



Certain subclasses of harmonic starlike functions associated with q -Mittag-Leffler function

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Abstract

A new subclass of harmonic starlike functions defined by an operator which is related to q -Mittag-Leffler function. By defining this class we obtain coefficient estimates, distortion theorems, extreme points, conditions for convolution functions and convex combination are determined. Moreover, the properties of the class preserving under certain operators like the generalized Bernardi- integral operator and the q -Jackson integral operator are studied.

Keywords

Harmonic function, analytic function, univalent function, starlike domain, convex domain, convolution.

AMS Subject Classification

30C45, 30C50.

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Contents

1	Introduction	988
2	Coefficient Bounds	990
3	Extreme points and Distortion bounds	993
4	Convolution and Convex Combinations	996
5	Class preserving integral operators	997
	References	999

1. Introduction

Let Ω be a complex domain. Let u and v be real and harmonic in Ω . Then a continuous complex valued harmonic function of the form $f = u + iv$ is in Ω . Let D be any simply-connected domain and subset of Ω , so $f = h + \bar{g}$, where h and g are analytic functions in D , where h and g are the analytic and co-analytic parts of f . From [8] it is observed that $|h'(z)| > |g'(z)|$ in D is a necessary and sufficient condition for f to be locally univalent and orientation preserving in D .

Let $\mathcal{S}_{\mathcal{H}}$ be the set of all harmonic, univalent and orientation preserving functions of the form $f = h + \bar{g}$ in

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}$$

normalized by $f(0) = h(0) = f_z(0) - 1 = 0$, where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k \quad \text{and} \quad |b_1| < 1 \quad (1.1)$$

so the function of the form $f = h + \bar{g}$, is

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad \text{and} \quad |b_1| < 1. \quad (1.2)$$

It is interesting to note that, if $g \equiv 0$ in $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$, then the class $\mathcal{S}_{\mathcal{H}}$ reduces to the class \mathcal{S} , where

$$\mathcal{S} := \{f : f \text{ is univalent with } f(0) = 0 \text{ and } f'(0) = 1\}.$$

Denote by $\mathcal{S}_{\mathcal{H}} \subset \mathcal{S}_{\mathcal{H}}$ the class consisting of functions of the form: $f = h + \bar{g}$, where

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \quad g(z) = \sum_{k=1}^{\infty} |b_k| z^k. \quad (1.3)$$

The class $\mathcal{S}_{\mathcal{H}}$ and its subclasses with various properties were investigated by Clunie and Sheil-Small [8], with this motivation there have been numerous investigations related to the subclasses of $\mathcal{S}_{\mathcal{H}}$. One can refer Ahuja [1–3], Al-Kharsani and Al-Khal [5], Dixit et al. [9–12], Frasin [15],

Frasin et al. [16], Jahangiri [19–21], Jahangiri et al. [22], Kim and Jahangiri [18], Silverman [28], Silverman and Silvia [29], Yalçın et al. [35] and others [4, 13, 14, 23–25] and the references therein for more details of different subclasses of \mathcal{S}_q and its properties.

Let \mathcal{A} denote the class of functions that are analytic in the open unit disc \mathbb{U} . For $0 < q < 1$, Jackson [17] defined the q -derivative of a function $h \in \mathcal{A}$ is defined as follows:

$$\partial_q h(z) = \begin{cases} \frac{h(z) - h(qz)}{z - qz}, & z \neq 0 \\ h'(z), & z = 0 \end{cases}, \quad z \in \mathbb{U}$$

and $\partial_q^2 h(z) := \partial_q(\partial_q h(z))$. Obviously,

$$\partial_q \left(\sum_{k=1}^{\infty} a_k z^k \right) = \sum_{k=1}^{\infty} [k]_q a_k z^{k-1}, \quad k \in \mathbb{N}, \quad z \in \mathbb{U},$$

where $[k]_q$ is defined by

$$[k]_q = \begin{cases} \frac{1 - q^k}{1 - q} & \text{if } k \in \mathbb{N} \\ 0 & \text{if } k = 0. \end{cases} \quad (1.4)$$

As $q \rightarrow 1$ and $k \in \mathbb{N}$, $[k]_q \rightarrow k$. In particular $h(z) = z^k$ for k , the q -derivative of $h(z)$ is given by

$$\partial_q(z^k) = \frac{z^k - (qz)^k}{(1 - q)z} = [k]_q z^{k-1}, \quad z \in \mathbb{U},$$

and

$$\lim_{q \rightarrow 1} \partial_q(h(z)) = \lim_{q \rightarrow 1} \partial_q(z^k) = kz^{k-1} = f'(z), \quad k \in \mathbb{N}, \quad z \in \mathbb{U}.$$

The Mittag-Leffler function $E_{\sigma}(z)$ is defined as

$$E_{\sigma}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\sigma k + 1)} z^k. \quad (1.5)$$

The initial two parametric generalizations for the function shown in (1.5) (see [32, 33]). It is defined in the following way:

$$E_{\sigma, \delta}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(\sigma k + \delta)} z^k, \quad \sigma, \delta \in \mathbb{C}, \quad \Re\{\sigma\} > 0, \quad \Re\{\delta\} > 0. \quad (1.6)$$

Recently, Sharma and Jain [27] introduced the q -analogue of generalized Mittag-Leffler function defined in the following way:

$$E_{\sigma, \delta}^v(z; q) = \sum_{k=0}^{\infty} \frac{(q^v; q)_k}{(q; q)_k} \frac{z^k}{\Gamma_q(\sigma k + \delta)}, \quad \sigma, \delta, v \in \mathbb{C},$$

$$\Re\{\sigma\} > 0, \quad \Re\{\delta\} > 0, \quad \Re\{v\} > 0, \quad |q| < 1, \quad (1.7)$$

where $\Gamma_q(z)$ is the q -gamma function and $\lim_{q \rightarrow 1} \Gamma_q(z) = \Gamma(z)$.

The q -analogue of the Pochhammer symbol (or q -shifted factorial) is defined by

$$(\lambda, q)_k = \begin{cases} (1 - \lambda)(1 - \lambda q) \cdots (1 - \lambda q^{k-1}), & k = 1, 2, 3, \dots \\ 1 & k = 0. \end{cases}$$

Further, the q -gamma function $\Gamma_q(z)$ satisfies the functional equation

$$\Gamma_q(z + 1) = \frac{1 - q^z}{1 - q} \Gamma_q(z) = [z]_q \Gamma_q(z).$$

Also,

$$(q^\lambda, q)_k = \frac{(1 - q)^k \Gamma_q(\lambda + k)}{\Gamma_q(\lambda)}, \quad k > 0.$$

In 2019 Elhaddad et al. [14] defined the function $\mathcal{Q}_{\sigma, \delta}^v(z)$ by

$$\begin{aligned} \mathcal{Q}_{\sigma, \delta}^v(z) &:= z \Gamma_q(z) E_{\sigma, \delta}^v(z; q) \\ &= z + \sum_{k=2}^{\infty} \frac{\Gamma_q(\delta) (q^v; q)_{k-1}}{\Gamma_q(\sigma(k-1) + \delta) (q; q)_{k-1}} z^k. \end{aligned} \quad (1.8)$$

For $f \in \mathcal{A}$ the differential operator

$$\mathcal{D}_{\lambda, q}^{v, m}(\sigma, \delta) : \mathcal{A} \rightarrow \mathcal{A}$$

defined by

$$\begin{aligned} \mathcal{D}_{\lambda, q}^{v, 0}(\sigma, \delta)h(z) &= h(z) * \mathcal{Q}_{\sigma, \delta}^v(z) \\ \mathcal{D}_{\lambda, q}^{v, 1}(\sigma, \delta)h(z) &= (1 - \lambda)h(z) * \mathcal{Q}_{\sigma, \delta}^v(z) + \\ &\quad \lambda z \partial_q(h(z) * \mathcal{Q}_{\sigma, \delta}^v(z)) \end{aligned}$$

in general

$$\begin{aligned} \mathcal{D}_{\lambda, q}^{v, m}(\sigma, \delta)h(z) &= \mathcal{D}_{\lambda, q}^{v, 1}(\mathcal{D}_{\lambda, q}^{v, m-1}(\sigma, \delta)h(z)) \\ &:= z + \sum_{k=2}^{\infty} [\lambda([k]_q - 1) + 1]^m \\ &\quad (\times) \frac{\Gamma_q(\delta) (q^v; q)_{k-1}}{\Gamma_q(\sigma(k-1) + \delta) (q; q)_{k-1}} a_k z^k, \quad z \in \mathbb{U}. \end{aligned}$$

For particular values of the parameters the defined operator $\mathcal{D}_{\lambda, q}^{v, m}(\sigma, \delta)$ reduces to various known operators for example we illustrate as follows:

1. $\lim_{q \rightarrow 1} \mathcal{D}_{\lambda, q}^{1, m}(\sigma, \delta)$ reduces to generalized differential operator of Mittag-Leffler function studied by Elhaddad et al. [13].
2. $\lim_{q \rightarrow 1} \mathcal{D}_{\lambda, q}^{1, m}(0, 1)$ reduces to Al-Oboudi operator introduced by Al-Oboudi [6].
3. $\lim_{q \rightarrow 1} \mathcal{D}_{1, q}^{1, m}(0, 1)$ reduces to Sălăgean operator introduced by Sălăgean [26].
4. $\lim_{q \rightarrow 1} \mathcal{D}_{\lambda, q}^{1, 0}(\sigma, \delta)$ reduces to integral operator involving Mittag-Leffler function studied by Srivastava et al. [30].



In 2019, Elhaddad et al.[14] defined the operator $\mathcal{D}_{\lambda,q}^{V,m}(\sigma, \delta) : h_t(z) = (1-t)z + th(z)$, and $g_t(z) = tg(z)$. Also, for functions $\mathcal{S}_{\mathcal{H}} \rightarrow \mathcal{S}_{\mathcal{H}}$ for functions of the form $f = h + \bar{g}$ given by (1.2) as $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$ where h and g are of the form (1.3) denote $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$ the subclass of $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$.

$$\begin{aligned} &\mathcal{D}_{\lambda,q}^{V,m}(\sigma, \delta) f(z) \\ &= \mathcal{D}_{\lambda,q}^{V,m}(\sigma, \delta) h(z) + \overline{\mathcal{D}_{\lambda,q}^{V,m}(\sigma, \delta) g(z)}, \quad z \in \mathbb{U}, \end{aligned} \tag{1.9}$$

where

$$\begin{aligned} &\mathcal{D}_{\lambda,q}^{V,m}(\sigma, \delta) h(z) \\ &= z + \sum_{k=2}^{\infty} [\lambda ([k]_q - 1) + 1]^m \Phi_{q,k}^{\delta,\sigma,v} a_k z^k \end{aligned} \tag{1.10}$$

and

$$\begin{aligned} &\mathcal{D}_{\lambda,q}^{V,m}(\sigma, \delta) g(z) \\ &= \sum_{k=1}^{\infty} [\lambda ([k]_q - 1) + 1]^m \Phi_{q,k}^{\delta,\sigma,v} b_k z^k, \quad m \in \mathbb{N}_0, \lambda \geq 0 \end{aligned} \tag{1.11}$$

for brevity, we write

$$\Phi_{q,k}^{\delta,\sigma,v} = \frac{\Gamma_q(\delta) (q^V; q)_{k-1}}{\Gamma_q(\sigma(k-1) + \delta) (q; q)_{k-1}}. \tag{1.12}$$

Inspired by the classes of Ahuja et al. [4], Elhaddad et al. [14], Frasin and Magesh [16], Jahangiri et al. [22], Magesh and Porwal [23], Magesh et al. [24] and Rosy et al. [25], the subclass $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$ defined for functions of the form (1.2) satisfying the criteria

$$\Re \left\{ (1 + \beta e^{i\alpha}) \frac{z \partial_q (\mathcal{D}_{\lambda,q}^{V,m}(\sigma, \delta) h(z)) - z \partial_q (\mathcal{D}_{\lambda,q}^{V,m}(\sigma, \delta) g(z))}{\mathcal{D}_{\lambda,q}^{V,n}(\sigma, \delta) h_t(z) + \mathcal{D}_{\lambda,q}^{V,n}(\sigma, \delta) g_t(z)} - \beta e^{i\alpha} \right\} > \gamma, \quad z \in \mathbb{U}, \tag{1.13}$$

where $m, n \in \mathbb{N}_0, \beta \geq 0, 0 \leq t \leq 1, 0 \leq \gamma < 1$, with $\alpha \in \mathbb{R}, \sigma, \delta, v \in \mathbb{C}, \Re\{\sigma\} > 0, \Re\{\delta\} > 0, \Re\{v\} > 0, 0 < q < 1$,

For the special values of parameters involved in the above defined class, one can get various subclasses of $\mathcal{S}_{\mathcal{H}}$. For illustration, we remark few here as follows:

1. $\mathcal{G}_{\mathcal{H}_{m,m,q}}^{\delta,\sigma,v,\lambda}(0, \gamma, 1) := \mathcal{S}_{\mathcal{H}_{m,q}}^{\delta,\sigma,v,\lambda}(\gamma)$ Elhaddad et al. [14].
2. (a) $\lim_{q \rightarrow 1} \mathcal{G}_{\mathcal{H}_{0,0,q}}^{1,0,1,0}(0, \gamma, 1) := \mathcal{S}_{\mathcal{H}}^*(\gamma)$ Jahangiri [20].
 (b) $\lim_{q \rightarrow 1} \mathcal{G}_{\mathcal{H}_{1,1,q}}^{1,0,1,0}(0, \gamma, 1) = K_{\mathcal{H}}(\gamma)$ Jahangiri [20].
 (c) $\lim_{q \rightarrow 1} \mathcal{G}_{\mathcal{H}_{0,0,q}}^{1,0,1,0}(0, 0, 1) := \mathcal{S}_{\mathcal{H}}^*$ Silverman and Silvia [29]; $b_1 = 0$ (see [7, 28]).
 (d) $\lim_{q \rightarrow 1} \mathcal{G}_{\mathcal{H}_{1,1,q}}^{1,0,1,0}(0, 0, 1) = K_{\mathcal{H}}$ Silverman and Silvia [29]; $b_1 = 0$ (see [7, 28]).
3. $\lim_{q \rightarrow 1} \mathcal{G}_{\mathcal{H}_{0,0,q}}^{1,0,1,0}(1, \gamma, 1) := G_{\mathcal{H}}(\gamma)$ Rosy et al.[25].
4. $\lim_{q \rightarrow 1} \mathcal{G}_{\mathcal{H}_{m,n,q}}^{1,0,1,0}(1, \gamma, 1) := G_{\mathcal{H}}(m, n, \gamma)$ Subramanian et al.[31].
5. $\lim_{q \rightarrow 1} \mathcal{G}_{\mathcal{H}_{0,0,q}}^{1,0,1,0}(\beta, \gamma, t) := G_{\mathcal{H}}(\beta, \gamma, t)$ Ahuja et al. [3].
6. $\lim_{q \rightarrow 1} \mathcal{G}_{\mathcal{H}_{m,n,q}}^{1,0,1,0}(0, \gamma, 1) := \mathcal{S}_{\mathcal{H}}(m, n, \gamma)$ Yalçin et al. [35].
7. $\lim_{q \rightarrow 1} \mathcal{G}_{\mathcal{H}_{n,n,q}}^{1,0,1,0}(0, \gamma, 1) := \mathcal{S}_{\mathcal{H}}(n, \gamma)$ Yalcin et al. [34], Dixit and Porwal [9].
8. $\mathcal{G}_{\mathcal{H}_{n,n,q}}^{1,0,1,0}(0, \gamma, 1) := \mathcal{S}_{\mathcal{H}}(n, \gamma, q)$ Jahangiri [21].

2. Coefficient Bounds

Our first theorem gives a sufficient condition for functions in $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$.

Theorem 2.1. *Let $f = h + \bar{g}$ be so that h and g are given by (1.1). If*

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta,\sigma,v}}{(1 - \gamma)} |a_k| \\ &+ \sum_{k=1}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta,\sigma,v}}{(1 - \gamma)} |b_k| \leq 1, \end{aligned} \tag{2.1}$$

where $m, n \in \mathbb{N}_0, \beta \geq 0, 0 \leq t \leq 1, 0 \leq \gamma < 1$, with $\alpha \in \mathbb{R}, \sigma, \delta, v \in \mathbb{C}, \Re\{\sigma\} > 0, \Re\{\delta\} > 0, \Re\{v\} > 0$ and $0 < q < 1$. Then $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$.



Proof. To prove that $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$, we only need to show that if (2.1) holds, then the required condition (1.13) is satisfied. For (1.13), we can write

$$\Re \left\{ (1 + \beta e^{i\alpha}) \frac{z \partial_q \left(\mathcal{D}_{\lambda,q}^{\nu,m}(\sigma, \delta) h(z) \right) - \overline{z \partial_q \left(\mathcal{D}_{\lambda,q}^{\nu,m}(\sigma, \delta) g(z) \right)}}{\mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) h_t(z) + \overline{\mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) g_t(z)}} - \beta e^{i\alpha} \right\} = \Re \left\{ \frac{A(z)}{B(z)} \right\} > \gamma.$$

Using the fact that $\Re\{\omega\} \geq \gamma$ if and only if $|1 - \gamma + \omega| \geq |1 + \gamma - \omega|$, it suffices to show that

$$|A(z) + (1 - \gamma)B(z)| - |A(z) - (1 + \gamma)B(z)| \geq 0, \tag{2.2}$$

where

$$A(z) = (1 + \beta e^{i\alpha}) \left[z \partial_q \left(\mathcal{D}_{\lambda,q}^{\nu,m}(\sigma, \delta) h(z) \right) - \overline{z \partial_q \left(\mathcal{D}_{\lambda,q}^{\nu,m}(\sigma, \delta) g(z) \right)} \right] - \beta e^{i\alpha} \left[\mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) h_t(z) + \overline{\mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) g_t(z)} \right]$$

and

$$B(z) = \mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) h_t(z) + \overline{\mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) g_t(z)},$$

where

$$z \partial_q \left(\mathcal{D}_{\lambda,q}^{\nu,m}(\sigma, \delta) h(z) \right) = z + \sum_{k=2}^{\infty} [k]_q [\lambda ([k]_q - 1) + 1]^m \Phi_{q,k}^{\delta,\sigma,\nu} a_k z^k$$

and

$$z \partial_q \left(\mathcal{D}_{\lambda,q}^{\nu,m}(\sigma, \delta) g(z) \right) = \sum_{k=1}^{\infty} [k]_q [\lambda ([k]_q - 1) + 1]^m \Phi_{q,k}^{\delta,\sigma,\nu} b_k z^k.$$

Further,

$$\mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) h_t(z) = z + \sum_{k=2}^{\infty} t [\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu} a_k z^k$$

and

$$\mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) g_t(z) = \sum_{k=1}^{\infty} t [\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu} b_k z^k.$$



Substituting for $A(z)$ and $B(z)$ in (2.2) and making use of (2.1), we obtain

$$\begin{aligned}
 & |A(z) + (1 - \gamma)B(z)| - |A(z) - (1 + \gamma)B(z)| \\
 &= \left| (1 + \beta e^{i\alpha}) \left[z \partial_q \left(\mathcal{D}_{\lambda,q}^{v,m}(\sigma, \delta) h(z) \right) - z \partial_q \left(\overline{\mathcal{D}_{\lambda,q}^{v,m}(\sigma, \delta) g(z)} \right) \right] - \beta e^{i\alpha} \left[\mathcal{D}_{\lambda,q}^{v,n}(\sigma, \delta) h_t(z) + \overline{\mathcal{D}_{\lambda,q}^{v,n}(\sigma, \delta) g_t(z)} \right] \right. \\
 &\quad \left. + (1 - \gamma) \left(\mathcal{D}_{\lambda,q}^{v,n}(\sigma, \delta) h_t(z) + \overline{\mathcal{D}_{\lambda,q}^{v,n}(\sigma, \delta) g_t(z)} \right) \right| - \left| (1 + \beta e^{i\alpha}) \left[z \partial_q \left(\mathcal{D}_{\lambda,q}^{v,m}(\sigma, \delta) h(z) \right) - z \partial_q \left(\overline{\mathcal{D}_{\lambda,q}^{v,m}(\sigma, \delta) g(z)} \right) \right] \right. \\
 &\quad \left. - \beta e^{i\alpha} \left[\mathcal{D}_{\lambda,q}^{v,n}(\sigma, \delta) h_t(z) + \overline{\mathcal{D}_{\lambda,q}^{v,n}(\sigma, \delta) g_t(z)} \right] \right. \\
 &\quad \left. - (1 + \gamma) \left(\mathcal{D}_{\lambda,q}^{v,n}(\sigma, \delta) h_t(z) + \overline{\mathcal{D}_{\lambda,q}^{v,n}(\sigma, \delta) g_t(z)} \right) \right| \\
 &\geq 2(1 - \gamma)|z| \\
 &\quad - \sum_{k=2}^{\infty} 2[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v} |a_k| |z|^k \\
 &\quad - \sum_{k=1}^{\infty} 2[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v} |b_k| |z|^k \\
 &\geq 2(1 - \gamma)|z| \left\{ 1 - \sum_{k=2}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v}}{1 - \gamma} |a_k| |z|^{k-1} \right. \\
 &\quad \left. - \sum_{k=1}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v}}{1 - \gamma} |b_k| |z|^{k-1} \right\} \\
 &\geq 2(1 - \gamma) \left\{ 1 - \sum_{k=2}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v}}{1 - \gamma} |a_k| \right. \\
 &\quad \left. - \sum_{k=1}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v}}{1 - \gamma} |b_k| \right\} \\
 &\geq 0
 \end{aligned}$$

which implies that $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, v, \lambda}(\beta, \gamma, t)$.

The coefficient bound (2.1) is sharp for the harmonic function

$$\begin{aligned}
 f(z) &= z + \sum_{k=2}^{\infty} \frac{1 - \gamma}{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v}} x_k z^k \\
 &\quad + \sum_{k=1}^{\infty} \frac{1 - \gamma}{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v}} \overline{y_k z^k},
 \end{aligned}$$

where $\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1$, shows that the coefficient bound given by (2.1) is sharp. □

Next, we show that the above sufficient condition is also necessary for functions in the class $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, v, \lambda}(\beta, \gamma, t)$.

Theorem 2.2. Let $f = h + \bar{g}$ be so that h and g are given by (1.3). Then $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, v, \lambda}(\beta, \gamma, t)$ if and only if

$$\begin{aligned}
 & \sum_{k=2}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v}}{1 - \gamma} |a_k| \\
 & \quad + \sum_{k=1}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, v}}{1 - \gamma} |b_k| \leq 1,
 \end{aligned}$$

where $m, n \in \mathbb{N}_0, \beta \geq 0, 0 \leq t \leq 1, 0 \leq \gamma < 1$, with $\alpha \in \mathbb{R}, \sigma, \delta, v \in \mathbb{C}, \Re\{\sigma\} > 0, \Re\{\delta\} > 0, \Re\{v\} > 0$ and $0 < q < 1$.



Proof. Since $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t) \subset \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$, we only need to prove the only if part of the theorem. To this end, for functions f of the form (1.3), we notice that the condition (1.13) is equivalent to

$$\Re \left\{ \frac{\left((1 + \beta e^{i\alpha}) \left[z \partial_q \left(\mathcal{D}_{\lambda,q}^{\nu,m}(\sigma, \delta) h(z) \right) - z \partial_q \left(\overline{\mathcal{D}_{\lambda,q}^{\nu,m}(\sigma, \delta) g(z)} \right) \right] \right)}{\mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) h_t(z) + \overline{\mathcal{D}_{\lambda,q}^{\nu,n}(\sigma, \delta) g_t(z)}} - \gamma \right\} \geq 0.$$

Upon choosing the values of z on the positive real axis where $0 \leq |z| = r < 1$, the above inequality reduces to

$$\Re \left\{ \frac{\left((1 - \gamma) - \sum_{k=2}^{\infty} [\lambda ([k]_q - 1) + 1]^m [k]_q - \gamma t [\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu} |a_k| r^{k-1} \right)}{1 - \sum_{k=2}^{\infty} t([\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu}) |a_k| r^{k-1} + \sum_{k=1}^{\infty} t([\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu}) |b_k| r^{k-1}} \right\} \\ - \Re \left\{ \beta e^{i\alpha} \frac{\left(\sum_{k=2}^{\infty} [\lambda ([k]_q - 1) + 1]^m [k]_q - t[\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu} |a_k| r^{k-1} \right)}{1 - \sum_{k=2}^{\infty} t([\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu}) |a_k| r^{k-1} + \sum_{k=1}^{\infty} t([\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu}) |b_k| r^{k-1}} \right\} \geq 0.$$

Since $\text{Re}(-e^{i\alpha}) \geq -|e^{i\alpha}| = -1$, the above inequality reduces to

$$\frac{\left((1 - \gamma) - \sum_{k=2}^{\infty} [(1 + \beta) [\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta) t [\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu} |a_k| r^{k-1}] \right)}{1 - \sum_{k=2}^{\infty} t([\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu}) |a_k| r^{k-1} + \sum_{k=1}^{\infty} t([\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta,\sigma,\nu}) |b_k| r^{k-1}} \geq 0. \tag{2.3}$$

If the condition (2.3) does not hold then the numerator in (2.3) is negative for r sufficiently close to 1. Thus there exists $z_0 = r_0$ in $(0, 1)$ for which the quotient in (2.3) is negative. This contradicts the condition for $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$. Hence the proof is complete. \square

3. Extreme points and Distortion bounds

In this section, our first theorem gives the extreme points of the closed convex hulls of $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$.

Theorem 3.1. *Let f be given by (1.3). Then $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)), \tag{3.1}$$

where



$$\begin{aligned}
 h_1(z) &= z, \\
 h_k(z) &= z - \frac{1 - \gamma}{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}} z^k, \quad k \geq 2, \\
 g_k(z) &= z + \frac{1 - \gamma}{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}} \bar{z}^k, \quad k \geq 1,
 \end{aligned}$$

$\sum_{k=1}^{\infty} (X_k + Y_k) = 1, X_k \geq 0, Y_k \geq 0$. In particular, the extreme points of $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$ are $\{h_k\}$ and $\{g_k\}$.

Proof. For functions f of the form (3.1), we have

$$\begin{aligned}
 f(z) &= \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)) \\
 &= \sum_{k=1}^{\infty} (X_k + Y_k) z \\
 &\quad - \sum_{k=2}^{\infty} \frac{1 - \gamma}{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}} X_k z^k \\
 &\quad + \sum_{k=1}^{\infty} \frac{1 - \gamma}{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}} Y_k \bar{z}^k.
 \end{aligned}$$

Then

$$\begin{aligned}
 &\sum_{k=2}^{\infty} \left(\frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}}{1 - \gamma} \right) \\
 &\quad \times \left(\frac{1 - \gamma}{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}} \right) X_k \\
 &+ \sum_{k=1}^{\infty} \left(\frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}}{1 - \gamma} \right) \\
 &\quad \times \left(\frac{1 - \gamma}{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}} \right) Y_k \\
 &= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1
 \end{aligned}$$

and so $f \in clco \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$. Conversely, suppose that $f \in clco \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$. Setting

$$X_k = \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}}{1 - \gamma} |a_k|, \quad k \geq 2,$$

and

$$Y_k = \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}}{1 - \gamma} |b_k|, \quad k \geq 1,$$

where $\sum_{k=1}^{\infty} (X_k + Y_k) = 1$. Then note that by Theorem 2.2, $0 \leq X_k \leq 1$ ($k = 2, 3, \dots$) and $0 \leq Y_k \leq 1$ ($k = 1, 2, 3, \dots$). We define

$$X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k \text{ and by Theorem 2.2, } X_1 \geq 0. \text{ Consequently, we obtain } f(z) = \sum_{k=1}^{\infty} (X_k h_k(z) + Y_k g_k(z)).$$

Using Theorem 2.2, it is easily seen that $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$ is convex and closed, so $clco \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t) = \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$.

In other words, the statement of Theorem 3.1 is really for $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$. □



The following theorem gives the distortion bounds for functions in $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$ which yields a covering result for this class.

Theorem 3.2. Let $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$ and

$$A \leq [(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}$$

and

$$A \leq [(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta,\sigma,\nu} \quad k \geq 2,$$

where

$$A = \min \left\{ \begin{aligned} &[(1 + \beta)([1 + ([2]_q - 1)\lambda]^m) [2]_q - (\gamma + \beta)t([1 + ([2]_q - 1)\lambda]^n)] \Phi_{q,2}^{\delta,\sigma,\nu}, \\ &[(1 + \beta)([1 + ([2]_q - 1)\lambda]^m) [2]_q + (\gamma + \beta)t([1 + ([2]_q - 1)\lambda]^n)] \Phi_{q,2}^{\delta,\sigma,\nu} \end{aligned} \right\}.$$

Then

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1 - \gamma}{A} - \frac{[(1 + \beta) + (\gamma + \beta)t] \Phi_{q,1}^{\delta,\sigma,\nu}}{A} |b_1| \right) r^2$$

and

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1 - \gamma}{A} - \frac{[(1 + \beta) + (\gamma + \beta)t] \Phi_{q,1}^{\delta,\sigma,\nu}}{A} |b_1| \right) r^2.$$

Proof. Let $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$. Taking the absolute value of f , we obtain

$$\begin{aligned} |f(z)| &\leq (1 + |b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|)r^k \\ &\leq (1 + |b_1|)r + r^2 \sum_{k=2}^{\infty} (|a_k| + |b_k|) \\ &= (1 + |b_1|)r + \frac{1 - \gamma}{A} r^2 \sum_{k=2}^{\infty} \left(\frac{A}{1 - \gamma} |a_k| + \frac{A}{1 - \gamma} |b_k| \right) \\ &\leq (1 + |b_1|)r \\ &\quad + \frac{1 - \gamma}{A} r^2 \sum_{k=2}^{\infty} \left(\frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1 - \gamma} |a_k| \right. \\ &\quad \left. + \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1 - \gamma} |b_k| \right) \\ &\leq (1 + |b_1|)r + \frac{1 - \gamma}{A} \left(1 - \frac{[(1 + \beta) + (\gamma + \beta)t] \Phi_{q,1}^{\delta,\sigma,\nu}}{1 - \gamma} |b_1| \right) r^2 \\ &= (1 + |b_1|)r + \left(\frac{1 - \gamma}{A} - \frac{[(1 + \beta) + (\gamma + \beta)t] \Phi_{q,1}^{\delta,\sigma,\nu}}{A} |b_1| \right) r^2 \end{aligned}$$

and similarly,

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1 - \gamma}{A} - \frac{[(1 + \beta) + (\gamma + \beta)t] \Phi_{q,1}^{\delta,\sigma,\nu}}{A} |b_1| \right) r^2.$$

The upper and lower bounds given in Theorem 3.2 are respectively attained for the following functions.

$$f(z) = z + |b_1|\bar{z} + \left(\frac{1 - \gamma}{A} - \frac{[(1 + \beta) + (\gamma + \beta)t] \Phi_{q,1}^{\delta,\sigma,\nu}}{A} |b_1| \right) \bar{z}^2$$



and

$$f(z) = (1 - |b_1|)z - \left(\frac{1 - \gamma}{A} - \frac{[(1 + \beta) + (\gamma + \beta)t] \Phi_{q,1}^{\delta, \sigma, \nu}}{A} |b_1| \right) z^2.$$

□

The following covering result follows from the left hand inequality in Theorem 3.2.

Corollary 3.3. Let f of the form (1.3) be so that $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$ and

$$A \leq [(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu},$$

and

$$A \leq (1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n \Phi_{q,k}^{\delta, \sigma, \nu}, \quad \text{for } k \geq 2,$$

where

$$A = \min \left\{ \begin{aligned} & [(1 + \beta) ([1 + ([2]_q - 1)\lambda]^m) [2]_q - (\gamma + \beta)t ([1 + ([2]_q - 1)\lambda]^n)] \Phi_{q,2}^{\delta, \sigma, \nu}, \\ & [(1 + \beta) ([1 + ([2]_q - 1)\lambda]^m) [2]_q + (\gamma + \beta)t ([1 + ([2]_q - 1)\lambda]^n)] \Phi_{q,2}^{\delta, \sigma, \nu} \end{aligned} \right\}.$$

Then

$$\left\{ \omega : |\omega| < \frac{A + 1 - \gamma}{A} + \frac{A - 1 + \gamma}{A} |b_1| \right\} \subset f(\mathbb{U}).$$

4. Convolution and Convex Combinations

In this section we show that the class $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$ is closed under convolution and convex combinations. Now we need the following definition of convolution of two harmonic functions. For $f(z) = z - \sum_{k=2}^{\infty} |a_k|z^k + \sum_{k=1}^{\infty} |b_k|\bar{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty} |A_k|z^k + \sum_{k=1}^{\infty} |B_k|\bar{z}^k$, we define the convolution of two harmonic functions f and F as

$$(f * F)(z) = f(z) * F(z) = z - \sum_{k=2}^{\infty} |a_k||A_k|z^k + \sum_{k=1}^{\infty} |b_k||B_k|\bar{z}^k. \tag{4.1}$$

Using the definition, we show that the class $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$ is closed under convolution.

Theorem 4.1. For $0 \leq \gamma < 1$, let $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$ and $F \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$. Then $f * F \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$.

Proof. Let $f(z) = z - \sum_{k=2}^{\infty} |a_k|z^k + \sum_{k=1}^{\infty} |b_k|\bar{z}^k$ and $F(z) = z - \sum_{k=2}^{\infty} |A_k|z^k + \sum_{k=1}^{\infty} |B_k|\bar{z}^k$ be in $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$. Then the convolution $f * F$ is given by (4.1). We wish to show that the coefficient of $f * F$ satisfy the required condition given in Theorem 2.2. For $F \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$, we note that $|A_k| \leq 1$ and $|B_k| \leq 1$. Now for the convolution function $f * F$, we obtain

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}}{1 - \gamma} |a_k||A_k| \\ & \quad + \sum_{k=1}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}}{1 - \gamma} |b_k||B_k| \\ & \leq \sum_{k=2}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q - (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}}{1 - \gamma} |a_k| \\ & \quad + \sum_{k=1}^{\infty} \frac{[(1 + \beta)[\lambda ([k]_q - 1) + 1]^m [k]_q + (\gamma + \beta)t[\lambda ([k]_q - 1) + 1]^n] \Phi_{q,k}^{\delta, \sigma, \nu}}{1 - \gamma} |b_k| \\ & \leq 1, \end{aligned}$$

since $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$. Therefore $f * F \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta, \sigma, \nu, \lambda}(\beta, \gamma, t)$.

□



Next, we show that the class $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$ is closed under convex combination of its members.

Theorem 4.2. *The class $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \dots$ let $f_i(z) \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$, where f_i is given by

$$f_i(z) = z - \sum_{k=2}^{\infty} |a_{ik}| z^k + \sum_{k=1}^{\infty} |b_{ik}| \bar{z}^k.$$

For $\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i(z) = z - \sum_{k=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i |a_{ik}| \right) z^k + \sum_{k=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i |b_{ik}| \right) \bar{z}^k.$$

Since,

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q - (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1-\gamma} |a_{ik}| \\ & + \sum_{k=1}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q + (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1-\gamma} |b_{ik}| \leq 1, \end{aligned}$$

from the above equation we obtain

$$\begin{aligned} & \sum_{k=2}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q - (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1-\gamma} \sum_{i=1}^{\infty} t_i |a_{ik}| \\ & + \sum_{k=1}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q + (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1-\gamma} \sum_{i=1}^{\infty} t_i |b_{ik}| \\ & = \sum_{i=1}^{\infty} t_i \left\{ \sum_{k=2}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q - (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1-\gamma} |a_{ik}| \right. \\ & \quad \left. + \sum_{k=1}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q + (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{k(1-\gamma)} |b_{ik}| \right\} \\ & \leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

This is the condition required by (2.3) and so $\sum_{i=1}^{\infty} t_i f_i(z) \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$. □

5. Class preserving integral operators

Finally, we consider the closure property of the class $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,\nu,\lambda}(\beta, \gamma, t)$ under the generalized Bernardi-Libera -Livingston integral operator $\mathcal{L}_c[f(z)]$ and the q -Jackson integral operator F_q .

1. The generalized Bernardi-Libera -Livingston integral operator $\mathcal{L}_c[f(z)]$ for $f(z) = h(z) + \overline{g(z)}$ given by (1.1) is

$$\mathcal{L}_c[f(z)] = \frac{c+1}{z^c} \int_0^z \xi^{c-1} f(\xi) d\xi, \quad c > -1.$$



2. For $f(z) = h(z) + \overline{g(z)}$ given by (1.1), the q -Jackson integral operator F_q is defined by the relation

$$F_q(z) = \frac{[c]_q}{z^{c+1}} \int_0^z u^c h(u) d_q u + \overline{\frac{[c]_q}{z^{c+1}} \int_0^z u^c g(u) d_q u}, \tag{5.1}$$

where $[c]_q$ is the q -number defined by (1.4).

Theorem 5.1. Let $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$, then $\mathcal{L}_c[f(z)] \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$.

Proof. From the representation of $\mathcal{L}_c[f(z)]$, it follows that

$$\begin{aligned} \mathcal{L}_c[f(z)] &= \frac{c+1}{z^c} \int_0^z \xi^{c-1} h(\xi) d\xi + \overline{\frac{c+1}{z^c} \int_0^z \xi^{c-1} g(\xi) d\xi} \\ &= \frac{c+1}{z^c} \int_0^z \xi^{c-1} \left(\xi - \sum_{k=2}^{\infty} |a_k| \xi^k \right) d\xi + \overline{\frac{c+1}{z^c} \int_0^z \xi^{c-1} \left(\sum_{k=1}^{\infty} |b_k| \xi^k \right) d\xi} \\ &= z - \sum_{k=2}^{\infty} A_k z^k + \sum_{k=1}^{\infty} B_k z^k, \end{aligned}$$

where $A_k = \frac{c+1}{c+k} |a_k|$ and $B_k = \frac{c+1}{c+k} |b_k|$. Hence

$$\begin{aligned} &\sum_{k=2}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q - (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,v} \left(\frac{c+1}{c+k} |a_k|\right)}{1-\gamma} \\ &+ \sum_{k=1}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q + (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,v} \left(\frac{c+1}{c+k} |b_k|\right)}{1-\gamma} \\ &\leq \sum_{k=2}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q - (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,v}}{1-\gamma} |a_k| \\ &+ \sum_{k=1}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q + (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,v}}{1-\gamma} |b_k| \\ &\leq 1, \end{aligned}$$

since $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$, therefore by Theorem 2.2, $\mathcal{L}_c(f(z)) \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$. □

In the next theorem, we show that the class $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$ is closed under the q -integral operator defined by (5.1).

Theorem 5.2. Let $f(z) = h(z) + \overline{g(z)}$ be given by (1.3) and $f \in \mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$ where $\lambda, \mu \in \mathbb{N}_0, \beta \geq 0, 0 \leq \gamma < 1, 0 \leq t \leq 1$ and $0 < q < 1$. Then F_q defined by (5.1) is also in the class $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$.

Proof. Let

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \bar{z}^k$$

be in $\mathcal{G}_{\mathcal{H}_{m,n,q}}^{\delta,\sigma,v,\lambda}(\beta, \gamma, t)$. Then by Theorem 2.2, the condition (2.3) is satisfied.

From the series representation (5.1) of F_q , it follows that,

$$F_q(z) = z - \sum_{k=2}^{\infty} \frac{[c]_q}{[k+c+1]_q} |a_k| z^k + \sum_{k=1}^{\infty} \frac{[c]_q}{[k+c+1]_q} |b_k| \bar{z}^k.$$



Since

$$[k+c+1]_q - [c]_q = \sum_{i=0}^{k+c+1} q^i - \sum_{i=0}^c q^i = \sum_{i=c}^{k+c+1} q^i > 0$$

$$[k+c+1]_q > [c]_q \quad (\text{or}) \quad \frac{[c]_q}{[k+c+1]_q} < 1.$$

Now,

$$\sum_{k=2}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q - (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1-\gamma} \frac{[c]_q}{[k+c+1]_q} |a_k|$$

$$+ \sum_{k=1}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q + (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1-\gamma} \frac{[c]_q}{[k+c+1]_q} |b_k|$$

$$\leq \sum_{k=2}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q - (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1-\gamma} |a_k|$$

$$+ \sum_{k=1}^{\infty} \frac{[(1+\beta)[\lambda([k]_q-1)+1]^m [k]_q + (\gamma+\beta)t[\lambda([k]_q-1)+1]^n] \Phi_{q,k}^{\delta,\sigma,\nu}}{1-\gamma} |b_k|$$

$$\leq 1.$$

Thus the proof of Theorem 5.2 is established. □

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