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On atom-based digraph of a rectangular skew lattice

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Abstract
In [8] Nimbhorkar and Borsarkar introduced the notion of atom based graph of a lattice. Here we introduce this notion to skew lattices and show that the atom-based digraph of a finite rectangular skew lattice is a complete symmetric digraph and vice versa. Some results on the direct product of atom-based digraphs of rectangular skew lattices are also given.

Keywords
Rectangular skew lattice, atom-based digraph, direct product of digraph.

AMS Subject Classification
06B99, 05C20, 05C25.

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Contents

1. Introduction

Skew lattices are noncommutative generalization of lattices. Pascual Jordan, a physicist, motivated by questions in quantum logic, initiated the study of noncommutative lattices in his 1949 paper, *Über Nichtkommutative Verbände*. A noncommutative lattice, generally speaking, is an algebra $(S, \vee, \wedge)$ where $\vee$ and $\wedge$ are associative, idempotent binary operations connected by absorption identities guaranteeing that $\vee$ in some way dualizes $\wedge$. The precise identities chosen depends upon the underlying motivation, with differing choices producing distinct varieties of algebras.

The study of graphs arising from algebraic structures has got great importance in recent years. The zero divisor graph of a commutative ring [1], Cayley graphs [2], graphs associated with lattices [4] etc, are some works in this direction. Nimbhorkar and Borsarkar have introduced the notion of atom-based graph of a lattice [8]. This work is an attempt to generalize this notion to skew lattices. Since skew lattices are noncommutative generalizations of lattices, the resulting graph is a diagraph. Here we provide a characterisation of rectangular skew lattices in terms of digraphs obtained from it.

2. Preliminaries

Here we recall the basic definitions, notations and results which are needed in the sequel. For details see [3, 6, 7].

A skew lattice is a non-empty set S with binary operations $\vee$ and $\wedge$ which are both idempotent and associative, satisfying the absorption laws $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ and $(x \vee y) \wedge y = y = (x \wedge y) \vee y$. The relation $\geq$ defined on S by: $x \geq y$ whenever $x \wedge y = y = y \wedge x$ or equivalently $x \vee y = x = y \vee x$ is a partial order, called natural partial order on S.

Let $(S, \vee, \wedge)$ be a skew lattice. S is called a rectangular skew lattice if $(S, \vee)$ and $(S, \wedge)$ are rectangular bands such that $x \vee y = y \wedge x$ for all $x, y \in S$.

The following result gives a characterisation for rectangular skew lattices.

Theorem 2.1 ([7], Theorem 1.5)**.** *Let $(S, \vee, \wedge)$ be a rectangular skew lattice. Then the following are equivalent:*

(i) $x \vee y = y \wedge x$.
(ii) $x \vee y \vee z = x \vee z$.
(iii) $x \wedge y \wedge z = x \wedge z$.
(iv) $x \vee y = y \vee x$ *if and only if* $x = y$.
(v) $x \wedge y = y \wedge x$ *if and only if* $x = y$.

(vi) $x \geq y$ *if and only if* $x = y$.

The skew lattice S^0 is obtained from a skew lattice $(S, \vee, \wedge)$ by adjoining 0, defined by $0 \vee x = x = x \vee 0$ and $0 \wedge x = 0 = x \wedge 0$ for all $x \in S$. Similarly, S^1 is obtained from a skew lattice $(S, \vee, \wedge)$ by adjoining 1, defined by $1 \vee x = 1 = x \vee 1$ and $1 \wedge x = x = x \wedge 1$. In this way we can make every skew lattice enriched.

An element a of a skew lattice S with least element 0 is an atom if $0 < a$ and there is no $b \in S$ such that $0 < b < a$. The set of all atoms in S is denoted by $\Omega(S)$.

The girth of a graph G is the length of the shortest cycle in G. The distance $d(u,v)$ between two vertices u and v in G is the length of a shortest path joining them. A shortest $u-v$ path is called a geodesic. The diameter $d(G)$ of a connected graph G is the length of any longest geodesic. A simple digraph is a digraph that has no self loops and parallel arcs. The outdegree $d^+(v)$ of a vertex v in a diagraph is the number of vertices adjacent from it and the indegree $d^-(v)$ is the number of vertices adjacent to it. A complete symmetric digraph is a simple digraph in which there is exactly one edge directed from every vertex to every other vertex.

The undefined terms related to skew lattice theory are from Leech [7] and terms related to graph theory are from Deo [3] and Harary [6]. Throughout this paper, all skew lattices are assumed to be finite.

3. Atom-based Digraph of a Skew lattice

In this section we define the atom-based digraph of a skew lattice similar to that of a lattice [8].

Definition 3.1. *Let $(S, \vee, \wedge)$ be a skew lattice. We can associate a digraph to S whose vertex set, V, is the set of all non-zero elements in S and for distinct vertices x and y in V, an ordered pair (x,y) is an edge if and only if $x \wedge y$ is an atom in S. We call this digraph as the atom-based digraph of S and denote it by $\Gamma_a(S^*)$.*

Clearly, atom-based digraph of a skew lattice is always a simple digraph.

Remark 3.2. *Let $\Gamma_a(S^*)$ be an atom-based digraph of a skew lattice $(S, \vee, \wedge)$. If (x,y) is an edge in $\Gamma_a(S^*)$ (ie, $x \wedge y$ is an atom), then (y,x) need not be an edge in $\Gamma_a(S^*)$. But if S is a lattice, then (x,y) is an edge in $\Gamma_a(S^*)$ if and only if (y,x) is an edge in $\Gamma_a(S^*)$, since $x \wedge y = y \wedge x$.*

Example 3.3. *Consider the skew lattice $(S, \vee, \wedge)$, where $S = \{0, a, b, c, 1\}$ with $\vee$ and $\wedge$ are given below:*

$\wedge$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	c	b
c	0	0	b	c	c
1	0	a	b	c	1

$\vee$	0	a	b	c	1
0	0	a	b	c	1
a	a	a	1	1	1
b	b	1	b	b	1
c	c	1	c	c	1
1	1	1	1	1	1

Its Atom-based digraph $\Gamma_a(S^*)$ is given in figure 1.

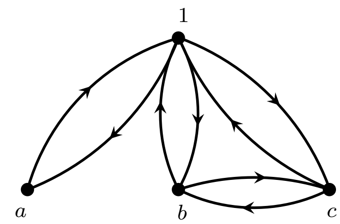


Figure 1

4. Atom-based Digraph of a Rectangular Skew lattice

Let S be a rectangular skew lattice and let S^0 denote the skew lattice obtained by adjoining 0. Also, we denote S^* for the set of nonzero elements of the skew lattice S^0.

Theorem 4.1. *If S^0 is a skew lattice with $(S, \vee, \wedge)$ rectangular, then all non-zero elements in S^0 are atoms. Conversely if all the non-zero elements in a skew lattice S are atoms then S^* will be a rectangular skew lattice.*

Proof. Let S^0 be a skew lattice with S rectangular. Assume that $x \in S^0$ is not an atom and $x \neq 0$. Then by the definition of an atom there exists some $y \in S$ such that $x > y > 0$, which gives $x \wedge y = y = y \wedge x$. By Theorem 2.1, $x = y$. Thus x is an atom.

Conversely, assume that all non-zero elements in S are atoms. That is, there exist no $x, y \in S$ such that $x > y > 0$. Thus for all $x, y \in S$, $x \geq y$ if and only if $x = y$ and hence S is rectangular, by Theorem 2.1. ☐

Corollary 4.2. *Let $(S, \vee, \wedge)$ be a rectangular skew lattice. Then $\Gamma_a(S^*)$ will always be connected. Moreover, $\Gamma_a(S^*)$ will be a complete symmetric digraph.*

Proof. By Theorem 4.1 all elements of S are atoms. Hence (a,b) is an edge for all $a, b \in S$, since $a \wedge b$ is an atom for any $a, b \in S$. Thus $\Gamma_a(S^*)$ is a complete symmetric digraph. ☐

Theorem 4.3. *Let $(S, \vee, \wedge)$ be a rectangular skew lattice. Then*

(i) $diam(\Gamma_a(S^*)) = 1$,

(ii) girth of $\Gamma_a(S^*)$ *is* 2,

(iii) If we adjoin 1 *to S, then* $d^+(1) = d^-(1) = |\Omega(S)|$.

Proof. **(i)** By Corollary 4.2, $\Gamma_a(S)$ is a complete symmetric digraph. Hence its diameter, $\mathrm{diam}(\Gamma_a(S^*)) = 1$.
(ii) Since $\Gamma_a(S^*)$ is a complete symmetric digraph, there exists edges (x,y) and (y,x) for every distinct vertices. Thus the length of a smallest cycle of $\Gamma_a(S^*)$, girth, is 2. **(iii)** By Theorem 4.1, every element of S is an atom. Also $x \wedge 1 = 1 \wedge x = x$ for all $x \in S$. Thus $(1,x)$ and $(x,1)$ are edges for every $x \in S$. Hence $d^+(1) = d^-(1) = |\Omega(S)|$. ☐

Theorem 4.4. *If D is a complete symmetric digraph, then there exists a rectangular skew lattice, $(S, \vee, \wedge)$, such that $\Gamma_a(S^*) = D$.*

Proof. Let D be a complete symmetric digraph with vertex set V. Let $S = V(D)$. Define $\vee$ and $\wedge$ on S as follows; $x \wedge y = y$ and $x \vee y = x$ for all $x, y \in S$ (see [7]). Clearly, $(S, \vee, \wedge)$ is a rectangular skew lattice. If we adjoin 0 to S, then $\Gamma_a(S^*)$ will be D itself. □

In view of Corollary 4.2 and Theorem 4.4 we have the following result.

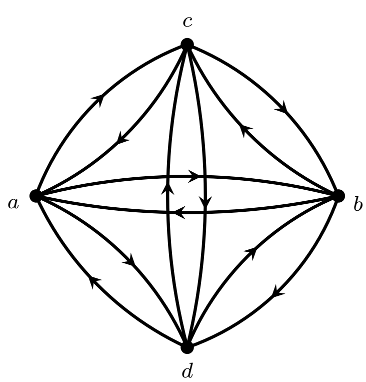
Theorem 4.5. *There exists a one-one correspondence between the class of finite rectangular skew lattices and the class of complete symmetric digraphs.*

Example 4.6. *Let $S = \{a, b, c, d\}$ be a rectangular skew lattice with $\vee$ and $\wedge$ are defined as follows:*

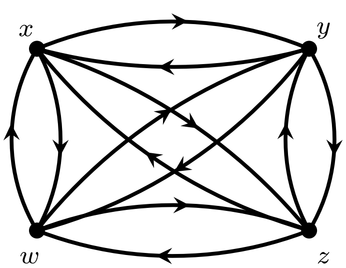
$\wedge$	a	b	c	d
a	a	b	c	d
b	a	b	c	b
c	a	b	c	d
d	a	b	c	d

$\vee$	a	b	c	d
a	a	a	a	a
b	b	b	b	b
c	c	c	c	c
d	d	d	d	d

Then $\Gamma_a(S^*)$ shown in figure 2(a), is a complete symmetric digraph.



(a)



(b)

Figure 2

Now, consider a complete symmetric digraph, D, shown in figure 2(b). Let $S_1 = \{x, y, z, w\}$. Define $\vee$ and $\wedge$ as follows:

$\wedge$	x	y	z	w
x	x	y	z	w
y	x	y	z	w
z	x	y	z	w
w	x	y	z	w

$\vee$	x	y	z	w
x	x	x	x	x
y	y	y	y	y
z	z	z	z	z
w	w	w	w	w

Clearly, S_1 is a rectangular skew lattice. Also, we can see that up to isomorphism $\Gamma_a(S_1^*) = D$.

5. The Direct product of Atom-based Digraphs on Skew lattices

Let D_1 and D_2 be simple digraphs. Their direct product is the digraph $D_1 \times D_2$ whose vertex set is the Cartesian product $V(D_1) \times V(D_2)$ and whose arcs are the pairs $((a,b),(a',b'))$ with $(a,a') \in E(D_1)$ and $(b,b') \in E(D_2)$ (see [5]). Thus,

$$V(D_1 \times D_2) = \{(a,b) \mid a \in V(D_1) \text{ and } b \in V(D_2)\},$$
$$E(D_1 \times D_2) = \{((a,b),(a',b')) \mid (a,a') \in E(D_1) \text{ and } (b,b') \in E(D_2)\}.$$

Since both D_1 and D_2 are simple digraphs, it follows that $D_1 \times D_2$ is a simple digraph.

Definition 5.1. *Let $(S, \vee, \wedge)$ be a skew lattice and $x \in S$. The atom-based right (left) annihilator of x, denoted by $Rann_\Omega(x)[Lann_\Omega(x)]$, is defined as $Rann_\Omega(x) = \{y \in S : x \wedge y \in \Omega(S)$ and $x \neq y\}$ $[Lann_\Omega(S) = \{y \in S : y \wedge x \in \Omega(S)$ and$x \neq y\}]$.*

Example 5.2. *Let $S = \{0, a, b, c, 1\}$ be a skew lattice with $\vee$ and $\wedge$ are defined in Example 3.3. Here $Rann_\Omega(0) = \{1\}$, $Rann_\Omega(b) = \{c, 1\}$, $Rann_\Omega(c) = \{b, 1\}$, $Rann_\Omega(1) = \{a, b, c\}$ and $Lann_\Omega(0) = \{1\}$, $Lann_\Omega(b) = \{c, 1\}$, $Lann_\Omega(c) = \{b, 1\}$, $Lann_\Omega(1) = \{a, b, c\}$.*

Theorem 5.3. *Let $(S_1, \vee, \wedge)$ and $(S_2, \vee, \wedge)$ be two skew lattices and $D_1 = \Gamma_a(S_1)$, $D_2 = \Gamma_a(S_2)$ be their atom-based digraphs. Then, for every $(a,b) \in V(D_1 \times D_2)$ $deg^+(a,b) = |Rann_\Omega(a)||Rann_\Omega(b)|$ and $deg^-(a,b) = |Lann_\Omega(a)||Lann_\Omega(b)|$.*

Proof. Assume that $((a,b),(c,d)) \in E(D_1 \times D_2)$, then $(a,c) \in E(D_1)$ and $(b,d) \in E(D_2)$. That is, $a \wedge c \in \Omega(S_1)$ and $b \wedge d \in \Omega(S_2)$. Thus $c \in Rann_\Omega(a)$, $a \in Lann_\Omega(c)$ and $b \in Rann_\Omega(d)$, $d \in Lann_\Omega(b)$. Since this holds for every edge $((a,b),(c,d))$ in $E(D_1 \times D_2)$, we see that $deg^+(a,b) = |Rann_\Omega(a)||Rann_\Omega(b)|$ and $deg^-(a,b) = |Lann_\Omega(a)|$ $|Lann_\Omega(b)|$. □

Theorem 5.4. *Let $(S_1, \vee, \wedge)$ and $(S_2, \vee, \wedge)$ be two skew lattices and $D_1 = \Gamma_a(S_1)$ and $D_2 = \Gamma_a(S_2)$ be their atom-based digraphs. Then*

$$\begin{aligned} |E(D_1 \times D_2)| &= \Sigma|Rann_\Omega(a_i)||Rann_\Omega(b_j)| \\ &\quad + \Sigma|Lann_\Omega(a_i)||Lann_\Omega(b_j)| \\ &= |E(D_1)||E(D_2)|, \end{aligned}$$

where, $a_i \in S_1$ and $b_j \in S_2$.

Proof. In the above theorem we have proved that $|Rann_{\Omega}(a)||Rann_{\Omega}(b)|$ is the number of edges incident out of a vertex (a,b) and the number of edges incident into a vertex (a,b) is $|Lann_{\Omega}(a)||Lann_{\Omega}(b)|$. Hence $|E(D_1 \times D_2)|$ is the sum of all these edges for every vertex (a,b) in $D_1 \times D_2$. Hence the proof. □

Definition 5.5. *If S_1 and S_2 are any two skew lattices, then (a,b) is covered by (c,d) in the skew lattice $S_1 \times S_2$, denoted by $(a,b) \prec (c,d)$ if and only if either $a = c$ and $b \prec d$ or $a \prec c$ and $b = d$.*

Similar to Nimbhorkar et.al [9], we will note that $\Omega(S_1 \times S_2) = \{(a,b) : a \in \Omega(S_1)$ and $b = 0$ or $a = 0$ and $b \in \Omega(S_2)\}$.

Theorem 5.6. *Let $(S_1, \vee, \wedge)$ and $(S_2, \vee, \wedge)$ be two skew lattices and $D_1 = \Gamma_a(S_1)$, $D_2 = \Gamma_a(S_2)$ be their atom-based digraphs. Then $D_1 \times D_2$ is not an induced subdigraph of $\Gamma_a(S_1 \times S_2)$.*

Proof. Clearly $V(D_1 \times D_2) \subseteq V(\Gamma_a(S_1 \times S_2))$. Now, assume that $((a,b),(c,d)) \in E(D_1 \times D_2)$, then $(a,c) \in E(D_1)$ and $(b,d) \in E(D_2)$, which implies $(a \wedge c) = p\ (let)\ \in \Omega(S_1)$ and $(b \wedge d) = q\ (let)\ \in \Omega(S_2)$. This gives p and q are non-zero, by definition. Hence $(p,q) \notin \Omega(S_1 \times S_2)$. But $(a,b) \wedge (c,d) = (a \wedge c, b \wedge d) = (p,q)$. Thus $((a,b),(c,d))$ cannot be an edge in $\Gamma_a(S_1 \times S_2)$.

Conversely, suppose that $((a,b),(c,d))$ is an edge in $\Gamma_a(S_1 \times S_2)$. Then $(a \wedge c, b \wedge d) \in \Omega(S_1 \times S_2)$ and so either $a \wedge c \in \Omega(S_1)$ and $b \wedge d = 0$ or $a \wedge c = 0$ and $b \wedge d \in \Omega(S_2)$. In either case only one of the edges (a,c) and (b,d) exists. Hence $((a,b),(c,d))$ can not be an edge in $D_1 \times D_2$. That is, there exists no common edge between $D_1 \times D_2$ and $\Gamma_a(S_1 \times S_2)$. Hence $D_1 \times D_2$ is not an induced subdigraph of $\Gamma_a(S_1 \times S_2)$. □

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