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Connected, regular and split liar domination on fuzzy graphs

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Abstract
Liar domination set in a fuzzy graph is the set to identify the intruder location in a computer network / communication network with minimum fuzzy cardinality of the nodes. In this paper we discussed Connected, Regular and Split liar domination on fuzzy graphs and also discussed some of their properties.

Keywords
Strong Edge, Open neighbourhood, closed neighbourhood, Domination, Liar Domination Set, Regular Fuzzy Graphs, Connected Liar Domination, Regular Liar Domination. Split Liar Domination.

AMS Subject Classification
05C72.

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Contents

1. Introduction

In 1736, Leonhard Euler invented the theory of graphs. Graph theory is very essential tool for solving combinatorial problems in several areas such as Algebra, Number Theory, Geometry, Topology and Operations Research and so on. There are many applications of graph theory in Computer Science, Linguistics, Electrical Engineering, Physics and Chemistry, Social Sciences and Biology, etc. In 1965 Zadeh L. A. [1] invented the concepts of fuzzy set of a set. The concepts of fuzzy graph theory is widely used in different fields including Medical and Life Sciences, Social Sciences, Engineering, Graph Theory, Management Science, Artificial Intelligence, Communication Networks, Computer Networks, Decision Making and Patent Recognition, etc. In 2008, P. J. Slater [16] introduced the concept of liar domination in graph theory. There are several methods for fault detection. This is one of the fuzzy logic method to identify the fault in a network. B.S. Panda et al.[2] discussed that liar domination set is used in deploying protection devices with minimum number of nodes so that the fault can be detected and reported correctly. In 1973, A. Kauffman[3] introduced the basic concepts of fuzzy graphs. In 1975, A. Rosenfeld[4] developed fuzzy graph theory. In 1987, A. P. Battacharya [5] discussed some remarks on fuzzy graphs. In 1994, J. M. Moderson and C.S. Peng [6] studied operations on fuzzy graphs. In 2002, M.S. Sunitha and A. Vijayakumar [7] discussed complement of fuzzy graphs. A. Nagoorgani [8] discussed the relationship between degree, size and order of fuzzy graphs. C. Y. Ponnapan[9] studied strong split dominating set of fuzzy graphs and investigated this with other parameters. A. Nagoorgani[10] stated some properties of regular fuzzy graphs and totally regular fuzzy graphs. O. T. Manjusha and M.S. Sunitha[11] discussed some characteristic properties of the existence of strong connected dominating set for a fuzzy graphs and its complements. S. Mathew[12] analysed the relationship between strong paths and strongest paths in a fuzzy graph. S. Narayanamoorthy and P. Karthick [13–15] studied gray level image threshold and intuitionistic fuzzy graph.

2. Preliminaries

Definition 2.1. *An edge uv is said to be strong if* $\mu^{\infty}(u,v) = \mu(u,v)$, *where* $\mu^{\infty}(u,v)$ *is maximum strength of all possible* $u-v$ *paths.*

Definition 2.2. *Open neighborhood of a vertex* $N(u)$ *is defined as,* $N(u) = \{v \in V(G) \backslash \mu^{\infty}(u,v) = \mu(u,v)\}$

Definition 2.3. *Closed neighborhood of a vertex* $N[\,u]$ *is defined as,* $N[\,u] = \{u\} \cup \{v \in V(G) : \mu^{\infty}(u,v) = \mu(u,v)\ \}$

Definition 2.4. *The vertex u is said to be dominated by the vertex v if* $u \in N[\,v]$ *, where*
$N[\,v] = \{v\} \cup \{u \in V, (u,v) is\ strong\ edge\}$.

Definition 2.5. *A fuzzy graph is connected if for every* x, y *in* $V, CONN_G(x,y) > 0$.

Definition 2.6. *Let* $G = <\sigma, \mu>$ *be a fuzzy graph on* V. *The set* $D \subseteq V$ *is called a liar dominating set if it satisfies the following conditions.*

1. *Each vertex* $u \in V(G)$ *is dominated by at least two vertices in D.*
2. *Every pair of vertices* $u, v \in V(G)$ *is dominated by at least three vertices in D.*

Example

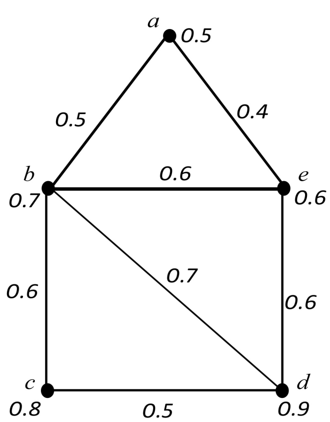

Figure 1

Liar domination sets are $\{a,b,e\}, \{b,e,d\}, \{b,c,d\}, \{c,d,e\}, \{a,e,d\}, \{a,b,c,d\}$

Definition 2.7. *The vertices in a liar domination set who have minimum fuzzy cardinality is called minimum liar domination set.*
The fuzzy cardinality of minimum liar domination set is called liar domination number.
In Figure 1, minimum liar dominating set = $\{a,b,e\}$
Liar domination number = 1.8
A liar dominating set D in a fuzzy graph is called minimal liar dominating set if no proper subset of D is a liar dominating set.

Example

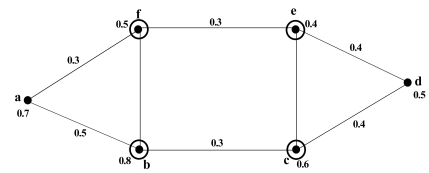

Figure 2

$\{b,c,e,f\}$ is called minimal liar dominating set.

3. Connected Liar Domination

Let G be a fuzzy graph. A liar domination set $D \subseteq V(G)$ is said to be connected liar domination set if the induced sub graph $<D>$ is connected.

Example

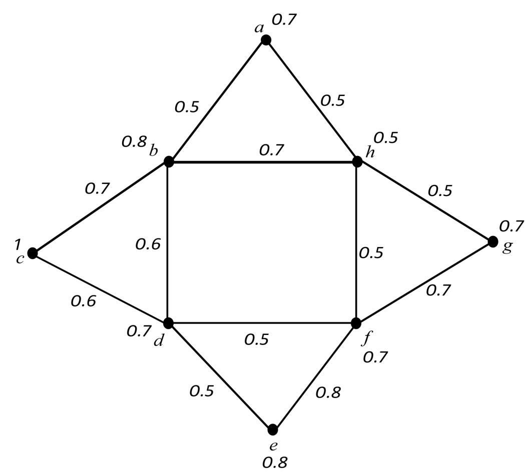

Figure 3

$<D> = \{b,d,f,h\}$ is connected liar domination set.
$<D> = \{a,b,d,e,f,h\}$ is also connected liar domination set.

A connected liar domination set is called minimum connected liar;s domination set if there is no connected liar domination set D' such that $|D| \geq |D'|$. The fuzzy cardinality of minimum connected liar domination set is called connected liar domination number and is denoted by λ_C.
In Figure 3, $<D> = \{b,d,f,h\}$ is minimum connected liar domination set and $\lambda_C(G) = 0.8 + 0.5 + 0.7 + 1 = 3$.

Theorem 3.1. *Let G be a liar domination set. Then the induced subgraph of D in any complete bipartite fuzzy graph is connected.*

Proof. Let G be a complete bipartite fuzzy graph with the vertex sets (V_1,V_2).
Then there is a strong edge between each vertex of V_1 and each vertex of V_2.
Let $u_1,u_2,...u_n$ be the vertices of V_1 and $v_1,v_2...v_m$ be the vertices of V_2.

Case 3.2. *Suppose there is no fuzzy path between u_i,u_j. Then u_iv_k, $v_ku_j \notin E(G)$.*
This is the contradiction to the choice of G.
Therefore, $\mu^\infty(u_i,u_j) > 0$ for all $u_i,u_j \in V_1$.

Case 3.3. *Suppose there is no fuzzy path between the vertices u_i,v_j. But $u_iv_j \in E(G)$.*
Therefore, there is a fuzzy path between u_i,v_j where $u_i \in V_1, v_j \in V_2$. That is, $\mu(u_i.v_j) > 0$, for all $u_i,v_j \in V(G)$.
Thus, the induced subgraph of D in any complete bipartite fuzzy graph is connected.

□

Corollary 3.4. *Liar domination number and connected liar domination number are equal for all complete bipartite graphs with $|V_1| \geq 3$ & $|V_2| \geq 3$.*

Proof. Let G be a complete bipartite fuzzy graph and D be minimum liar domination set of G.
Then by previous theorem, the induced subgraph of D is connected. Then the fuzzy cardinality of D is liar domination number as well as connected liar domination number. □

Theorem 3.5. *Connected liar domination number of a fuzzy cycle is $p - max(\sigma(u_i)), u_i \in V(G), i = 1,2,....n-1$.*

Proof. Let C_n be the fuzzy cycle with n nodes, namely $u_0,u_1,...u_{n-1} = u_0$, where $\mu(u_i,u_{i+1}) > 0, u_i \in V(G)$.
Let D be a liar domination set of G.
Then the induced subgraph of D is connected only if $u_i,u_{i+1},....u_j,u_{j+1},...u_{i-3},u_{i-2} \in D, u_i$ is vertex of $V(G)$.
The set D forms minimum connected liar domination set if $\sigma(u_k) \notin D$, where $\sigma(u_k) = max(\sigma(u_i)), u_i \in V(G)$.
Thus, $\lambda(C_n) = p - max[\sigma(u_i)]$ □

Theorem 3.6. *Connected liar domination number of a fuzzy path is p.*

Proof. Let P_n be a fuzzy path with n nodes, namely $v_1,v_2...v_n$.
D is said to be connected liar domination set if induced subgraph of D is connected.
In liar domination set, every vertex must be dominated by at least two vertices in G. Since $v_1,v_2 \in N[v_1]$, (i.e.) there are only two vertices, which are dominating v_1. One is v_1 itself, other one is v_2. Therefore, to form liar domination set, v_1,v_2 must be in D.
Since D is taken as a connected liar domination set, $v_3,v_4...v_{n-1}$ are the vertices of D. Similarly, v_{n-2} and v_n are only two strong neighbours of v_n.
Thus, $v_1,v_2....v_{n-1},v_n$ form connected liar domination set.
Therefore, connected liar domination number is p. □

Definition 3.7. *A connected liar domination set D is minimal if no proper subset of D is connected liar domination set.*

Theorem 3.8. *Every minimum connected liar domination set is minimal connected liar domination set.*

Proof. Let D be a minimum connected liar domination set and D' be a minimal connected liar domination set.
Let $u \in D$, then $D - \{u\}$ is not a minimum connected liar domination set, since D is called minimum connected liar domination set only if $|D| \ngtr |D'|$.
This implies that it is also a minimal connected liar dominated set. Because, the set S is called minimal connected liar dominated set if no proper subset of S is connected liar dominated set.
Note: But, the converse need not be true .

□

Theorem 3.9. *If the vertices of connected liar dominating set D forms a cycle C, then it is a minimal connected liar domination set, only if more than one vertex is removed from C.*

Proof. Let D be the connected liar domination set of a fuzzy graph G, which forms a cycle.
Let $v_k \in D$ for some k. Let v_k be removed from C, then $CONN_G(v_i,v_j) > 0$, for every $v_i,v_j \in D$. Therefore D is still connected liar dominating set.
Let $v_{k_1},v_{k_2}, \in D$, for some k_1,k_2. Let v_{k_1},v_{k_2} be removed from C.
Then, $CONN_G(v_i,v_j) \ngtr 0$, for every $v_i,v_j \in D$.
Thus, D is minimal connected liar domination set, only if more than one vertex is removed from C. □

4. Regular Liar Domination

Definition 4.1. *A liar domination set D of a fuzzy graph is said to be regular liar domination set if all the vertices of D have same degree.*
Example

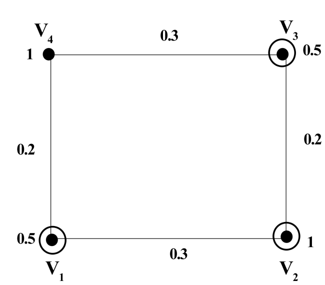

Figure 4

$\{v_1,v_2,v_3\}$ is Regular liar domination set.
Note: Regular liar domination number $\lambda_R(G)$ is minimum fuzzy cardinality of regular liar domination set.

In Figure 4, $\{v_1, v_2, v_3, v_4\}$ *and* $\{v_2, v_3, v_4\}$ *are regular domination set.*
$\lambda_R(G) = 2$

Theorem 4.2. *Liar domination number of R-regular fuzzy graph is,* $\lambda_R(G) \leq \lceil \frac{6P}{2+3R} \rceil$

Proof. Let $D \subseteq V(G)$ be liar domination set of R-regular fuzzy graph G.
Let $\lambda_R(G)$ be liar domination number of R-regular fuzzy graph and $\lambda_r(G)$ be liar domination number of r-regular crisp graph.
w.k.t $\lambda_R(G) \leq \lambda_r(G)$ and $R \leq r$
Let $p = \sum_{x \in V(G)} \sigma(x)$
$q = \sum_{(x,y) \in E(G)} \mu(x,y)$
$p_1 = \sum_{x \in L(G)} \sigma(x)$
$p_2 = \sum_{x \in V-L(G)} \sigma(x)$
$q_1 = \sum_{(x,y) \in E(L)} \mu(x,y)$
$q_2 = \sum_{(x,y) \in E(G) \setminus E(L)} \mu(x,y)$
Then $q_1 \geq \frac{0.2}{0.3} \lambda_R(G)$
$q_1 \geq \frac{2}{3} \lambda_R(G)$
$|E(G) \setminus E(L)| \leq \lambda_R(G)R - \frac{4}{3}\lambda_R(G) \leq \lambda_r(G)r - \frac{4}{3}\lambda_r(G)$
$p_2 \geq \frac{1}{2}[\, \lambda_R(G)R - \frac{4}{3}\lambda_R(G)]$ (since $2p \geq q$)
$p - p_1 \geq \frac{1}{2}[\, \lambda_R(G)R - \frac{4}{3}\lambda_R(G)]$
$p - \lambda_R(G) \geq \frac{1}{2}[\, \lambda_R(G)R - \frac{4}{3}\lambda_R(G)]$
$p \geq \lambda_R(G) + \frac{\lambda_R(G)R}{2} - \frac{2}{3}\lambda_R(G)$
$p \geq \lambda_R(G)[\, 1 + \frac{R}{2} - \frac{2}{3}]$
$p \geq \lambda_R(G)[\, \frac{2+3R}{6}]$
$\lambda_R(G) \leq \frac{6p}{2+3R}$
without loss of generality,
$\lambda_R(G) \leq \lceil \frac{6P}{2+3R} \rceil$ □

Theorem 4.3. *Regular liars domination set exists in a fuzzy cycle, only if one of the following conditions hold.*

1. $\mu(u,v) = k$ *for every two adjacent nodes*
2. $\mu(u_i, u_{i+1}) = \mu(u_{i+2}, u_{i+3}), \forall i = 1,2,...n-4$
   $\mu(u_{i+1}, u_{i+2}) = \mu(u_{i+3}, u_{i+4}), \forall i = 0,1,2,...n-3$

Proof. Let C be a fuzzy cycle.

(i) If $\mu(u,v) = k$, for every two adjacent nodes in c. Then obviously, this fuzzy cycle is regular. Therefore regular liar domination set exists in c.

(ii) Let,
$\mu(u_i, u_{i+1}) = \mu(u_{i+2}, u_{i+3}), \forall i = 1,2,...n-4$
$\mu(u_{i+1}, u_{i+2}) = \mu(u_{i+3}, u_{i+4}), \forall i = 0,1,2,...n-3$

The above two conditions indicates that the alternative edges are having same membership values. Then, all vertices in c have same degree. Therefore c is regular implies that regular liar domination set exists in c. □

5. Split Liar Domination

A liar domination set s is called split liar domination set, if $V - S$ is disconnected.

The minimum fuzzy cardinality of split liar domination set is called split liar domination number and is denoted by $\lambda_S(G)$.
Split liar domination set s is called minimal split liar domination set if no proper subset of s is split liar domination set.
Example

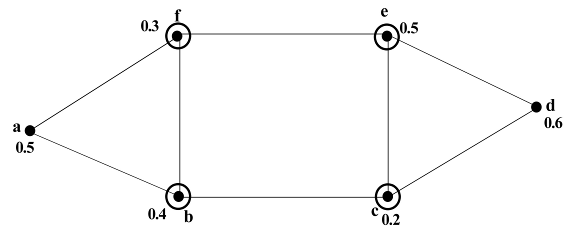

Figure 5

$\{b, c, e, f\}$ is split liar domination set.
$\lambda_S(G) = 0.3 + 0.4 + 0.2 + 0.5 = 1.4$

Theorem 5.1. *A liar domination set D is a split liar domination set iff there exists two vertices* $u, v \in V \setminus D$ *such that* $\mu^\infty(u,v) = 0$ *in* $G \setminus < D >$.

Proof. Let D be a liar domination set of a fuzzy graph G.
Suppose $u, v \in V \setminus D$ such that $\mu^\infty(u,v) = 0$ in $G - < D >$.
Then there is no connectedness between the vertices u, v in $G \setminus < D >$.
That is $G \setminus < D >$ is disconnected. Therefore D must be a split liar domination set of G.
Conversely, Let D be a split liar domination set of D, then removal of $< D >$ will disconnect the graph G. That is, if $u, v \in V \setminus D$, then $\mu^\infty(u,v) = 0$ □

Theorem 5.2. *A split liar domination set is minimal iff for any two vertices* $a, b \in s$*, at least one of the following condition hold.*

(i) $N(a) \cap S = \{b\}$

(ii) There is a vertex $c \in V \setminus S$ *such that*
$N(C) \cap S = \{a, b\}$

Proof. Let S be a split liar domination set. Let S satisfy the given two conditions. We prove that S is minimal.
Let $a, b \in S$. If $N(a) \cap S = \{b\}$, then $S' = S - a$ is not a liar domination set, since a is not dominated by two vertices in S'.
Suppose there is a vertex $C \in V \setminus S$ such that $N(C) \cap S = \{a, b\}$.
Let $S' = S - b$, then S' is not a liar domination set, since c is not dominated by two vertices in S'.
Hence the split liar domination set S is minimal.
Conversely, split liar domination set S is minimal. Let $a \in S$.

Then $S' = S - a$ is not liar domination set. This means that some vertex $z \in V \backslash S'$ is not dominated by S'.
Suppose $z = a$, $N(a) \cap S = \{b\}$. If not, then there is some $z \in V \backslash S'$. Such that $N(z) \cap S = \{a, b\}$.
Hence proved. □

Theorem 5.3. *If S is a minimal split liar domination set of G, Then there is a fuzzy path between any two vertices of $V \backslash S$ containing at least three vertices of S.*

Proof. Let S be a split liar domination set of a fuzzy graph G. Then every vertices of $V(G)$ are dominated by at least two vertices in S.
That is, vertices of S are also dominated by two vertices in S and every pair of vertices in S are dominated by at least three vertices in S. Suppose s_i, s_j, s_k are the vertices of S.
Then, $N(s_i) = \{s_j\}$ and $N(s_j) = \{s_i, s_k\}$ that is, there is a fuzzy path $s_i - s_k$ in S. Every vertex $c_i, c_j \in V \backslash S$ is dominated by at least two vertices in S. Suppose c_i is dominated by $s_i \& s_j$ and c_j is dominated by s_j & s_k.
Then there exists fuzzy path $c_i - c_j$ containing three vertices in S, namely s_i, s_j & s_k.
Therefore, Fuzzy path between any two vertices of $V \backslash S$ contains three vertices of S. □

Theorem 5.4. *For any fuzzy graph G, split liar domination number is*

(i) $\lambda_S(G) \geq \lambda(G) - 2p$

(ii) $\lambda_S(G) \geq q - 2p$

Proof. $p = O(G) = \sum_{u \in V} \sigma(u)$
$q = S(G) = \sum_{u,v \in E(G)} \mu(u, v)$
$\sum_{uv \in E(G)} \mu(u, v) \geq \vee(d(v) | u \in V)$, since $\Delta(G) = \vee(d(v) | v \in V)$
That is, $\sum_{uv \in E(G)} \mu(u, v) \geq \Delta(G)$
We know that , $\sum_{u \in v} d(v) = 2 \sum \mu(u, v)$
This implies that, $\sum d(v) \geq \sum \mu(u, v)$
Thus, $2p \geq \sum_{v \in V} d(v) \geq \sum_{uv \in E(G)} \mu(u, v) \geq \Delta(G)$.
(since $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$)
$2p + \lambda_S(G) \geq \Delta(G)$
$\lambda_S(G) \geq \lambda(G) - 2p$
$2p + \lambda_S(G) \geq q$
$\lambda_S(G) \geq q - 2p$ □

6. Conclusion

The concepts of fuzzy graph theory is widely used in different field including Medical and Life Sciences, Social Sciences, Engineering, Graph Theory, Management Science, Artificial Intelligence, Communication Networks, Computer Networks, Decision Making and Patent Recognition, etc. In this paper we discussed Connected, Regular, Split liar domination on fuzzy graphs and their properties.

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