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# Some aspects of 2-fuzzy 2-metric projection operator of 2-fuzzy 2-Banach spaces

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**Abstract**

In this paper, continuous homogeneous selection and continuity for the set valued 2-fuzzy 2-generalized inverse in 3-strictly 2-fuzzy 2-convex space are investigated using fuzzy continuity of metric projection. Hence approximative compactness of 2-fuzzy 2-Banach space is not necessary for the 2-fuzzy 2-upper semi continuity of the set valued 2-fuzzy 2-metric generalized inverse.

**Keywords**

2-fuzzy 2-H-Property, 2-fuzzy 2-Continuous Selections, 2-fuzzy 2-Chebyshev Subspace, 2-fuzzy 2-Metric Generalized Inverse.

**AMS Subject Classification**:

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## Contents

1 Introduction .......... 1012

2 Preliminaries .......... 1012

3 Continuity of 2-fuzzy 2-metric projection operator and 2-fuzzy 2-approximative compactness .......... 1014

4 2-Fuzzy 2-Continuous selections and 2-Fuzzy 2-Continuity of the set valued Metric Generalized Inverse .......... 1015

References .......... 1020

## 1. Introduction

The concept of fuzzy set was first introduced by L.A. Zadeh [13] in 1965. Many mathematicians considered fuzzy metric in different views [3, 6–8, 13]. George and Veeramani [6] defined fuzzy metric space in a new way. Various definitions of fuzzy norms on a linear space were introduced by different authors [1, 2, 4, 9, 10]. Rano and Bag [11] introduced the definition of fuzzy norm following the notion introduced by Bag and Samanta[1].

A satisfactory theory of 2-norm on a linear space has been introduced and developed by Gahler[5]. Somasundaram and Thangaraj Beaula [12] introduced the concept of 2-fuzzy 2-normed linear space and gave the notion of $\alpha$-2-norm using the ideas of Bag and Samanta [1].

In this paper, continuous homogeneous selection and continuity for the set valued 2-fuzzy 2-generalized inverse in 3-strictly 2-fuzzy 2-convex space are investigated using fuzzy continuity of metric projection. Hence approximative compactness of 2-fuzzy 2-Banach space is not necessary for the 2-fuzzy 2-upper semi continuity of the set valued 2-fuzzy 2-metric generalized inverse.

## 2. Preliminaries

**Definition 2.1.** *Let $X$ be a universe of discourse a fuzzy set is defined as $A=\{x,\mu_A(x): x\in X\}$ which is characterized by a membership function $\mu_A(x): X\to[0,1]$ where $\mu_A(x)$ denotes the degree of membership of the element $x$ to the set $A$.*

**Definition 2.2.** *Let $X$ be a non empty and $F(X)$ be the set of all fuzzy sets in $X$. If $f\in F(X)$ then $f=\{(x,\mu)/x\in X$ and $\mu\in(0,1]\}$. Clearly $f$ is bounded function for $|f(x)|\leq 1$. Let $K$ be the space of real numbers then $F(X)$ is a linear space over the field $K$ where the addition and scalar multiplication are defined by*
*$f+g=\{(x,\mu)+(y,\eta)\}=\{(x+y),(\mu,\eta)/(x,\mu)\in f$ and $(y,\eta)\in g\}$*
*and*

$$kf=\{(kf,\mu)/(x,\mu)\in f\}$$

*where $k\in K$.*
*The linear space $F(X)$ is said to be normed space if for every*

*$f \in F(X)$ there is associated a non-negative real number $\|f\|$ called the norm of $f$ in such a way,*
*(1) $\|f\| = 0$ if and only if $f = 0$*
*For,*

$$\|f\| = 0 \Leftrightarrow \{\|(x,\mu)\| / (x,\mu) \in f\} = 0$$
$$\Leftrightarrow x = 0, \mu \in (0,1] \Leftrightarrow f = 0$$

*(2)$\|kf\| = |k|\|f\|, k \in K$*
*For,*

$$\|kf\| = \{\|k(x,\mu)\| / (x,\mu) \in f, k \in K\}$$
$$= \{|k|\|x,\mu\| / (x,\mu) \in f\} = |k|\|f\|$$

*(3) $\|f+g\| < \|f\| + \|g\|$ for every $f, g \in F(X)$*
*For,*

$$\|f+g\| = \{\|(x,\mu)+(y,\eta)\| / x,y \in X, \mu,\eta \in (0,1]\}$$
$$= \{\|(x+y),(\mu \wedge \eta)\| / x,y \in X, \mu,\eta \in (0,1]\}$$
$$\leq \{\|(x,\mu \wedge \eta)\| + \|(y,\mu \wedge \eta)\| / (x,\mu) \in f \text{ and } (y,\eta) \in g\}$$
$$= \|f\| + \|g\|$$

*Then $(F(X), \|.\|)$ is a normed linear space.*

**Definition 2.3.** *A 2-fuzzy set on X is a fuzzy set on $F(X)$.*

**Definition 2.4.** *Let $F(X)$ be a linear space over the real field K. A fuzzy subset N of $F(X) \times F(X) \times R$ (R, the set of real numbers) is called a 2-fuzzy 2-norm on X (or fuzzy 2-norm on $F(X)$) if and only if,*
***(N1)** for all $t \in R$ with $t \leq 0, N(f_1, f_2, t) = 0$.*
***(N2)** for all $t \in R$ with $t \geq 0, N(f_1, f_2, t) = 1$, if and only if $f_1$ and $f_2$ are linearly dependent.*
***(N3)** $N(f_1, f_2, t)$ is invariant under any permutation of $f_1, f_2$.*
***(N4)** for all $t \in R$, with $t \geq 0, N(f_1, cf_2, t) = N(f_1, f_2, t/|c|)$ if $c \neq 0, c \in K$ (field).*
***(N5)** for all $s,t \in R$, $N(f_1, f_2 + f_3, s+t) \geq min\{N(f_1, f_2, s), N(f_1, f_3, t)\}$.*
***(N6)** $N(f_1, f_2, .) : (0,\infty) \to [0,1]$ is continuous.*
***(N7)** $\lim_{t\to\infty} N(f_1, f_2, t) = 1$.*
*Then $(F(X), N)$ is a fuzzy 2-normed linear space or (X,N) is a 2-fuzzy 2-normed linear space.*

**Definition 2.5.** *A sequence $\{f_n\}$ in a 2-fuzzy normed linear space $(F(X), N)$ is said to be a convergent sequence if for a given $t > 0$ and $0 < r < 1$ there exist a positive number $n_0 \in N$ such that*
*$N(f_n - f, g, t) > 1 - r$ for $g \in F(X)$ and for every $n \geq n_0$.*

**Definition 2.6.** *A sequence $\{f_n\}$ is said to be a Cauchy sequence in a 2-fuzzy normed linear space $F(X)$ if for a given $r > 0$ with $0 < r < 1, t > 0$ there exist a positive number $n_0$ such that*
*$N(f_n - f_m, g, t) > 1 - r$ for every $n, m \geq n_0$ and for $g \in F(X)$.*

**Definition 2.7.** *A 2-fuzzy 2-normed linear space (X, N) is said to be complete if every Cauchy sequence in X converge to some point in X.*

**Definition 2.8.** *A complete 2-fuzzy 2-normed linear space is a 2-fuzzy 2-Banach space.*

**Definition 2.9.** *Let $F(X)$ be a linear space over the real field K. A fuzzy subset M of $F(X) \times F(X) \times R$,( R the set of real numbers) is called a 2-fuzzy 2-metric space on X if and only if*
*(M1) for all $t \in R$ with $t \leq 0, M(f_1, f_2, h, t) = 0$.*
*(M2) for all $t \in R$ with $t > 0, M(f_1, f_2, h, t) = 1$ if and only if $f_1, f_2$ are linearly dependent.*
*(M3) $M(f_1, f_2, h, t) = M(f_2, f_1, h, t)$.*
*(M4) $M(f_1, f_2, h, t) * M(f_2, f_3, h, s) \leq M(f_1, f_3, h, t+s)$.*
*(M5) $M(f_1, f_2, h, .) : (0,\infty) \to [0,1]$ is continuous.*
*Then $(F(X), M, *)$ is a 2-fuzzy metric space or $(X, M, *)$ is 2-fuzzy 2-metric space for all $f_1, f_2, f_3, h \in F(X)$.*

**Definition 2.10.** *Let $(\mathfrak{F}(X), N)$ be a real 2-fuzzy 2-Banach space. Let $S(\mathfrak{F}(X))$ and $B(\mathfrak{F}(X))$ denote the unit sphere and the unit ball of $\mathfrak{F}(X)$, respectively. Let $[\mathfrak{F}(X)]^*$ denote the dual space of $\mathfrak{F}(X)$ and T be a linear bounded operator from $\mathfrak{F}(X)$ to $\mathfrak{F}(Y)$.*
*Let D(T),R(T)and N(T) denote the domain, range and null space of T, respectively. The 2-fuzzy 2-chebyshev radius and 2-fuzzy 2-chebyshev center of subset A of $\mathfrak{F}(X)$ are defined as*

$$R(A) = \inf_{g\in A} \sup_{f\in A} \{\inf\{t : N(f-g, h, t) \geq \alpha, \alpha \in (0,1)\}\} \quad (2.1)$$

$$C(A) = \{g \in A : \sup_{f\in A} \{\inf\{t : N(f-g, h, t) \geq \alpha, \alpha \in (0,1)\}\}\} = R(A) \quad (2.2)$$

*Moreover, if A is 2-fuzzy 2-convex then $C(A)$ is a 2-fuzzy 2-convex set. Then the 2-fuzzy 2-metric projection from $\mathfrak{F}(X)$ onto $\mathscr{C}$ is a mapping $\mathscr{P}_{\mathscr{C}} : \mathfrak{F}(X) \to \mathscr{C}$ is defined by*

$$\mathscr{P}_{\mathscr{C}}(f) = \{g \in \mathscr{C} : \mathrm{M}(f, g, h, t) = \inf_{g\in\mathscr{C}} \{inf\{t : N(f-g, h, t) \geq \alpha, \alpha \in (0,1)\}\}\}.$$

**Definition 2.11.** *A non empty set $\mathscr{C}$ is said to be a 2-fuzzy 2-chebyshev set if $\mathscr{P}_{\mathscr{C}}(\mathfrak{F}(X))$ is one point for all $f \in \mathfrak{F}(X)$. A non empty set $\mathscr{C}$ is said to be 2-fuzzy 2-proximinal if $\mathscr{P}_{\mathscr{C}}(f) \neq \varnothing$ for all $f \in \mathfrak{F}(X)$.*

**Definition 2.12.** *A 2-fuzzy 2-Banach space $\mathfrak{F}(X)$ is said to be k-strictly 2-fuzzy 2-convex if for any $k+1$ elements $f_1, f_2, \ldots f_{k+1} \in \mathscr{S}(\mathfrak{F}(X))$,*
*and if $N(f_1 + f_2 + \ldots + f_{k+1}, g, k+1) = 1$ then $f_1, f_2, \ldots f_{k+1}$ are linearly dependent. It is well known that $\mathfrak{F}(X)$ is a 1-strictly 2-fuzzy 2-convex space if and only if $\mathfrak{F}(X)$ is a strictly 2-fuzzy 2-convex space.*

**Definition 2.13.** *A non empty subset $\mathscr{C}$ of $\mathfrak{F}(X)$ is said to be appoximatively 2-fuzzy 2-compact if for any $\{g_n\} \subset \mathscr{C}$ and $f \in \mathfrak{F}(X)$ satisfying*
*$\inf\{t : N(f - g_n, h, t) \geq \alpha, \alpha \in (0,1)\}$ converges to*
*$\inf_{g\in\mathscr{C}}\{inf\{t : N(f-g, h, t) \geq \alpha, \alpha \in (0,1)\}\}$ then the sequence $\{g_n\}$ has a subsequence converging to an element in $\mathscr{C}$.*

**Definition 2.14.** *Set-valued mapping $G : \mathfrak{F}(X) \to \mathfrak{F}(Y)$ is said to be 2-fuzzy 2-upper semi continuous at $f_0$, if for each 2-fuzzy 2-open set W with $G(f_0) \subset W$, there exists a 2-fuzzy 2-neighborhood U of $f_0$ such that $G(f) \subset W$ for all $f$ in*

*U. G is called 2-fuzzy 2-lower semi continuous at $f_0$, if for any $g \in W(f_0)$ and any $\{f_n\}$ in $\mathfrak{F}(X)$ with $f_n$ converges to $f_0$, there exists $g_n \in W(f_n)$ such that $g_n$ converges to g. W is called 2-fuzzy 2-continuous at $f_0$, if W is 2-fuzzy 2-upper semicontinous and is 2-fuzzy 2-lower semi continuous at $f_0$.*

**Definition 2.15.** *A point $f \in \mathscr{S}(\mathfrak{F}(X))$ is said to be 2-fuzzy 2-H-point if for a sequence $\{f_n\}$ in $\mathscr{S}(\mathfrak{F}(X))$ and $f_n \xrightarrow{\omega} f$, then $f_n convergesto f$. Moreover, if the set of all 2-fuzzy 2-H-points is equal to $\mathscr{S}(\mathfrak{F}(X))$ then $\mathfrak{F}(X)$ is said satisfy the 2-fuzzy 2-H-property.*

**Definition 2.16.** *A point $f_0 \in D(T)$ is said to be the 2-fuzzy 2-best approximative solution to the fuzzy operator equation $Tf = g$, if*

$$\inf\{t : N(Tf_0 - g, h, t) \geq \alpha, \alpha \in (0,1)\} = \inf\{\inf\{t : N(Tf - g, h, t) \geq \alpha, \alpha \in (0,1)\} : f \in D(T)\}$$

$$\inf\{t : N(f_0, h, t) \geq \alpha, \alpha \in (0,1)\} = \min\{\inf\{t : N\left(f^{'}, h, t\right) \geq \alpha, \alpha \in (0,1)\} : f^{'} \in D(T)\} \tag{2.3}$$

$$\inf\left\{t : N\left(Tf^{'} - g, h, t\right) \geq \alpha, \alpha \in (0,1)\right\} = \inf_{f \in D(T)}\{\inf\{t : N(Tf - g, h, t) \geq \alpha, \alpha \in (0,1)\}\}$$

**Definition 2.17.** *Let $\mathfrak{F}(X), \mathfrak{F}(Y)$ be a 2-fuzzy 2-Banach spaces and T be a linear operator from $\mathfrak{F}(X)$ to $\mathfrak{F}(Y)$. The mapping $T^{'} : \mathfrak{F}(Y) \to \mathfrak{F}(X)$ defined by*

$$T^{'}(g) = \{f_0 \in D(T) : f_0 \ \ is \ \ a \ \ 2-fuzzy \ \ 2-best \ \ approximative \ \ solution \ \ to \ \ T(f) = g\} \tag{2.4}$$

*for any $g \in D\left(T^{'}\right)$ is said to be the 2-fuzzy 2-metric generalized inverse of T, where*

$$D\left(T^{'}\right) = \{g \in \mathfrak{F}(Y) : T(f) = g \ \ has \ \ a \ \ 2-fuzzy \ \ 2-best \ \ approximative \ \ solution \ \ in \ \ \mathfrak{F}(X)\} \tag{2.5}$$

**Definition 2.18.** *A 2-fuzzy 2-normed linear space is 2-fuzzy 2-compact if every sequence has a convergent subsequence.*

**Definition 2.19.** *A 2-fuzzy 2-metric space $\mathfrak{F}(X)$ is 2-fuzzy 2-sequencially compact if every sequence of points in $\mathfrak{F}(X)$ has a convergent subsequence.*

## 3. Continuity of 2-fuzzy 2-metric projection operator and 2-fuzzy 2-approximative compactness

**Theorem 3.1.** *Let $\tau \in \mathscr{S}\left(\mathfrak{F}(X)^*\right)$, $H = \{f \in \mathfrak{F}(X), \tau(f) = \tilde{0}\}$ and the set $A_\tau = \{f \in \mathfrak{F}(X), \tau(f) = \tilde{1}\}$ is a non - empty 2-fuzzy 2-compact set. Then*

*(i) $\mathscr{P}_H(f) = f - \tau(f)A_\tau$ for any $f \in \mathfrak{F}(X)$*

*(ii) The 2- fuzzy 2-metric projector $\mathscr{P}_H$ is 2-fuzzy 2-continuous.*

*Proof.* (i) Let $f \in \mathfrak{F}(X)$, choose $h \in H$ and $g \in \mathscr{S}(\mathfrak{F}(X))$ there exists a scalar $\alpha$ such that $f - h = \alpha g$. On applying $\tau$ on both sides,

$$\tau(f) - \tau(h) = \alpha\tau(g)$$

which implies $\alpha = \frac{\tau(f)}{\tau(g)}$ since $\tau(h) = \tilde{0}$ as $h \in H$, it is obvious that

$$f - h = \left(\frac{\tau(f)}{\tau(g)}\right)g$$

Again
$\inf\{t : N(f - h, g, t) \geq \alpha, \alpha \in (0,1)\}$
$\geq \frac{\inf\{t:N(\tau(f),g,t)\geq\alpha,\alpha\in(0,1)\}}{\inf\{t:N(\tau(g),g,t)\geq\alpha,\alpha\in(0,1)\}} \inf\{t : N(g, f, t) \geq \alpha, \alpha \in (0,1)\}$
$\geq \frac{\inf\{t:N(\tau(f),g,t)\geq\alpha,\alpha\in(0,1)\}}{\inf\{t:N(\tau(g),g,t)\geq\alpha,\alpha\in(0,1)\}}$
Thus $h \in \mathscr{P}_H(f)$ if and only if $g \in A_\tau$, (i.e) $\tau(g) = 1$. Hence $\mathscr{P}_H(f) = f - \tau(f)A_\tau$ for any $f \in \mathfrak{F}(X)$.
(ii) Suppose $\mathscr{P}_H$ is not 2-fuzzy 2-upper semi continuous at $f_0$, then there exist a sequence $\{f_n\}$ in $\mathfrak{F}(X)$ and an open set $W$ containing $\mathscr{P}_H(f_0)$ such that $\mathscr{P}_{N(T)}(f_n)$ not a subset of W, where $N(T)$ is the null space of the operator $T : \mathfrak{F}(X) \to \mathfrak{F}(Y)$ and $f_n$ *converging* $f_0$. Then there exists $h_n \in \mathscr{P}_{N(T)}(f_n)$ such that $h_n \notin W$. By (i), it follows $h_n = f_n - \tau(f_n)g_n$ where $g_n \in A_\tau$. By hypothesis $A_\tau$ is 2-fuzzy 2-compact, it is a 2-fuzzy 2-sequencially compact, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $g_{n_k}$ *converges to* $g_0$ *in* $A_\tau$.
Fix $h_0 = f_0 - \tau(f_0)g_0$ then $h_0 \in \mathscr{P}_H(f_0)$ and

$$\begin{aligned}\lim_{k\to\infty} h_{n_k} &= \lim_{n\to\infty}\left(f_{n_k} - \tau\left(f_{n_k}\right)g_{n_k}\right) \\ &= f_0 - \tau(f_0)g_0 = h_0\end{aligned} \tag{3.1}$$

leads to a contradiction since $h_n \notin W$ hence it implies that $\mathscr{P}_H$ is 2-fuzzy 2-upper semicontinuous.
Now assume $\{f_n\}$ converges to $f_0$, let $h_0 \in \mathscr{P}_H(f_0)$. Then by (i), there exists $g_0 = f - \tau(f_0)g_0$. Again from (i), $g_n = f_n - \tau(f_n)g_0 \in \mathscr{P}_{N(T)}(f_n)$ and

$$\begin{aligned}\lim_{n\to\infty} h_n &= \lim_{n\to\infty}(f_n - \tau(f_n)g_0) \\ &= f_0 - \tau(f_0)g_0 = h_0\end{aligned} \tag{3.2}$$

This leads to the desired requirement that $\mathscr{P}_H$ is 2-fuzzy 2-lower semicontinuous at $f_0$. □

**Definition 3.2.** *The space A is said to be 2-fuzzy 2-approxima tively compact with respect to B if every sequence $\{f_n\}$ in A satisfies the condition*
*$M(g, f_n, t) \to M(g, A, t)$ for some $g \in B$ has a convergent subsequence.*

**Definition 3.3.** *The space A subset of $\mathfrak{F}(X)$ is 2-fuzzy 2-relatively compact if its closure is 2-fuzzy 2-compact.*

**Theorem 3.4.** *Suppose that every 2-fuzzy 2-proximinal hyperplane of $\mathfrak{F}(X)$ is 2-fuzzy 2-approximatively compact. Then $\mathfrak{F}(X)$ has the 2-fuzzy 2-H-property.*

*Proof.* Let $f_n \xrightarrow{\omega} f$, where $\{f_n\}$ is a sequence in $\mathscr{S}(\mathfrak{F}(X))$ and $f \in \mathscr{S}(\mathfrak{F}(X))$. Then there exists $f^* \in \mathscr{S}\left(\mathfrak{F}(X)^*\right)$ such that $f^*(f) = \tilde{1}$ and the hyperplane $H_{f^*} = \{f \in \mathfrak{F}(X) : f^*(f) = \tilde{1}\}$ is 2-fuzzy 2-proximinal. Suppose that the sequence $\{f_n\}$ does not converge to $f$. Without loss of generality assume that $\inf\{t : N(f_n - f, h, t) \geq \alpha, \alpha \in (0,1)\} > \varepsilon$ for every $n \in N$. Since $H_{f^*}$ is a proximinal set, there exists $g_n \in H_{f^*}$ such that $M\left(f_n, H_{f^*}, h, t\right) = \inf\{t : N(f_n - g_n, h, t) \geq \alpha,\ \alpha \in (0,1)\}$.
Since
$\lim_{n\to\infty}\{\inf\{t : N(f_n - g_n, h, t) \geq \alpha, \alpha \in (0,1)\}\}$
$= \lim_{n\to\infty}\{M(f_n, H_{f^*}, h, t)\} = 1$.
It implies that

$$\begin{aligned} &M\left(0, H_{f^*}, h, t\right) = 1 \\ &= \lim_{n\to\infty}\{\inf\{t : N(f_n, h, t) \geq \alpha, \alpha \in (0,1)\}\} \\ &= \lim_{n\to\infty}\{\inf\{t : N(0 - g_n, h, t) \geq \alpha, \alpha \in (0,1)\}\} \end{aligned} \tag{3.3}$$

This implies that the sequence $\{g_n\}$ is 2-fuzzy 2-relatively compact. Hence the sequence $\{f_n\}$ is 2-fuzzy 2-relatively compact. Then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}\}$ is a Cauchy sequence. Since $f_{n_k} \xrightarrow{\omega} f$, then $f_{n_k}$ converges to $f$, a contradiction. Hence,$f_n$ converges to $f$. This implies that $\mathfrak{F}(X)$ has the 2-fuzzy 2-H-Property. □

**Theorem 3.5.** *Let $H_1$ be a closed subspace of $\mathfrak{F}(X_1)$ and $H_2$ be a closed subspace of $\mathfrak{F}(X_2)$ ; $\mathscr{P}_{H_1}$ is a 2-fuzzy 2-lower semi continuous on $\mathfrak{F}(X_1)$, $\mathscr{P}_{H_2}$ is a 2-fuzzy 2-lower semi continuous on $\mathfrak{F}(X_2)$. Then the metric projection operator $\mathscr{P}_{H_1\times H_2}$ is 2-fuzzy 2-lower semi continuous on $(\mathfrak{F}(X_1)\times\mathfrak{F}(X_2), N)$ where*

$$N((f_1, f_2), h, t) = \min\{N(f_1, h, t), N(f_2, h, t)\}$$

*Proof.* Let $(f_{1,n}, f_{2,n})$ be a sequence in $H_1 \times H_2$ converging to $(f_1, f_2)$ which implies $f_{1,n}$ converges to $f_1$ and $f_{2,n}$ converges to $f_2$.
Consider $\mathscr{P}_{H_1\times H_2}(f_1, f_2)$

$$\begin{aligned} &= \left\{ \begin{array}{c} (g_1, g_2) \in H_1 \times H_2 : \inf\{t : N(((f_1, f_2) - (g_1, g_2)), h, t) \geq \alpha\} \\ = M((f_1, f_2), (g_1, g_2), h, t) = \inf\{\inf\{t : N((f_1, f_2) - (h_1, h_2), g, t) \geq \alpha\}\} \end{array} \right\} \\ &= \left\{ \begin{array}{c} (g_1, g_2) \in H_1 \times H_2 : \inf\{t : N(((f_1 - g_1), (f_2 - g_2)), h, t) \geq \alpha\} \\ = M((f_1, g_1), (f_2, g_2), h, t) = \inf\{\inf\{t : N((f_1 - h_1), (f_2 - h_2), g, t) \geq \alpha\}\} \end{array} \right\} \\ &= \left\{ \begin{array}{c} g_1 \in H_1 : \inf\{t : N((f_1 - g_1), h, t) \geq \alpha\} \\ = M(f_1,\ g_1, h, t) = \inf\{\inf\{t : N((f_1 - h_1), g, t) \geq \alpha\}\} \end{array} \right\} \\ &= \left\{ \begin{array}{c} g_1 \in H_1 : \inf\{t : N((f_1 - g_1), h, t) \geq \alpha\} \\ = M(f_1,\ g_1, h, t) = \inf\{\inf\{t : N((f_1 - h_1), g, t) \geq \alpha\}\} \end{array} \right\} \\ &\qquad \times \left\{ \begin{array}{c} g_2 \in H_2 : \inf\{t : N((f_2 - g_2), h, t) \geq \alpha\} \\ = M(f_2,\ g_2, h, t) = \inf\{\inf\{t : N((f_2 - h_2), g, t) \geq \alpha\}\} \end{array} \right\} \end{aligned}$$

$= \mathscr{P}_{H_1}(f_1) \times \mathscr{P}_{H_2}(f_2)$ for any $(f_1, f_2) \in \mathfrak{F}(X_1)\times\mathfrak{F}(X_2)$
Then $f_{1,n}$ converges to $f_1$ and $f_{2,n}$ converges to $f_2$. Let $(h_1, h_2) \in \mathscr{P}_{H_1\times H_2}(f_1, f_2)$. Since $\mathscr{P}_{H_1\times H_2}(f_1, f_2) = \mathscr{P}_{H_1}(f_1) \times \mathscr{P}_{H_2}(f_2)$, by $f_{1,n}$ converges to $f_1$ and $f_{2,n}$ converges to $f_2$ there exists $h_{1,n} \in \mathscr{P}_{H_1}(f_{1,n})$ and $h_{2,n} \in \mathscr{P}_{H_2}(f_{2,n})$ such that $h_{1,n}$ converges to $h_1$ and $h_{2,n}$ converges to $h_2$. Hence $(h_{1,n}, h_{2,n})$ converges to $(h_1, h_2)$. Hence $\mathscr{P}_{H_1\times H_2}$ is a 2-fuzzy 2-lower semicontinuous.
Let $\mathfrak{F}(X)$ be k-strictly convex and $H = \{f \in \mathfrak{F}(X) : \tau(f) = \tilde{0}, \tau \in \mathscr{S}\left(\mathfrak{F}(X)^*\right)\}$. Then $A_\tau = \{f \in \mathfrak{F}(X) : \tau(f) = \tilde{1}\}$ is a non-empty compact set. Then by theorem (3.1) the metric projector operator $\mathscr{P}_H$ is 2-fuzzy 2-lower semicontinuous. Let $\mathfrak{F}(Y)$ be strictly convex and $M$ is an approximately compact closed subspace of $\mathfrak{F}(Y)$. Then the metric projector operator $\mathscr{P}_M$ is 2-fuzzy 2-continuous. Therefore, by theorem (3.5), it implies that $\mathscr{P}_{H\times M}$ is 2-fuzzy 2-lower semicontinuous on $(\mathfrak{F}(X), \mathfrak{F}(Y), N)$, with norm $N$. □

## 4. 2-Fuzzy 2-Continuous selections and 2-Fuzzy 2-Continuity of the set valued Metric Generalized Inverse

**Theorem 4.1.** *Let $\mathfrak{F}(X)$ be a 3-strictly 2-fuzzy 2-convex space. $\mathfrak{F}(Y)$ be a 2-fuzzy 2-Banach space, D(T) be a closed subspace of $\mathfrak{F}(X)$ and R(T) be an 2-fuzzy 2-approximatively compact and 2-fuzzy 2-chebyshev subspace of $\mathfrak{F}(Y)$. Then*

*(i) $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-upper semicontinuous if and only if $T'$ is 2-fuzzy 2-upper semicontinuous.*

*(ii) $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-continuous if and only if $T'$ is 2-fuzzy 2-continuous.*

*(iii) If $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-continuous,then there exists a homogeneous selection $T^*$ of $T'$ such that $T^*$ is 2-fuzzy 2-continuous on $\{h \in \mathfrak{F}(Y) :$ $\liminf_{k\to h}\left[diam\left(C\left(T'(k)\right)\right)\right]_\alpha$ $\geq \left[diam\left(C\left(T'(h)\right)\right)\right]_\alpha$ for $\alpha \in (0,1)\}$.*

*Proof.* (i) Let $g_0 \in \mathfrak{F}(Y)$. To prove that $T'$ is 2-fuzzy 2-upper semicontinuous at $g_0$ consider a sequence $\{g_n\}$ in $\mathfrak{F}(Y)$ converging to $g_0$ and open set $W$ with $T'(g_0) \subset W$ then there exists a positive number $N_0$ such that $T'(g_n) \subset W$ whenever $n > N_0$.
Take $f_0 \in T'(\mathscr{P}_{R(T)}(g_0))$, by the definition of 2-fuzzy 2-metric generalized inverse, it is obvious that $T'(g_0) = f_0 - \mathscr{P}_{N(T)}(f_0)$ . Since T is a bounded linear operator, it is obvious that N(T) is a closed subspace of D(T).
Let $\overline{T} : \frac{D(T)}{N(T)} \to R(T)$ defined as

$$\bar{T}[f] = Tf \tag{4.1}$$

It is clear that $R(\overline{T}) = R(T)$ and $R(\bar{T}) = R(T)$. Suppose that $R(\bar{T}) \neq R(T)$, then there exists $g' \in R(\bar{T})$ such that $g' \notin R(T)$. It is easy to see that $\{g \in R(T) : inf\{t : N(g' - g, h, t) \geq \alpha, \alpha \in (0,1)\}\} = M(g', R(T), h, t) = \varnothing$.
This implies that $R(T)$ is not a 2-fuzzy 2-chebyshev subspace of $\mathfrak{F}(Y)$, a contradiction to the hypothesis. since $R(\bar{T}) = R(T)$, it follows that $R(T)$ is a 2-fuzzy 2-Banach space and $\bar{T}$ is a bounded linear operator with $N(\bar{T}) = \{0\}$. Further implies that the bounded linear operator $\bar{T}$ is both injective and

surjective. Hence $\bar{T}^{-1}$ is a bounded linear operator. Take $f_n \in \bar{T}^{-1}(\mathscr{P}_{R(T)}(g_n))$. Since $\mathfrak{F}(Y)$ is 2-fuzzy 2-approximatively compact and $R(T)$ is a 2-fuzzy 2-chebshev subspace of $\mathfrak{F}(Y)$, it implies that the metric projection operators $\mathscr{P}_{R(T)}$ is 2-fuzzy 2-continuous and so $\mathscr{P}_{R(T)}(g_n)$ converges to $\mathscr{P}_{R(T)}(g_0)$.Since $\bar{T}^{'}$ is a bounded linear operator, it follows that

$$\lim_{n\to\infty}\{inf\{t : N([f_n - f_0], h, t) \geq \alpha, \alpha \in (0,1)\}\} = \lim_{n\to\infty}\{inf\{t : N([f_n] - [f_0], h, t) \geq \alpha, \alpha \in (0,1)\}\} = 0 \tag{4.2}$$

Without loss of generality assume that $f_n$ converges to $f_0$. Since $T^{'}(g_0) = f_0 - \mathscr{P}_{N(T)}(f_0)$, it implies that $f_0 - \mathscr{P}_{N(T)}(f_0) \subset W$. Hence for any $h \in \mathscr{P}_{N(T)}(f_0)$, it follows that $f_0 - h \in W$. So there exist $\delta_h \in (0,1)$ and $r_h \in (0,1)$ such that $B(f_0, \delta_h, t) - B(h, r_h, t) \subset W$ with $t \in R$. Since $\mathfrak{F}(X)$ is a 3-strictly 2-fuzzy 2-convex space, it follows that $\mathscr{P}_{N(T)}(f_0)$ is 2-fuzzy 2-compact.
Since

$$\mathscr{P}_{N(T)}(f_0) \subset \bigcup_{\substack{h \in \mathscr{P}_{N(T)}(f_0) \\ t \in R}} B(h, r_h, t) \tag{4.3}$$

there exist $h_1 \in \mathscr{P}_{N(T)}(f_0)$, $h_2 \in \mathscr{P}_{N(T)}(f_0)$,$h_K \in \mathscr{P}_{N(T)}(f_0)$ such that

$$\mathscr{P}_{N(T)}(f_0) \subset \bigcup_{i=1}^{k} B\left(h_i, r_{h_i}, t\right) \tag{4.4}$$

Let $2\delta = \min\{\delta_{h_1}, \delta_{h_2}, \ldots \delta_{h_k}\}$ . Since $B(f_0, \delta_h, t) - B(h, r_h, t) \subset W$ for any $h \in \mathscr{P}_{N(T)}(f_0)$,

$$f_0 - \mathscr{P}_{N(T)}(f_0) \subset B(f_0, \delta, t) - \bigcup_{\substack{i=1 \\ t \in R}}^{k} B\left(h_i, r_{h_i}, t\right) \subset W \tag{4.5}$$

This implies that

$$\mathscr{P}_{N(T)}(f_0) = f_0 - \left(f_0 - P_{N(T)}(f_0)\right) \subset f_0 - B(f_0, \delta, t) - \bigcup_{\substack{i=1 \\ t \in R}}^{K} B\left(h_i, r_{h_i}, t\right) \tag{4.6}$$

Since $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-upper semicontinuous, there exists $n_0 \in N$ such that $\inf\{t : N(f_n - f_0, h, t) \geq \alpha, \alpha \in (0,1)\} < \delta$ for $\delta \in (0,1)$ and

$$\mathscr{P}_{N(T)} f_n \subset f_0 - \left( B(f_0, \delta, t) - \bigcup_{\substack{i=1 \\ t \in R}}^{k} B\left(h_i, r_{h_i}, t\right) \right) \quad when \quad n > n_0 \tag{4.7}$$

Since $2\delta = min\{\delta_{h_1}, \delta_{h_2}, \ldots \delta_{h_k}\}$ and $B(f_0, \delta_h, t) - B(h, r_h, t) \subset W$ for any $h \in \mathscr{P}_{N(T)}(f_0)$, it is obvious that $B(f_0, \delta) - \bigcup^k_{i=1} B\left(h_i, r_{h_i}\right) \subset W$. Hence

$$T^{'}(g_n) = f_n - P_{N(T)}(f_n) \subset f_n - \left( f_0 - B(f_0, \delta, t) - \bigcup_{\substack{i=1 \\ t \in R}}^{k} B\left(h_i, r_{h_i}, t\right) \right) \tag{4.8}$$

$$= f_n - f_0 + B(f_0, \delta, t) - \bigcup_{\substack{i=1 \\ t \in R}}^{k} B\left(h_i, r_{h_i}, t\right)$$

$$\subset B(f_0, 2\delta, t) - \bigcup_{\substack{i=1 \\ t \in R}}^{k} B\left(h_i, r_{h_i}, t\right) \subset W$$

This implies that $T^{'}$ is 2-fuzzy 2-upper semicontinuous at $g_0$. Hence $T^{'}$ is 2-fuzzy 2-upper semicontinuous. Conversely assume as a contrary that $\mathscr{P}_{N(T)}$ is not 2-fuzzy 2-continuous at $g_0$. Then there exists $\{f_n\}$ in $\mathfrak{F}(X)$ with $f_0 \in \mathfrak{F}(X)$, and an open set W such that $\{f_n\}$ converges to $f_0$ and $\mathscr{P}_{N(T)}(f_n) \subset W$ with $\mathscr{P}_{N(T)} \not\subset (f_n)W$. Hence there exists $\pi_{N(T)}(f_n) \in \mathscr{P}_{N(T)}(f_n)$ such that $\pi_{N(T)}(f_n) \notin W$. It asserts to show that there exists $\delta \in (0,1)$ such that

$$\bigcup_{\substack{h \in \mathscr{P}_{N(T)}(f_0) \\ t \in R}} B(h, 2\delta, t) \subset W \tag{4.9}$$

Or else there exists $h_n \in \mathscr{P}_{N(T)}(f_0)$ such that $B\left(h_n, \frac{1}{n}, t\right) \not\subset W$. Since $\mathscr{P}_{N(T)}(f_0)$ is 2-fuzzy 2-compact, assume that $h_n$ converges to $h_0$ in $\mathscr{P}_{N(T)}(f_0)$ so choose $\eta \in (0,1)$ such that $B(h_0, 4\eta, t) \subset W$. Hence there exists $n_0 \in N$ such that $\frac{1}{n_0} > 2\eta$ and $N\left(h_{n_0} - h_0, g, t\right) > 2\eta$. Hence for any $h \in B\left(h_{n0}, \frac{1}{n_0}, t\right)$, then $N\left(h_{n_0} - h_0, g, t\right) \geq \frac{1}{n_0}$

$$\begin{aligned} &\inf\left\{t_1 + t_2 : N\left(h - h_{n_0}, g, t_1 + t_2\right) \geq \alpha, \alpha \in (0,1)\right\} \\ &\quad \geq \inf\left\{t_1 : N\left(h - h_{n_0}, g, t_1\right) \geq \alpha, \alpha \in (0,1)\right\} + \\ &\quad\quad \inf\left\{t_2 : N\left(h_{n_0} - h_0, g, t_2\right) \geq \alpha, \alpha \in (0,1)\right\} \end{aligned}$$

$$> 2\eta + 2\eta = 4\eta \tag{4.10}$$

This implies that $h \in W$. Hence $B\left(h_{n_0}, \frac{1}{n_0}, t\right) \subset W$, a contradiction, Let $g_n = T f_n$ and $g_0 = T f_0$. Then $T^{'}(g_n) = f_n - \mathscr{P}_{N(T)}(f_n)$,

$$T^{'}(g_0) = f_0 - P_{N(T)}(f_0) \tag{4.11}$$

$\lim g_n = g_0$
Since $\mathscr{P}_{N(T)}(f_0) \subset W$, it follows that $T^{'}(g_0) = f_0 - \mathscr{P}_{N(T)}(f_0) \subset f_0 - W$. It asserts to show that

$$f_n - \pi_{N(T)}(f_n) \notin f_0 - \bigcup_{\substack{h \in \mathscr{P}_{N(T)}(f_0) \\ t \in R}} B(h,\delta,t) \quad (4.12)$$

whenever $\inf\{t : N(f_n - f_0, h, t) \geq \alpha, \alpha \in (0,1)\} < \delta$.
Suppose that $f_n - \pi_{N(T)}(f_n) \notin f_0 - \bigcup_{\substack{h \in \mathscr{P}_{N(T)}(f_0) \\ t \in R}} B(h,\delta,t)$
Whenever $\inf\{t : N(f_n - f_0, h, t) \geq \alpha, \alpha \in (0,1)\} < \delta$. Then

$$\pi_{N(T)}(f_n) = f_n - \left(f_n - \pi_{N(T)}(f_n)\right) \in f_n - \left(f_0 - \bigcup_{\substack{h \in \mathscr{P}_{N(T)}(f_0) \\ t \in R}} B(h,\delta,t)\right) \quad (4.13)$$

$= \cup_{\substack{h \in \mathscr{P}_{N(T)}(f_0) \\ t \in R}} B(h,\delta,t) + (f_n - f_0) \subset \cup_{h \in \mathscr{P}_{N(T)}(f_0)} B(h,\delta,t) \subset W$
a contradiction. Since,
$f_n - \pi_{N(T)}(f_n) \notin f_0 - \cup_{h \in \mathscr{P}_{N(T)}(f_0)} B(h,\delta,t)$
whenever $\inf\{t : N(f_n - f_0, h, t) \geq \alpha, \alpha \in (0,1)\} < \delta$, it follows that $T^{'}$ is not 2-fuzzy 2-upper semicontinuous at $g_0$, a contradiction.
(ii) Let $g_0 \in \mathfrak{F}(Y)$ and $g_n$ converges to $g_0$. Then, by the previous argument there exists $f_0 \in \mathfrak{F}(X)$ and a sequence $\{f_n\}$ in $\mathfrak{F}(X)$ such that $\mathscr{P}_{R(T)}(g_0) = Tf_0$, $\mathscr{P}_{R(T)}(g_n) = Tf_n$, and $f_n$ converges to $f_0$. Then $T^{'}(g_0) = f_0 - \mathscr{P}_{N(T)}(f_0)$ and $T^{'}(g_n) = f_n - \mathscr{P}_{N(T)}(f_n)$. Since $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-continuous, it follows that for any $h \in \mathscr{P}_{N(T)}(f_0)$, there exists $h \in \mathscr{P}_{N(T)}(f_n)$ such that $h_n$ converges to $h_0$. Hence, for any $f_0 - h \in f_0 - \mathscr{P}_{N(T)}(f_0)$, there exists $f_n - h_n \in f_n - \mathscr{P}_{N(T)}(f_n)$ such that $f_n - h_n$ converges to $f_0 - h$. Hence $T^{'}$ is 2-fuzzy 2-lower semicontinuous at $g_0$. Hence by (i), $T^{'}$ is 2-fuzzy 2-continuous at $g_0$.
Conversely let $g_0 \in \mathfrak{F}(Y)$ and $g_n$ converges to $g_0$. Then by the previous argument, there exist $f_0 \in \mathfrak{F}(X)$ and $\{f_n\} \subset \mathfrak{F}(X)$ such that $\mathscr{P}_{R(T)}(g_0) = Tf_0$, $\mathscr{P}_{R(T)}(g_n) = Tf_n$, and $f_n$ converges to $f_0$. Then $T^{'}(g_0) = f_0 - \mathscr{P}_{N(T)}(f_0)$ and $T^{'}(g_n) = f_n - \mathscr{P}_{N(T)}(f_n)$ since $T^{'}$ is 2-fuzzy 2-continuous, it follow that for any $f_0 - h \in f_0 - \mathscr{P}_{N(T)}(f_0)$, so there exists $f_n - h_n \in f_n - \mathscr{P}_{N(T)}(f_n)$ such that $f_n - h_n$ converges to $f_0 - h$. Hence for any $h \in \mathscr{P}_{N(T)}(f_0)$, there exists $h_n \in \mathscr{P}_{N(T)}(f_n)$ such that $h_n$ converges to $h_0$. This implies that $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-continuous at $g_0$ . Next to prove the condition (iii) is true.
(iii)(a) Define a mapping $G : \mathfrak{F}(Y)$ converges to $\mathfrak{F}(X)$ such that $G(g) = \mathscr{C}\left(T^{'}(g)\right)$, if $\{g_n\}$ converges to $g$ then to prove

$$\lim_{n\to\infty} \sup_{h \in T^{'}(g)} \inf_{h_n \in T^{'}(g_n)} \{\inf\{t : N(h_n - h, g, t) \geq \alpha, \alpha \in (0,1)\}\} = 0 \quad (4.14)$$

where $g_n \in \mathfrak{F}(Y)$ and $g \in \mathfrak{F}(Y)$. Without loss of generality assume that

$$\sup_{h \in T^{'}(g)} \inf_{h_n \in T^{'}(g_n)} \{\inf\{t : N(h_n - h, g, t) \geq \alpha, \alpha \in (0,1)\}\} \geq 2\eta$$

for all $n \in N$. Then there exists $h(n) \in T^{'}(g)$ such that $\inf_{h_n \in T^{'}(g_n)} \{\inf\{t : N(h_n - h(n), g, t) \geq \alpha, \alpha \in (0,1)\}\} \geq \eta$. Since $\mathfrak{F}(X)$ is a 3-strictly 2-fuzzy 2-convex space, it follows that $\mathscr{P}_{N(T)}(f)$ is 2-fuzzy 2-compact. From the previous argument, there exists $f \in \mathfrak{F}(X)$ such that $T^{'}(g) = f - \mathscr{P}_{N(T)}(f)$. This implies that $T^{'}(g)$ is 2-fuzzy 2-compact. Hence without loss of generality assume that $h(n)$ converges to $h_0$. This implies that $h_0 \in T^{'}(g)$ and without loss of generality that

$$\inf_{h_n \in T^{'}(g_n)} \{\inf\{t : N(h_n - h_0, \text{g}, \text{t}) \geq \alpha, \alpha \in (0,1)\}\} \geq \frac{1}{2}\eta \quad (4.15)$$

for all $n \in N$. Since $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-continuous, by $h_0 \in T(g)$, there exists $k_n \in T^{'}(g_n)$ such that $k_n$ converges to $h_0$, which contradicts (23) Next to prove that G is 2-fuzzy 2-upper semicontinuous. Suppose that G is not 2-fuzzy 2-upper semicontinuous. Then there exist $\{g_n\}$ in $\mathfrak{F}(X)$, $g_0 \in \mathfrak{F}(Y)$ and a norm open set W, such that $\mathscr{C}\left(T^{'}(g_0)\right) \subset W, \mathscr{C}\left(T^{'}(g_n)\right) \not\subset W$, $g_n$ converges to $g_0$. Hence there exists $f_n \in \mathscr{C}\left(T^{'}(g_n)\right)$ such that $f_n \notin W$. Since $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-continuous, it follows that $T^{'}$ is 2-fuzzy 2-upper semicontinuous.
Hence, for any $\varepsilon > 0$, there exists $n_0 \in N$ such that

$$f_n \in \bigcup_{f \in T^{'}(g_0)} B(f,\varepsilon) \quad (4.16)$$

whenever $n > n_0$. This implies that $M\left(\{f_n\}, T^{'}(g_0,h,t)\right) = 0$. Hence there exists $f(n) \in \mathscr{C}\left(T^{'}(g_0)\right)$ such that $M(\{f_n\}, f(n), h, t) < \frac{1}{n}$. Since $T^{'}(g_0)$ is 2-fuzzy 2-compact, assume without loss of generality that $\{f_n\}$ converges to $f_0$. To prove that $f_0 \in \mathscr{C}\left(T^{'}(g_0)\right)$. Suppose that $f_0 \notin \mathscr{C}\left(T^{'}(g_0)\right)$. Let $r_0 = r\left(T^{'}(g_0)\right)$, then there exists $k_0 \in T^{'}(g_0)$ and $\delta \in (0,1)$ such that

$$\inf\{t : N(k_0 - f_0, \text{g}, \text{t}) \geq \alpha, \alpha \in (0,1)\} \geq r_0 + \delta$$

It asserts to show that

$$\lim_{n\to\infty} \sup_{h_n \in T^{'}(g_n)} \inf_{h \in T^{'}(g)} \{\inf\{t : N(h - h_n, \text{g}, \text{t}) \geq \alpha, \alpha \in (0,1)\}\} = 0 \quad (4.17)$$

Otherwise, assume that there exist $k_n \in T^{'}(g_n)$ and $\eta \in (0,1)$ such that
$\inf_{h \epsilon T^{'}(g)} \{inf\{t : N(h - j_n, g, t) \geq \alpha, \alpha \in (0,1)\}\} \geq \eta$

Since$P_{N(T)}$ is 2-fuzzy 2-continuous, it implies that T is 2-fuzzy 2-continuous.
Assume without loss if generality that $k_n$ converges to $k_0$ in $T^{'}(g)$, a contradiction. Therefore, by (4.14) and (4.17), assume that

$$\sup_{h\in T^{'}(g)}\inf_{h_n\in T^{'}(g_n)}\{\inf\{t:N(h_n-h,g,t)\geq\alpha,\alpha\in(0,1)\}\}>\frac{1}{64}\delta \tag{4.18}$$

$$\sup_{h\in T^{'}(gn)}\inf_{h\in T^{'}(g)}\{\inf\{t:N(h-h_n,g,t)\geq\alpha,\alpha\in(0,1)\}\}>\frac{1}{64}\delta$$

for every $h_n\in N$. Therefore, by (4.18) and $\{f_n\}$ converges to $f_0$ and assume that there exist $k_n\in T^{'}(g_n)$ such that

$$\inf\{t:N(k_n-k_0,g,t)\geq\alpha,\alpha\in(0,1)\}>\frac{1}{64}\delta \tag{4.19}$$

$$\inf\{t:N(k_n-k_0,g,t)\geq\alpha,\alpha\in(0,1)\}>\frac{1}{64}\delta$$

for every $n\in N$ .

$$inf\{t_1+t_2+t_3:N(f_n-k_n,g,t_1+t_2+t_3)\geq\alpha\}$$
$$\geq\inf\{t_1:N(f_0-k_0,g,t_1)\geq\alpha\}$$
$$-\inf\{t_2:N(f_n-f_0,g,t_2)\geq\alpha\}$$
$$-\inf\{t_3:N(k_n-k_0,g,t_3)\geq\alpha\}$$

$$\geq r_0+\delta-\frac{1}{16}\delta-\frac{1}{16}\delta\geq r\left(T^{'}(g_0)\right)+\frac{3}{4}\delta \tag{4.20}$$

for every $n\in N$. since $f_n\in\mathscr{C}(T^{'}(g_n))$, it follows that

$$r\left(T^{'}(g_n)\right)>r\left(T^{'}(g_0)\right)+\frac{3}{4}\delta \tag{4.21}$$

for every $n\in N$. Let $h\in\mathscr{C}(T^{'}(g_0))$. Therefore, by (4.18) there exists $h_n\in\mathscr{C}(T^{'}(g_n))$ such that
$\inf\{t:N(h-h_n,g,t)\geq\alpha\}>\frac{\delta}{64}$. Since the set $T^{'}(g_n)$ is 2-fuzzy 2-compact, there exist $\omega_{n\in}T^{'}(g_n)$ such that
$\inf\{t:N(\omega_n-h_n,g,t))\geq\alpha\}>r\left(T^{'}(g_n)\right)$
From (4.18), there exists $\omega(n)\in\mathscr{C}(T^{'}(g_0))$ such that
$\inf\{t:N(\omega(n)-\omega_n,g,t))\geq\alpha\}>\frac{\delta}{64}$
Since the set $T^{'}(g_0)$ is 2-fuzzy 2-compact, assume without loss of generality that $\omega(n)$ converges to $\omega$. Without loss of generality assume that
$\inf\{t:N(\omega-\omega_n,g,t)\geq\alpha\}>\frac{\delta}{60}$.
Therefore, by $h\in\mathscr{C}(T^{'}(g_0))$, it implies that
$r(T'(g_n))\geq\inf\{t_1+t_2:N(\omega_n-h_n,g,t_1+t_2)\geq\alpha\}$

$$>\inf\{t_1:N(h-h_n,g,t_1)\geq\alpha\}+\inf\{t_2:N(\omega-h,g,t_2)\geq\alpha\} \tag{4.22}$$
$$>\frac{\delta}{64}+\frac{\delta}{60}+r(T^{'}(g_0))$$
$$>r\left(T^{'}(g_0)\right)+\frac{1}{16}\delta,$$

Which contradicts (4.21). This implies that G is 2-fuzzy 2-upper semi continuous.
(iii)b To prove that if $\mathfrak{F}(X)$ is a 3-strictly convex space and $h\in\mathfrak{F}(X)$, then there exists $f_n\in\mathfrak{F}(X)$ and a 2-dimensional space $\mathfrak{F}(X)_h$ such that $\mathscr{P}_{N(T)}(h)\subset f_h+\mathfrak{F}(X)$
Assume that $h=0$, let $f_1,f_2,f_3,f_4\in\mathscr{P}_{N(T)}(0)$ such that $f_1,f_2,f_3$ are linearly independent. Then $(f_1+f_2+f_3+f_4)/4\in N(T)$. Therefore, by the Hahn-Banach theorem, there exists $f^*\in\mathscr{S}\left(\mathfrak{F}(X)^*\right)$ such that $f^*(f_1+f_2+f_3+f_4)/4$. Then $inf\{s+t+u+v:N(f_1+f_2+f_3+f_4,g,s+t+u+v)\geq\alpha\}=4$,

$$\begin{aligned}inf\{s:N(f_1,h,s)\geq\alpha\}&=inf\{t:N(f_2,h,t)\geq\alpha\}\\&inf\{u:N(f_3,h,u)\geq\alpha\}\\&inf\{v:N(f_4,h,v)\geq\alpha\}=\tilde{1}\end{aligned} \tag{4.23}$$

Without loss of generality assume that $f_4=\alpha_1f_1+\alpha_2f_2+\alpha_3f_3$.
Then $f^*(f_4)=f^*(\alpha_1f_1+\alpha_2f_2+\alpha_3f_3)=\tilde{1}$. Hence $\alpha_1+\alpha_2+\alpha_3=1$. Since $f_1,f_2,f_3$ are linearly independent, that for any $f\in\mathscr{P}_{N(T)}(0)$, if
$f_3=\alpha_1f_1+\alpha_2f_2+\alpha_3f_3$, then $\alpha_3\neq0$. Hence

$$\begin{aligned}f=\left(-\frac{\alpha_1}{\alpha_3}\right)f_1+\left(-\frac{\alpha_1}{\alpha_3}\right)f_2+\frac{1}{\alpha_3}f_3\\\left(-\frac{\alpha_1}{\alpha_3}\right)+\left(-\frac{\alpha_1}{\alpha_3}\right)+\frac{1}{\alpha_3}=1\end{aligned} \tag{4.24}$$

This implies that, for any $f\in\mathscr{P}_{N(T)}(0), f=\lambda_1f_1+\lambda_2f_2+\lambda_3f_3$, where $\lambda_1+\lambda_2+\lambda_3=\tilde{1}$. Then

$$f=\lambda_1(f_1-f_3)+\lambda_2(f_2-f_3)+f_3 \tag{4.25}$$

This implies that $\mathscr{P}_{N(T)}(0)\subset span\{f_1-f_3,f_2-f_3\}+f_3$.
Hence, if $\mathfrak{F}(X)$ is a 3-strictly 2-fuzzy 2-convex space and $h\in\mathfrak{F}(X)$, then there exist $f_2\in\mathfrak{F}(X)$ and a two-dimensional space $\mathfrak{F}(X)_h$ such that $\mathscr{P}_{N(T)}(h)\subset f_n+\mathfrak{F}(X)_h$.
Moreover for any $g\in\mathfrak{F}(Y)$ , there exists $f\in\mathfrak{F}(X)$ such that $T^{'}(g)=f-\mathscr{P}_{N(T)}(f)$. Hence, for any $g\in\mathfrak{F}(Y)$, there exists $f_h\in\mathfrak{F}(X)$ and a two-dimensional space $\mathfrak{F}(X)_h$ such that $T^{'}(g)\subset f_h+\mathfrak{F}(X)_h$.
(iii)c To prove that, for any $g\in\mathfrak{F}(Y)$, the set $\mathscr{C}(T^{'}(g))$ is a line segment. In fact, suppose that $\{h_1,h_2,h_3\}\subset T^{'}(g)-f_h$ and $h_1\notin[h_2,h_3]$. Then there exists $\eta>0$ such that

$$\begin{aligned}B\left(\frac{1}{3}(h_1+h_2+h_3),\eta\right)\cap\mathfrak{F}(X)_h&\subset\mathscr{C}_0\{h_1,h_2,h_3\}\\&\subset\mathscr{C}\left(T^{'}(g)\right)-f_h\end{aligned} \tag{4.26}$$

Since $(h_1+h_2+h_3)/3\in\mathscr{C}(T'(g))-f_h$, there exists $h\in\mathscr{C}(T'(g))-f_h$ such that

$$\inf\{t:N(\frac{1}{3}(h_1+h_2+h_3)+f_h-(h+f_h),g,t)\geq\alpha\}=r(T^{'}(g))$$

$$(4.27)$$

Moreover, by (4.26) there exists $t \in (0,1)$ such that
$\alpha(\frac{1}{3}(h_1+h_2+h_3))+(1-\alpha)h+f_h \in \mathscr{C}(T^{'}(g))$,

$$\inf\{t : N\left(\left(\alpha\left(\frac{1}{3}(h_1+h_2+h_3)\right)+(1-\alpha)h+f_h\right)-(h+f_h),g,t\right)\geq\alpha\} \tag{4.28}$$

$=\{\inf\{t : N((\frac{1}{3}(h_1+h_2+h_3))+f_h-(h+f_h),g,t/\alpha)\geq\alpha\}\}$
$>r\left(T^{'}(g)\right)$,

a contradiction. This implies that the set $T^{'}(g)-f_h$ is a line segment. Hence the set $\mathscr{C}(T^{'}(g))$ is a line segment.
(iii)d From the proof of (iii)c, it is obvious that the set $\mathscr{C}(T'(h))$ is a line segment for all $h \in \mathfrak{F}(X)$. Let

$$\mathscr{C}(T^{'}(h))[f1,h),f(2,h)]$$

Define

$$T^{'}(h)=\frac{1}{2}[f(1,h),f(2,h)] \tag{4.29}$$

for any $h \in \mathfrak{F}(Y)$. To prove that $T^{'}$ is 2-fuzzy 2-continuous at g, where

$$\begin{aligned} g\varepsilon\{h\in\mathfrak{F}(Y)\} : lim_{j\to h}\inf\left[diam\left(\mathscr{C}\left(T^{'}(j)\right)\right)\right]_{\alpha} \\ \geq\left[diam\left(\mathscr{C}\left(T^{'}(g)\right)\right)\right]_{\alpha}\end{aligned} \tag{4.30}$$

Let $g_n$ converges to $g$. Then

$$\lim_{n\to\infty}\inf\left[diam\left(\mathscr{C}\left(T^{'}(g_n)\right)\right)\right]_{\alpha}\geq\left[diam\left(\mathscr{C}\left(T^{'}(g)\right)\right)\right]_{\alpha} \tag{4.31}$$

Since the $\mathscr{C}(T'(g))$ is a line segment for any $g \in \mathfrak{F}(Y)$, there exist two sequences $\{f(1,g_n)\}$ and $\{f(2,g_n)\}$. such that

$$\mathscr{C}(T^{'}(g_n))=[f(1,g_n),f(2,g_n)] \tag{4.32}$$

$\mathscr{C}(T'(g))[f(1,g),f(2,g)]$
Since $g\in\{h\in\mathfrak{F}(Y)\} : \lim_{k\to h}\left[diam\left(\mathscr{C}\left(T^{'}(j)\right),g,t\right)\right]_{\alpha}\geq\left[diam\left(\mathscr{C}\left(T^{'}(h)\right),g,t\right)\right]_{\alpha}$
It implies that,

$$\begin{aligned}\lim_{n\to\infty}\inf\{t : N(f(1,g_n)-f(2,g_n),h,t)\geq\alpha\} \\ \geq inf\{t : N(f(1,g)-f(2,g),h,t)\geq\alpha\}\end{aligned} \tag{4.33}$$

It asserts to show that

$$\begin{aligned}\lim\sup_{n\to\infty}\{t : N(f(1,g_n)-f(2,g_n),h,t)\geq\alpha\} \\ \geq inf\{t : N(f(1,g)-f(2,g),h,t)\geq\alpha\}\end{aligned}$$

Or else there exists a subsequence $\{n_k\}$ of $\{n\}$ such that

$$\begin{aligned}\lim_{n\to\infty}\{\inf\{t : N\left(f\left(1,g_{n_k}\right)-f\left(2,g_{n_k}\right),h,t\right)\geq\alpha\}\} \\ < inf\{t : N(f(1,g)-f(2,g),h,t)\geq\alpha\}\end{aligned} \tag{4.34}$$

Since G is 2-fuzzy 2-upper semicontinuous, by the proof of (iii)a, without loss of generality assume that

$$\lim_{k\to\infty}f\left(1,g_{n_k}\right)=f_1\in[f(1,g),f(2,g)] \tag{4.35}$$

$$\lim_{k\to\infty}f\left(2,g_{n_k}\right)=f_2\in[f(1,g),f(2,g)]$$

This implies that

$$\lim_{k\to\infty}\{\inf\{t : N\left(f\left(1,g_{n_k}\right)-f\left(2,g_{n_k}\right),h,t\right)\geq\alpha\}\}$$

$$=\inf\{t : N(f_1-f_2,h,t)\geq\alpha\} \tag{4.36}$$

$$\leq inf\{t : N(f(1,g)-f(2,g),h,t)\geq\alpha\}$$

which contradicts

$$\lim_{k\to\infty}\{\inf\{t : N\left(f\left(1,g_{n_k}\right)-f\left(2,g_{n_k}\right),h,t\right)\geq\alpha\}\}$$

$$> inf\{t : N(f(1,g)-f(2,g),h,t)\geq\alpha\}$$

Therefore, by

$$lim\ sup_{n\to\infty}\{t : N(f(1,g_n)-f(2,g_n),h,t)\geq\alpha\}$$

$$\geq inf\{t : N(f(1,g)-f(2,g),h,t)\geq\alpha\}$$

and from (4.33), it follows that

$$\lim_{k\to\infty}\{\inf\{t : N(f(1,g_n)-f(2,g_n),h,t)\geq\alpha\}\}$$

$$= inf\{t : N(f(1,g)-f(2,g),h,t)\geq\alpha\} \tag{4.37}$$

and $[f_1,f_2]=[f(1,g),f(2,g)]$. Suppose that $T^*$ is not 2-fuzzy 2-continuous at g. Then assume that there exists $\delta\in(0,1)$ such that

$$\inf\{t : N(T^*(g_n)-T^*(g),h,t)\geq\alpha\}<\delta$$

for all $n\in N$. Moreover, since

$$f(1,g_{nk})\to f_1\in[(1,g),(2,g)],$$
$$f(2,g_{nk})\to f_2\in[(1,g),(2,g)],$$

and $[f_1,f_2]=[f(1,g),f(2,g)], f_1=f(1,g)$ and $f_2=f(2,g)$. This implies that

$$\begin{aligned}\lim_{k\to\infty}T^*(g_{nk}) &= \lim_{k\to\infty}\frac{1}{2}\left(f\left(1,g_{n_k}\right)-f\left(2,g_{n_k}\right)\right) \\ &= \frac{1}{2}(f_1+f_2) \\ &= \frac{1}{2}[f(1,g)+f(2,g)] \\ &= T^*(g)\end{aligned} \tag{4.38}$$

which contradicts $\inf\left\{t : N\left(T^{'}(g_n)-T^{'}(g),h,t\right)\geq\alpha\right\}\geq\delta$
For all $n\in N$. Hence it follows that $T^{'}$ is 2-fuzzy 2-continuous

on $\{h \in \mathfrak{F}(Y) : \lim_{k \to h} \left[diam\left(\mathscr{C}\left(T^{'}(j)\right)\right)\right]_{\alpha} \geq \left[diam\left(\mathscr{C}\left(T^{'}(h)\right)\right)\right]_{\alpha}\}$

(iii)e To prove that $T^{*}$ is a homogeneous selection of $T^{'}$. Pick $g \in \mathfrak{F}(Y)$.
Then, by the previous argument, there exists $f \in \mathfrak{F}(X)$ such that
$T(f) = \mathscr{P}_{R(T)}(g)$ and $T^{'}(g) = f - \mathscr{P}_{N(T)}(f)$.
Since

$$\lambda \mathscr{P}_{R(T)}(g) = \lambda\{h \in \mathfrak{F}(Y) : \inf_{h \in R(T)}\{\inf\{t : N(g-h,f,t) \geq \alpha\}\}\} \tag{4.39}$$

$$= \{h \in \mathfrak{F}(Y)\} : \inf_{h \in R(T)} \inf\{t : N(\lambda g - h, f, t) \geq \alpha\}$$

$$= \mathscr{P}_{R(T)}(\lambda g)$$

and $T(\lambda f) = \lambda T f = \lambda \mathscr{P}_{R(T)}(\lambda g)$. Therefore, by the definition of 2-fuzzy 2-metric generalized inverse,
$T(\lambda g) = \lambda f - \mathscr{P}_{R(T)}(\lambda f)$.
Let $\mathscr{C}\left(f - \mathscr{P}_{N(T)}(f)\right) = [f_1, f_2]$.
Then $\mathscr{C}\left(\mathscr{P}_{N(T)}(f)\right) = [f - f_1, f - f_2]$. Let

$$\mathfrak{F}(X)_0 = \{\beta f + h : h \in N(T), \beta \in R\} \tag{4.40}$$

Then $\mathfrak{F}(X)_0$ is a closed subspace of $\mathfrak{F}(X)$. Since $\mathfrak{F}(X)$ is a 3-strictly 2-fuzzy 2-convex space it follows that $\mathfrak{F}(X)_0$ is a 3-strictly 2-fuzzy 2-convex space. Moreover, by the Hahn-Banach theorem, there exists $\tau_f \in \mathscr{S}\left(\mathfrak{F}(X)_0^{*}\right)$ such that

$$N(T) = \{h \in \mathfrak{F}(X)_0 : \tau_f(h) = \tilde{1}\} \tag{4.41}$$

Since $\mathfrak{F}(X)_0$ is a 3-strictly 2-fuzzy 2-convex space, it implies that $A_{\tau_f}$ is compact. Therefore, by thorem(3.1), it implies that $\mathscr{P}_{N(T)}(f) = f - A_{\tau_f}$, where

$$A_{\tau_f} = \{h \in S(\mathfrak{F}(X)_0) : \tau_f(h) = \tilde{1}\}$$

Since $\mathscr{C}\left(\mathscr{P}_{N(T)}(f)\right) = [f - f_1, f - f_2]$ and
$\mathscr{P}_{N(T)}(f) = f - A_{\tau_f}$, we have
$\mathscr{C}\left(A_{\tau_f}\right) = [f_1, f_2]$. Then $\mathscr{C}\left(\lambda A_{\tau_f}\right) = [\lambda f_1, \lambda f_2]$
Therefore, by $\mathscr{P}_{N(T)}(\lambda f) = \lambda\left(f - A_{\tau_f}\right)$,
It implies that

$$\mathscr{C}\left(\mathscr{P}_{N(T)}(\lambda f)\right) = \mathscr{C}\left(\lambda\left(f - A_{\tau_f}\right)\right) = \lambda \mathscr{C}\left(f - A_{\tau_f}\right) \tag{4.42}$$

$$= [\lambda f - \lambda f_1, \lambda f - \lambda f_2]$$

This implies that

$$\mathscr{C}\left(T^{'}(\lambda g)\right) = \mathscr{C}\left(\lambda f - \mathscr{P}_{N(T)}(\lambda f)\right) = [\lambda f_1, \lambda f_2] \tag{4.43}$$

Therefore, by $\mathscr{C}\left(f - \mathscr{P}_{N(T)}(f)\right) = [f_1, f_2]$ and from (4.43), $T^{*}(\lambda g) = [\lambda f_1 + \lambda f_2]/2$ and $T^{*}(g) = [f_1 + f_2]/2$ It reduces to $T^{*}(\lambda g) = \lambda T^{*}(g)$ and hence there exists a homogeneous selection $T^{*}$ of $T^{'}$ such that $T^{'}$ is a 2-fuzzy 2-continuous on $\{h \in \mathfrak{F}(Y) : \lim_{k \to h} \left[diam\left(\mathscr{C}\left(T^{'}(j)\right)\right)\right]_{\alpha}$
$\geq \left[diam\left(\mathscr{C}\left(T^{'}(h)\right)\right)\right]_{\alpha}\}$ □

**Corollary 4.2.** *Let $\mathfrak{F}(X)$ be a 2-strictly 2-fuzzy 2-convex space, $\mathfrak{F}(Y)$ be a 2-fuzzy 2-Banach space,$D(T)$ be a closed subspace of $\mathfrak{F}(X)$ and $R(T)$ be an approximatively 2-fuzzy 2-compact chebyshev subspace of $\mathfrak{F}(Y)$. Then*

*(i) $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-upper semicontinous if and only if $T^{'}$ is 2-fuzzy 2-upper semincontinous.*

*(ii) $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-continuous if and only if $T^{'}$ is 2-fuzzy 2-continuous.*

*(iii) If $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-continuous, then there exist a homogeneous selection $T^{*}$ of $T^{'}$ such that $T^{*}$ is 2-fuzzy 2-continuous on $\mathfrak{F}(Y)$.*

*Proof.* By theorem(3.1) it is obvious that (i) and (ii) are true. Since is a 2-strictly 2-fuzzy 2-convex space, it follows that $\mathscr{P}_{N(T)}(f)$ is a line segment for all $\mathfrak{F}(X)$. Then $\mathscr{C}(T^{'}(g))$ is a singleton for all $g \in \mathfrak{F}(Y)$. Therefore by theorem(4.1) it is clear that corollary (4.2) is true. □

**Corollary 4.3.** *Let $\mathfrak{F}(X)$ be a strictly 2-fuzzy 2-convex space, $\mathfrak{F}(Y)$ be a 2-fuzzy 2-Banach space, $D(T)$ be a closed subspace of $\mathfrak{F}(X)$, and $R(T)$ be an approximative 2-fuzzy 2-compact chebyshev subspace of $\mathfrak{F}(Y)$. Then the following statements are equivalent:*

*(i) $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-upper semicontinuous.*

*(ii) $\mathscr{P}_{N(T)}$ is 2-fuzzy 2-continuous.*

*(iii) $T^{'}$ is a 2-fuzzy 2-continuous homogenous single-valued mapping.*

*Proof.* By corollary (4.1) it is obvious that corollary is true. □

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