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Positive solutions of an initial value problem of a delay-self-reference nonlinear differential equation

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Abstract

In this paper we study the existence of positive solutions for an initial value problem of a delay-state-dependent nonlinear differential equation. The continuous dependence of the unique solution on the initial data and the delay-state-dependent function will be proved. Some especial cases and examples will be given.

Keywords

Delay-state-dependent, nonlinear differential equation, existence of solutions, continuous dependence, Arzela-Ascoli Theorem, Schauder fixed point Theorem.

AMS Subject Classification

2010 MSC: 34A12, 34A30, 34D20.

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Contents

1. Introduction

Many authors studied the the differential and integral equations with deviating arguments only the in time itself, however, the case of the deviating arguments depend on both the state variable x and the time t is important in theory and practice, see for example [1]-[4], [7], [10], [11], [13]-[19].

In [4], the author studied the existence of a unique solution $x \in C[a,b]$ and its continuous dependence on the initial data of the intimal value problem of the self-refereed differential equation

$$\frac{d}{dt}x(t) = f(t,x(x(t))), \quad t \in (0,T] \text{ and } x(0) = x_o$$

where $f \in (C[a,b], C[a,b])$.

In [8], the authors studied the existence of solutions $x \in C[0,T]$ and the continuous dependence of the unique solution on the initial data and the function g of the initial value problem of functional integro-differential equation of self-reference ($\phi(t) = t$) and state-dependence ($\phi(t) \leq t$)

$$\frac{d}{dt}x(t) = f(t,x(\int_0^{\phi(t)} g(t,x(t)))ds)), \quad t \in (0,T]$$

$$\text{and } x(0) = x_o$$

where $g : [0,T] \times R^+ \to [0,T]$ is continuous and $g(t,x(t)) \leq 1$. The authors in [9] proved, when $g : [0,T] \times R^+ \to [0,T]$ is continuous and $g(t,x(t)) \leq t$, the existence of positive solutions $x \in C[0,T]$ and the continuous dependence of the unique solution on the initial data and the function g of the initial value problem

$$\frac{d}{dt}x(t) = f(t,x(g(t,x(t)))), \ a.e., \ t \in (0,T], \tag{1.1}$$

$$x(0) = x_0 \in [0,T]. \tag{1.2}$$

Let $C[0,T]$ be the class of continuous functions defined on $[0,T]$ with norm

$$||x|| = \sup_{t \in [0,T]} |x(t)|, \quad x \in C[0,T].$$

Let $g : [0,T] \times R^+ \to [0,T]$ be continuous and $g(t,x(t)) \le x(t)$. Consider the initial value problem of the delay-state-dependent nonlinear differential equation

$$\frac{d}{dt}x(t) = f(t,x(g(t,x(t)))), \ a.e., \ t \in (0,T] \quad (1.3)$$

$$x(0) = x_0 \in [0,T]. \quad (1.4)$$

Our aim in this work is to prove the existence of positive solutions $x \in C[0,T]$ of the initial value problem (1.3)-(1.4). The continuous dependence of the unique solution on the initial data x_o and the delay-state-dependent function g will be studied.

2. Main Results

In this section, we deal with the existence and uniqueness of solution for the initial value problem (1.3)- (1.4). Also we prove that the solution depends continuously on the initial data and the function g. Now, we consider the following assumptions to establish the existence results:

(1) $f : [0,T] \times [0,T] \to R$ satisfies Carathéodory condition i.e. f is measurable in t for all $x \in C[0,T]$ and continuous in x for almost all $t \in [0,T]$

(2) There exists a measurable bounded function $m(t)$ and a constant $b \ge 0$ such that $|f(t,x)| \le m(t) + b|x|$

(3) $g : [0,T] \times R^+ \to [0,T]$ is continuous such that $g(t,x(t))) \le x(t)$

(4) $L = M + b\, T < 1$.

(5) $LT + |x(0)| \le T$.

Some examples for the function g

(1) $g(t,x(t)) = \frac{x(t)}{1+ax^2(t)}, \quad a \ge 0$

(2) $g(t,x(t)) = \frac{x(t)\ e^{-ax^2(t)}}{1+b\sin^2 x(t)}, \quad a,b \ge 0$

(3) $g(t,x(t)) = \frac{x(t)}{1+ae^{-x^2(t)}}, \quad a \ge 0.$

2.1 Existence theorem

Theorem 2.1. *Let the assumptions $(1)-(5)$ be satisfied, then the initial value problem (1.3), (1.4) has at least one solution $x \in S_L \subset C[0,T]$.*

Proof. Let x be a solution of the problem (1)-(2). Integrating the differential equation (1) we obtain the corresponding integral equation

$$x(t) = x_0 + \int_0^t f(s,x(g(t,x(s)))\, ds > 0, \quad t \in [0,T]. \quad (2.1)$$

Define the set S_L by

$$S_L = \left\{x \in C[0,T] : |x(t_2) - x(t_1)| \le L|t_2 - t_1|\right\} \subset C[0,T].$$

where $L = M + b\, T$.

It clear that S_L is nonempty, closed, bounded and convex subset of $C[0,T]$.
Define the operator F associated with equation (1.3) by

$$Fx(t) = x_0 + \int_0^t f(s,x(g(s,x(s))))ds \quad t \in [0,T]$$

Firstly, we prove that F is uniformly bounded. Let $x \in C[0,T]$, then we get

$$\begin{aligned} |Fx(t)| &\le |x_0| + \int_0^t |f(s,x(g(s,x(s))))|ds \quad t \in [0,T] \\ &\le |x_0| + \int_0^t \{m(s) + b|x(g(s,x(s))))|\}ds \\ &\le |x_0| + \int_0^t \{M + b|x(g(s,x(s))))|\}ds. \end{aligned}$$

But

$$|x(\phi(x(t))))| - |x_0| \le L|x(g(t,x(t)))) - x(0)| \le L|g(t,x(t))|,$$

then

$$|x(\phi(x(t))))| \le L|g(t,(t))| + |x_0|. \quad (2.2)$$

Using (2.2) we obtain

$$\begin{aligned} |Fx(t)| &\le |x_0| + \int_0^t \{M + b(L|x(s)| + |x_0|)\}ds \\ &\le |x_0| + \int_0^t \{M + b\ (LT + |x_0|)\}ds \\ &\le |x_0| + (M + b(LT + |x_0|))t \\ &\le |x_0| + (M + bT)T \\ &\le LT + |x_0| \ \le T. \end{aligned}$$

This proves that the class functions $\{Fx\}$ is uniformly bounded. Secondly, we will show that $F : S_L \to S_L$ and the class of functions $\{Fx\}$ is equi-continuous.
Let $x \in S_L$ and $t_1,\ t_2 \in [0,T]$ with $t_1 < t_2$ such that $|t_2, -t_1| < \delta$, then

$$\begin{aligned} |Fx(t_2) - Fx(t_1)| &= |\int_{t_1}^{t_2} f(s,x(g(s,x(s))))ds| \\ &\le \int_{t_1}^{t_2} |f(s,x(g(s,x(s))))|ds \\ &\le \int_{t_1}^{t_2} \{M + b|x(g(s,x(s))))|\}ds \\ &\le \int_{t_1}^{t_2} \{M + b(L|g(s,x(s))| + |x_0|)\}ds \\ &\le \int_{t_1}^{t_2} (M + b(L|x(s)| + |x_0|))ds \\ &\le \int_{t_1}^{t_2} (M + bT)ds \quad \le L|t_2 - t_1|. \end{aligned}$$

This proves that $F : S_L \to S_L$ and the class of functions $\{Fx\}$ is equi-continuous.
Now by Arzela-Ascoli Theorem [5] F is compact.
Finally, let $\{x_n\} \subset S_L$ such that $x_n \to x$ on $[0,T]$, then we have

$$\begin{aligned} & |x_n(g(t,x_n(t))) - x(g(t,x(t)))| \\ = & |x_n(g(t,x_n(t))) - x_n(g(t,x(t))) \\ + & x_n(g(t,x(t))) - x(g(t,x(t)))| \\ \leq & |x_n(g(t,x_n(t))) - x_n(g(t,x(t)))| \\ + & |x_n(g(t,x(t))) - x(g(t,x(t)))| \\ \leq & L|g(t,x_n(t)) - g(t,x(t))| \\ + & |x_n(g(t,x(t))) - x(g(t,x(t)))| \end{aligned}$$

which implies that

$$x_n(g(t,x_n(t))) \to x(g(t,x(t))) \text{ in } S_L.$$

Also, from the continuity the function f we obtain

$$f\big(t,x_n(g(t,x_n(t)))\big) \to f\big(t,x(g(t,x(t)))\big).$$

Using assumption (2) and Lebesgues dominated convergence theorem [6] we deduce that

$$\begin{aligned} \lim_{n\to\infty} x_n(t) &= x_0 + \lim_{n\to\infty} \int_0^t f\big(s,x_n(g(t,x_n(t)))\big)ds \\ &= x_0 + \int_0^t f\big(s,x(g(t,x(t)))\big)ds. \end{aligned}$$

Then F is continuous.
Now all conditions of Schauder fixed point Theorem [5], are satisfied, then the operator F has at least one fixed point $x \in S_L$. Consequently there exist at leat one solution $x \in C[0,T]$ of the integral equation equation (2.1).

Now, to complete the proof, differentiating the integral equation (3) we obtain the differential equation (1).
Also letting $t = 0$ in (3) we obtain the initial data (2).
This completes the proof of the equivalence between the initial value problem (1)-(2) and the integral equation (3).
Hence the initial value problem (1)-(2) has at least one positive solution $x \in C[0,T]$ which completes the proof. □

Now, we have the following corollary which generalize the results in [4].

Corollary 2.2. *Let the assumptions of Theorem 1.3 be satisfied, if $g(t,x(t)) = x(t)$, then the integral equation*

$$x(t) = x_0 + \int_0^t f(s,x(x(s)))ds, \qquad t \in [0,T],$$

has at least one solution $x \in C[0,T]$. Consequently the initial value problem

$$\begin{aligned} \frac{dx(t)}{dt} &= f(t,x(x(t))) \text{ a.e. } \quad t \in (0,T] \\ x(0) &= x_0 \end{aligned}$$

has at least one solution $x \in AC[0,T]$.

2.2 Uniqueness of the solution

In this section we prove the uniqueness of the solution for the integral equation (2.1). For this aim we assume that:

(1′) $|f(t,x) - f(t,y)| \leq b\,|x-y|$

(2′) $|f(t,0)| \leq M$,

(3′) $|g(t,x) - g(t,y)| \leq k|x-y|$

Theorem 2.3. *Let the assumptions (1),(3),(4) of Theorem 2.1 and (1′), (2′) and (3′) be satisfied, if $bT(L\,k+1) < 1$, then the solution of equation (2.1) is unique.*

Proof. Assumption (2) of Theorem 2.1 can be deduced from assumptions (1′) *and* (2′). By putting $y = 0$ in (1′) we get

$$|f(t,x)| \quad \leq \quad b\,|x| + |f(t,0)|$$

hence we deduce that all assumptions of Theorem 2.1 are satisfied. Then the solution of equation (2.1) exists. Now let $x,\ y$ be two solutions of (2.1), then

$$\begin{aligned} & |x(t) - y(t)| \\ = & \Big|\int_0^t f(s,x(g(s,x(s)))ds \\ - & \int_0^t f(s,y(g(s,y(s))))ds\Big| \\ \leq & \int_0^t |f(s,x(g(s,x(s)))) - f(s,y(g(s,y(s))))|ds \\ \leq & b\int_0^t |x(g(s,x(s))) - y(g(s,y(s))|ds \\ \leq & b\int_0^t |x(g(s,x(s))) - x(g(s,y(s)))|ds \\ + & b\int_0^t |x(g(s,y(s))) - y(g(s,y(s)))|ds \\ \leq & bL\int_0^t |g(s,x(s)) - g(s,y(s))|ds \\ + & b\int_0^t |x(g(s,y(s))) - y(g(s,x(s)))|ds \\ \leq & b\,L\,k\int_0^t |x(s) - y(s)|ds \\ + & b\int_0^t |x(g(s,y(s))) - y(g(s,y(s)))|ds \\ \leq & b\,L\,k\,T\|x-y\| + b\,T\|x-y\| \\ = & b\,T\,(L\,k+1)\|x-y\|, \end{aligned}$$

then we obtain

$$\|x-y\| \quad \leq \quad b\,T\,(L\,k+1)\,\|x-y\|.$$

Since $b\,T\,(Lk+1) < 1$, then we deduce that $x(t) = y(t)$ and hence the solution of (2.1) is unique. □

2.3 Continuous dependence

Here we prove that the solution of equation (2.1) depends continuously of the initial data x_0.

Definition 2.4.

The solution of the integral equation (2.1) depends continuously on the initial data x_0
if $\forall\, \varepsilon > 0\, \exists\, \delta(\varepsilon) > 0$ such that,

$$|x_0 - x_0^*| \leq \delta \Rightarrow \|x - x^*\| \leq \varepsilon \tag{2.3}$$

where x^* is the unique solution of the equation where x^* is the unique solution of the integral equation

$$x^*(t) = x_0^* + \int_0^t f(s, x^*(g(s,(x^*(s))))ds, \quad t \in [0,T]. \tag{2.4}$$

Theorem 2.5. *Let the assumptions of Theorem 2.3 be satisfied, then the solution of (2.1) depends continuously on the initial data x_0.*

Proof. Let x, x^* be the solution of the integral equation (3) and (2.4). Then, we have

$$\begin{aligned}
& |x(t) - x^*(t)| \\
= \ & |x_0 + \int_0^t f(s, x(g(s,x(s))))ds - x_0^* \\
+ \ & \int_0^t f(s, x^*(g(s,x^*(s))))ds| \\
\leq \ & |x_0 - x_0^*| \\
+ \ & \int_0^t |f(s, x(g(s,x(s))) - f(s, x^*(g(s,x^*(s))))|ds \\
\leq \ & |x_0 - x_0^*| + b \int_0^t |x(g(s,x(s))) - x^*(g(s,x^*(s)))|ds \\
\leq \ & |x_0 - x_0^*| + b\, L \int_0^t |g(s,x(s)) - g(s,x^*(s))|ds \\
+ \ & b \int_0^t |x(g(s,x^*(s))) - x^*(g(s,x^*(s)))|ds \\
\leq \ & |x_0 - x_0^*| + b\, L\, k \int_0^t |x(s) - x^*(s)|ds \\
+ \ & b \int_0^t |x(g(s,(x^*(s))) - x^*(g(s,x^*(s)))|ds \\
\leq \ & \delta + b\, L\, k\, \|x - x^*\|\, T + b\|x - x^*\|\, T
\end{aligned}$$

and

$$\|x - x^*\| \leq \delta + b\, T(L\, k\, + 1)\|x - x^*\|,$$

then

$$\|x - x^*\| \leq \frac{\delta}{(1 - b\, T(L\, k\, + 1))}.$$

Since $b\, T(L\, k\, + 1) < 1$ it follows that the solution of (2.1) depends continuously on the initial data x_0.

Definition 2.6.

The solution of the integral equation (2.1) depends continuously on the function g
$if,\ \forall\, \varepsilon > 0.\ \exists\, \delta(\varepsilon) > 0$ such that

$$|g(t,x(t) - g^*(t,x(t)| \leq \delta \Rightarrow \|x - x^*\| \leq \varepsilon \tag{2.5}$$

where x^* is the unique solution of the integral equation

$$x^*(t) = x_0 + \int_0^t f(s, x^*(g^*(s,(x^*(s))))ds, \quad t \in [0,T]. \tag{2.6}$$

Theorem 2.7. *Let the assumptions of Theorem 2.3 be satisfied, then the solution of (2.1) depends continuously on the function g.*

Proof. Let x, x^* be the solution of the integral equation (3) and (2.6). Then, we have Let x, x^* be the solution of the integral equation (3) and (2.4). Then, we have

$$\begin{aligned}
& |x(t) - x^*(t)| \\
= \ & |x_0 + \int_0^t f(s, x(g(s,x(s))))ds \\
- \ & x_0 + \int_0^t f(s, x^*(g^*(s,x^*(s))))ds| \\
\leq \ & \int_0^t |f(s, x(g(s,x(s))) - f(s, x^*(g^*(s,x^*(s))))|ds \\
\leq \ & b \int_0^t |x(g(s,x(s))) - x^*(g^*(s,x^*(s)))|ds \\
\leq \ & b \int_0^t |x(g(s,x(s))) - x(g^*(s,x^*(s)))|ds \\
+ \ & b \int_0^t |x(g^*(s,x^*(s))) - x^*(g^*(s,x^*(s)))|ds \\
\leq \ & b\, L \int_0^t |g(s,x(s)) - g^*(s,x^*(s))|ds \\
+ \ & b \int_0^t |x(g^*(s,x^*(s))) - x^*(g^*(s,x^*(s)))|ds \\
\leq \ & b\, L \int_0^t |g(s,x(s)) - g(s,x^*(s))|ds \\
+ \ & b\, L \int_0^t |g(s,x^*(s)) - g^*(s,x^*(s))|ds \\
+ \ & b\, T\|x - x^*\| \\
\leq \ & b\, L\, T\, k\, \|x - x^*\| + b\, L\, T\delta \\
+ \ & b\, T\, \|x - x^*\|
\end{aligned}$$

and

$$\|x - x^*\| \leq b\, L\, T\, \delta + b\, T(L\, k\, + 1)\|x - x^*\|,$$

then

$$\|x - x^*\| \leq \frac{b\, L\, T\, \delta}{(1 - b\, T(L\, k\, + 1))}.$$

Since $b\, T(L\, k\, + 1) < 1$ it follows that the solution of (2.1) depends continuously on the function g. □

3. Example

Example 3.1. *Consider the nonlinear differential equation*

$$\frac{dx}{dt} = \frac{1}{5}(1+t) + \frac{1}{7}\,x\left(\frac{x(t)\,e^{-x^2(t)}}{1+\sin^2 x(t)}\right), \quad t \in (0,2], \quad (3.1)$$

with the initial condition

$$x(0) = \frac{1}{5}. \quad (3.2)$$

Set

$$f(t, x(\phi(x(t)))) = \frac{1}{5}(1+t) + \frac{1}{7}\,x\left(\frac{x(t)\,e^{-x^2(t)}}{1+\sin^2 x(t)}\right)$$

thus

$$|f(t,x)| \le \frac{1}{5}(1+t) + \frac{1}{7}\,|x|$$

here we have $m(t) = \frac{1}{5}(1+t)$ *which is measurable and bounded function with bound* $M = 3/5$ *and* $b = 1/7$, $x(0) = \frac{1}{5}$,
then $L_2 = M + b\,T = 31/35 < 1$ *and* $LT + |x(0)| \approx 1.97 < T = 2$.
Therefore, by applying to Theorem 2.1, the initial value problem (3.1)-(3.2) has a continuous solution.

Example 3.2. *Consider the nonlinear differential equation*

$$\frac{dx}{dt} = \frac{1}{9}t^3\sin(t^2) + \frac{1}{4}\,x\left(\frac{x(t)}{1+x^2(t)}\right), \quad t \in (0,1], \quad (3.3)$$

with the initial condition

$$x(0) = \frac{1}{2}. \quad (3.4)$$

Set

$$f(t, x(\phi(x(t)))) = \frac{1}{9}t^3\sin(t^2) + \frac{1}{4}\,x\left(\frac{x(t)}{1+x^2(t)}\right)$$

thus

$$|f(t,x)| \le \frac{1}{9}t^3|\sin(t^2)| + \frac{1}{4}\,|x|$$

here we have $m(t) = \frac{1}{9}t^3|\sin(t^2)|$ *which is measurable and bounded function with bound* $M = 1/9$ *and* $b = 1/4$, $x(0) = \frac{1}{2}$,
then $L_2 = M + b\,T = 13/36 < 1$ *and* $LT + |x(0)| = \frac{31}{36} < T = 1$.
Therefore, by applying to Theorem 2.1, the initial value problem (3.3)-(3.4) has a continuous solution.

4. Conclusion

In this paper, we prove the existence, the uniqueness and the continuous dependence of positive continuous solution $x \in C[0,T]$ of an initial value problem of a delay-self-reference nonlinear differential equation under a suitable assumptions. Here we relax the assumptions and generalize the results in [4,8], also we introduced some examples and applications to indicate our results.

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