(S)$ is homeomorphism for every angelic compact subspace $S \subseteq M$.*

Definition 4.2. *A space M is termed as Angelic countably normal if there exists a normal space N and a bijective $p : M \to N \ni$ the restriction $p|_S : S \to p(S)$ is a homeomorphism for each angelic countable subspace $S \subseteq M$.*

Example 4.3. *An $\mathscr{AC}$ normal space is not an $\mathscr{AC}_2$-paracompact.*

Proof. Suppose $\mathbb{R}$ with $L = \{\emptyset, \mathbb{R}\} \cup \{(-\infty, x) : x \in \mathbb{R}\}$. In this space $(\mathbb{R}, L)$, any two nonempty closed sets are intersect; thus, $(\mathbb{R}, L)$ is normal and thus $\mathscr{AC}$-normal. $(\mathbb{R}, L)$ is not Hausdorff as any two nonempty open sets must intersect. A subset $S \subset \mathbb{R}$ is angelic compact iff it has a maximum element. Suppose that $(\mathbb{R}, L)$ is $\mathscr{AC}_2$-paracompact. Take N is Hausdorff paracompact space and $p : R \to Y$ be a bijection $\ni$ $p|_S : S \to p(S)$ is homeomorphism for every angelic compact subspace S of $\mathbb{R}$. Let $S = (-\infty, 0]$; then S is an angelic compact in $(\mathbb{R}, L)$ and S as a subspace is not Hausdorff because any two nonempty open sets in S must intersect. However, S will be homeomorphic to $p(S)$ and $p(S)$ is Hausdorff, being a subspace of a Hausdorff space, and this is a contradiction. Thus, $(\mathbb{R}, L)$ cannot be $\mathscr{AC}_2$-paracompact space. □

Example 4.4. *An infinite $\mathscr{AC}$ normal space is not an $\mathscr{AC}$-paracompact space.*

Proof. Let $M = [0, \infty)$. Define $\tau = \{\emptyset, M\} \cup \{[0, x) : x \in \mathbb{R}, 0 < x\}$. Consider $(\mathbb{R}, L)$ is just the angelic subspace of $(\mathbb{R}, L)$. (i.e), $\tau = L_M = L_{[0,\infty)}$. Now consider (M, τ_0), where τ_0 is the particular point topology. We have that τ is coarser than τ_0 because any nonempty open set in τ must contain 0. Thus, (M, τ_0) cannot be an angelic paracompact. Observe that (M, τ) is normal because there are no two nonempty closed disjoint subsets. Thus, (M, τ) is an $\mathscr{AC}$-normal. Now, a subset S of M is an angelic compact iff S has maximal element. If S has maximal element, then any open cover for S will be covered by one member of the open cover, the one that contains the maximal element. If S has no maximal element, then S cannot be finite. If S is unbounded above, then $\{[0, n) : n \in N\}$ would be an open cover for S has no finite subcover. If S is bounded above, let $y = supS$ and pick an increasing sequence $(c_n) \subseteq S \ni c_n \to y$, where the convergence is taken in the usual metric topology on M. Then $\{[0, c_n) : n \in N\}$ would be an open cover for S that has no finite subcover. Thus, S would not be an angelic compact. (M, τ) is Frechet. That is because M is first countable. If $x \in M$, then $B(x) = \{[0, x + 1n) : n \in N\}$ is a countable local base for M at x.

Now, suppose that M is an $\mathscr{AC}$-paracompact. Choose an angelic paracompact space Y and a bijective mapping $p : M \to N \ni p|_S : S \to p(S)$ is a homeomorphism for each angelic compact subspace A of M. By Corollary 3.8, p is continuous. Thus, for some nonempty open subset U of N we have that $p^{-1}(U)$ is open in M. By reason of p is a bijective, M is infinite. For each $y \in M$, pick an open neighborhood U_y of $y \ni$ the family $\{U_y : y \in Y\}$ is an infinite open cover for M. By reason of each U_y contains the element $p(0)$, then the open cover $\{U_y : y \in Y\}$ cannot have any locally finite open refinement and thus Y is not paracompact, which is a contradiction. Therefore, M is $\mathscr{AC}$-normal but not an $\mathscr{AC}$-paracompact. □

Lemma 4.5. *If $p : M \to N$ is a bijective function, M is an $\mathscr{AC}$-normal space and any finite subset of M is discrete, then N is T_1.*

Proof. By reason of $p : M \to N$ is a bijective function $\ni p|_S : S \to p(S)$ is a homeomorphism for each angelic compact subspace $S \subseteq M$. Assume M has more than one element and take a, b are distinct elements of N. Let c and d be the unique elements of $M \ni p(c) = a$ and $p(d) = b$. Then $p|_{\{c,d\}} : \{c, d\} \to \{a, b\}$ is a homeomorphism and $\{c, d\}$ is a discrete subspace of M. Thus, $p(\{c\}) = \{a\}$ and $p(\{d\}) = \{b\}$ are both open in $\{a, b\}$ as a subspace of N. Thus, $\exists$ an open neighborhood $U_a \subseteq N$ of a $\ni U_a \cap \{a, b\} = \{a\}$; hence, $b \notin U_a$, and similarly $\exists$ an open neighborhood $U_b \subseteq N$ of $b \ni a \notin U_b$. Thus, N is T_1. □

Example 4.6. $\mathbb{R}$ *With $\tau_p(\mathbb{R}, \tau_p)$ is not an $\mathscr{AC}$-normal space.*

Proof. $\mathbb{R}$ with τ_p, here the particular point is $p \in \mathbb{R}$, is not $\mathscr{AC}$-normal. By reason of $\tau_p = \{\emptyset\} \cup \{U \subseteq \mathbb{R} : p \in U\}$. We known that $(\mathbb{R}, \tau_p)$ is neither T_1 nor normal space and if $A \subseteq \mathbb{R}$, then $\{\{x, p\} : x \in A\}$ is an open cover for S, thus a subset S of R is an angelic compact iff it is finite. To show $(\mathbb{R}, \tau_p)$ is not $\mathscr{AC}$-normal, suppose that $(\mathbb{R}, \tau_p)$ is $\mathscr{AC}$-normal. Take N is a normal space, $p : M \to N$ be a function $\ni$ the restriction $p|_S : S \to p(S)$ is a homeomorphism for every angelic compact subspace S of $(\mathbb{R}, \tau_p)$. Consider the ensuing two cases for the space N,

Case (i): M is T_1. Take $S = \{a, b\}$, where $a \neq b$; then S is an angelic compact subspace of $(\mathbb{R}, \tau_p)$. By assumption $p|_S : S \to p(S) = \{p\{a\}, p\{b\}\}$ is a homeomorphism. By reason of $p(S)$ is a finite subspace of M and M is T_1, then $p(S)$ is a discrete subspace of M. Hence, $p|_S$ is not continuous which is contradiction as $p|_S$ is a homeomorphism.

Case (ii): M is not T_1. To prove the topology on M is the particular point topology with $p(b)$ as its particular point. Assume that N is not the particular point topology then $\exists$ a non-empty open set $U \subset N \ni p(b) \notin U$. Choose $y \in U$ and take $x \in \mathbb{R}$ is the unique real number $\ni p(x) = y$. Suppose $\{a, b\}$ and $a \notin b$ because $p(x) = y \in U$, $p(b) \notin U$, and p is 1-1. Take $p|_{a,b} : \{a, b\} \to \{y, p(b)\}$. Currently, $\{y\}$ is open in the subspace $\{y, p(b)\}$ of N as $\{y\} = U \cap \{y, p(b)\}$, but $p^{-1}(\{y\}) = \{x\}$ and $\{x\}$ is not open in the subspace $\{a, b\}$ of $(\mathbb{R}, \tau_p)$, which means $p|_{\{a,b\}}$ is not continuous. Any particular point space consisting of more than one point cannot be normal, so which contradiction as N is normal. Hence, $(\mathbb{R}, \tau_p)$ is not an $\mathscr{AC}$-normal. □

Theorem 4.7. *Every angelic compact non-normal space is not an $\mathscr{AC}$-normal.*

Proof. Consider M is an angelic compact non-normal space. Assume M is an $\mathscr{AC}$-normal, then $\exists$ a normal space N and a bijective mapping $p : M \to N \ni$ the restriction $p|_S : S \to p(S)$ is homeomorphism for every angelic compact subspace $S \subseteq M$. By reason of M is an angelic compact, then $M \cong N$, and this is a contradiction as N is normal and M is not an angelic compact non-normal space. Hence M cannot be an $\mathscr{AC}$-normal. □

Theorem 4.8. *Let M be an $\mathscr{AC}$-normal space. If every countable subspace of M is included in an angelic compact subspace, then M is an angelic countably normal.*

Proof. Take M is any $\mathscr{AC}$-normal space $\ni$ if S is any countable subspace of M, then $\exists$ an angelic compact subspace E $\ni S \subseteq E$. Take N is a normal space and $p : M \to N$ be a bijective mapping $\ni p|_S : S \to p(S)$ is homeomorphism for every angelic compact subspace S of M. Presently, take S is some countable subspace of M. Choose an angelic compact subspace E of $M \ni S \subseteq E$, next $p|_E : E \to p(E)$ is homeomorphism, hence $p|_S : S \to p(S)$ is homeomorphism as $(p|_E)|_S = p|_S$. □

Theorem 4.9. *Let M be an $\mathscr{AC}$-normal. If M is a Frechet Lindelöf space $\ni$ any finite subspace of M is discrete, then M is an $\mathscr{AC}_2$-paracompact.*

Proof. Consider M is $\mathscr{AC}$-normal, then $\exists$ a normal space N and a bijection function $p : M \to N \ni$ the restriction $p|_S : S \to p(S)$ is homeomorphism for every angelic compact subspace $S \subseteq M$. By Lemma 4.1, Y is T_1 and hence T_4. By reason of M is Frechet, then p is continuous. By reason of M is Lindelöf and p is continuous and onto, then N is Lindelöf. By reason of any T_3 Lindelöf space is an angelic paracompact, then N is T_2 angelic paracompact. Therefore, M is an $\mathscr{AC}_2$-paracompact space. □

Theorem 4.10. *If (M,τ) is Lindelöf epinormal space then (M,τ) is an $\mathscr{AC}_2$-paracompact.*

Proof. Suppose (M,τ) is some Lindelöf epinormal space. Choose a coarser topology τ' on $M \ni (M,\tau')$ is T_4. By reason of (M,τ) is Lindelöf and τ' is coarser than τ we have (M,τ') is T_3 and Lindelöf, and hence Hausdorff paracompact. Therefore, (M,τ') is an $\mathscr{AC}_2$-paracompact as the identity function $id : (M,\tau) \to (M,\tau')$. Hence (M,τ) is an $\mathscr{AC}_2$-paracompact. □

Theorem 4.11. *Let (M,τ) be an $\mathscr{AC}_2$-paracompact Frechet space. Then (M,τ) is an epinormal.*

Proof. Take (M,τ) is some $\mathscr{AC}_2$-Paracompact Frechet space and (M,τ) is normal. Suppose that (M,τ) is not normal. Let (N,τ') be a T_2 angelic paracompact space and $p : (M,\tau) \to (N,\tau')$ be a bijective mapping $\ni$ the restriction $p|_S : S \to p(S)$ is homeomorphism for every angelic compact subspace $S \subseteq M$. By reason of M is Frechet, p is continuous; Theorem 3.5, Define $\tau* = \{p^{-1}(U) : U \in \tau'\}$. It is clear $\tau*$ is a topology on M coarser than $\tau \ni p : (M,\tau*) \to (N,\tau')$ is continuous. If $W \in \tau*$, then W is $W = p^{-1}(U)$ here $U \in \tau'$. Thus, $p(W) = p(p^{-1}(U)) = U$, gives p is open and homeomorphism. hence, $(M,\tau*)$ is T_4. hence, (M,τ) is an epinormal. □

Corollary 4.12. *Let (M,τ) be an $\mathscr{AC}_2$-paracompact Frechet space, then (M,τ) is completely Hausdorff.*

Example 4.13. *Any $\mathscr{AC}_2$-paracompact Frechet space is an epinormal.*

Proof. Suppose that two countably infinite sets are termed as almost disjoint if their intersection is finite. Consider a subfamily of $[\omega_0]^{\omega_0} = \{A \subset \omega_0 :$ A is infinite $\}$ a mad family on ω_0 if it is a maximal (with respect to inclusion) pairwise almost disjoint subfamily. Take A is a pairwise almost disjoint subfamily of $[\omega_0]^{\omega_0}$. The Mrowka space $\psi(A)$ is describe by $\omega_0 \cup A$, every point of ω_0 is isolated, and a basic open neighborhood of $W \in A$ has $\{W\} \cup (W/F)$, with $F \in [\omega_0] < \omega_0 = \{B \subseteq \omega_0 : Bis finite\}$. since $\exists$ an almost disjoint family $A \subset [\omega_0]^{\omega_0} \ni |A| > \omega_0$ and the Mrowka space $\psi(A)$ is a Tychonoff, separable, first countable, and locally angelic compact space that is neither countably angelic compact, angelic paracompact, nor normal. A is a mad family iff $\psi(A)$ is pseudocompact. The Mrowka space $\psi(A)$ is an $\mathscr{AC}_2$-paracompact, being T_2 locally angelic compact. $\psi(A)$ is also Frechet, being first countable. Hence Mrowka space is an epinormal. □

Remark 4.14. *Any minimal Hausdorff $\mathscr{AC}_2$-paracompact Frechet space is an angelic compact.*

Theorem 4.15. *Let M be a minimal Hausdorff second countable space. The ensuing are equivalent.*

(i) M is an $\mathscr{AC}_2$-paracompact.

(ii) M is locally angelic compact.

(iii) M is an angelic compact

(iv) M is an epinormal.

(v) M is metrizable.

(vi) M is lower compact.

(vii) M is minimal T_4.

Proof. $(i) \Rightarrow (ii)$ By reason of any second countable space is first countable and any first countable space is Frechet, then Theorem 4.5, gives that M is T_2 angelic compact and hence locally angelic compact. $(ii) \Rightarrow (iii)$ By reason of any T_2 locally angelic compact space is Tychonoff, by the minimality, M is an angelic compact. $(iii) \Rightarrow (iv)$ Any T_2 angelic compact space is T_4. $(iv) \Rightarrow (v)$ Any epinormal space is $T_{21/2}$. By minimilaity, M is angelic compact and hence T_3. By reason of any T_3 second countable space is metrizable, the result follows. $(v) \Rightarrow (vi)$ By minimality, M is $T_{21/2}$ angelic compact and hence lower angelic compact. $(vi) \Rightarrow (vii)$ Again, by minimality, M is T_2 angelic compact and hence T_4. By reason of any minimal T_4 space is an angelic compact. $(vii) \Rightarrow (i)$ By reason of any minimal TT_4 space is angelic compact, M will be T_2 angelic paracompact and hence $\mathscr{AC}_2$-paracompact. □

Example 4.16. *A minimal Hausdorff second countable $\mathscr{AC}$-paracompact space is Cannot be $\mathscr{AC}_2$-paracompact.*

Proof. Let $M = \{a, b, c_j, a_{ij}, b_{ij} : i \in \mathbb{N}, j \in \mathbb{N}\}$ here all these elements are distinct. Characterize the ensuing neighborhood system on M:

For each $i, j \in N$, a_{ij} is isolated and b_{ij} is isolated.

For each $i \in N, B(c_i) = \{V^n(c_i)\} = \{c_i, a_{ij}, b_{ij} : j \geq n : n \in \mathbb{N}\}$.

$\mathscr{B}(a) = \{V^n(a) = \{a, a_{ij} : i \geq n\} : n \in \mathbb{N}\}$.

$\mathscr{B}(a) = \{V^n(b) = \{b, b_{ij} : i \geq n\} : n \in \mathbb{N}\}$.

Denote the unique topology on M caused by the above neighborhood system by τ. Next τ is minimal Hausdorff and (M, τ) is cannot compact. By reason of M is countable and each local base is countable, then the neighborhood system is a countable base for (M, τ), so it is second countable but not $\mathscr{AC}_2$-paracompact because it is not $T_{21/2}$ as the closure of any open neighborhood of a must intersect the closure of any open neighborhood of b. □

5. Conclusion

Our primary outcomes incorporates the two new ideas of $\mathscr{AC}$-Paracompact spaces and $\mathscr{AC}_2$-Paracompact spaces. Likewise demonstrated that, each $\mathscr{AC}$-Paracompactness and $\mathscr{AC}_2$-Paracompactness has a topological property. We likewise explored the $\mathscr{AC}$-normal and its properties.

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