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On Berezin radius inequalities via Cauchy-Schwarz type inequalities

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Abstract. A functional Hilbert space is the Hilbert space of complex-valued functions on some set $\Theta \subseteq \mathbb{C}$ that the evaluation functionals are continuous for each $\tau \in \Theta$ on $\mathcal{H}$. The Berezin transform $\widetilde{S}$ and the Berezin radius of an operator S on the functional Hilbert space (or reproducing kernel Hilbert space) over some set Θ with the reproducing kernel k_τ are defined, respectively, by

$$\widetilde{S}(\tau) = \left\langle S\widehat{k}_\tau, \widehat{k}_\tau \right\rangle, \ \tau \in \Theta \text{ and } \operatorname{ber}(S) := \sup_{\tau \in \Theta} \left|\widetilde{S}(\tau)\right|.$$

Using this limited function $\widetilde{S}$, we investigate several novel inequalities that include improvements to some Berezin radius inequalities for operators working on the functional Hilbert space.

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Contents

1. Introduction

Let $\mathbb{L}(\mathcal{H})$ be the Banach algebra of all bounded linear operators defined on a complex Hilbert space $(\mathcal{H}, \langle ., . \rangle)$. Throughout the paper, we work on functional Hilbert space (FHS), which are complete inner product spaces made up of complex-valued functions defined on a non-empty set Θ with bounded point evaluation. Recall that a functional Hilbert space $\mathcal{H} = \mathcal{H}(\Theta)$ is a complex Hilbert space on a (nonempty) Θ, which has the property that point evaluatians are continuons for each $\tau \in \Theta$ there is an unique element $k_\tau \in \mathcal{H}$ such that $f(\tau) = \langle f, k_\tau \rangle$, for all $f \in \mathcal{H}$. The family $\{k_\tau : \tau \in \Theta\}$ is called the reproducing kernel $\mathcal{H}$. If $\{e_n\}_{n \geq 0}$ is an orthonormal basis for FHS, the reproducing kernel is showed by $k_\tau = \sum_{n=0}^{\infty} \overline{e_n(\tau)} e_n(z)$. For $\tau \in \Theta$, $\widehat{k}_\tau = \frac{k_\tau}{\|k_\tau\|_{\mathcal{H}}}$ is called the normalized reproducing kernel.

Definition 1.1. *(i) For $S \in \mathbb{L}(\mathcal{H})$, the function $\widetilde{S}$ defined on Θ by*

$$\widetilde{S}(\tau) = \left\langle S\widehat{k}_\tau, \widehat{k}_\tau \right\rangle_{\mathcal{H}}$$

is the Berezin symbol (or Berezin transform) of S.

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(ii) The Berezin range of S (or Berezin set of S) is

$$\operatorname{Ber}(S) := \operatorname{Range}(\widetilde{S}) = \left\{ \widetilde{S}(\tau) : \tau \in \Theta \right\}.$$

(iii) The Berezin radius of S (or Berezin number of S) is

$$\operatorname{ber}(S) := \sup \left\{ \left| \widetilde{S}(\tau) \right| : \tau \in \Theta \right\}.$$

The Berezin transform $\widetilde{S}$ is a bounded real-analytic function on for each bounded operator S on $\mathcal{H}$. The Berezin transform $\widetilde{S}$ frequently reflects the characteristics of the operator S. A key tool in operator theory is the Berezin transform, which Berezin first described in [10]. This is because the Berezin transforms of many significant operators include information on their fundamental characteristics. The Berezin range and Berezin radius of the operator were defined by Karaev in [25].

Recall that the numerical range and numerical radius number of $S \in \mathbb{L}(\mathcal{H})$ are denoted respectively, by

$$W(S) = \{ \langle Su, u \rangle : u \in \mathcal{H} \text{ and } \|u\| = 1 \} \text{ and,}$$

$$w(S) = \sup \{ |\langle Su, u \rangle| : u \in \mathcal{H} \text{ and } \|u\| = 1 \}.$$

The absolute value of positive operator is denoted by $|S| = (S^*S)^{\frac{1}{2}}$. The numerical range has several intriguing features. For example, it is usually assumed that an operator's spectrum is confined in the closure of its numerical range. For an illustration of how this and other numerical radius inequalities were addressed in those sources, we urge the reader read [1, 14, 28, 29]. For $S, T \in \mathbb{L}(\mathcal{H})$ it is clear from the definition of the Berezin number and the Berezin norm that the following properties hold:

(B1) $\operatorname{ber}(zS) = |z| \operatorname{ber}(S)$ for all $z \in \mathbb{C}$,

(B2) $\operatorname{ber}(S+T) \leq \operatorname{ber}(S) + \operatorname{ber}(T)$,

(B3) $\operatorname{ber}(S) \leq \|S\|_{\text{ber}}$,

(B4) $\|zS\|_{\text{ber}} = |z| \|S\|_{\text{ber}}$ for all $z \in \mathbb{C}$,

(B5) $\|S+T\|_{\text{ber}} \leq \|S\|_{\text{ber}} + \|T\|_{\text{ber}}$.

It is clear from the definition that $\operatorname{Ber}(S) \subseteq W(S)$ and so

$$\operatorname{ber}(S) \leq w(S) \leq \|S\| \tag{1.1}$$

for any $S \in \mathbb{L}(\mathcal{H}(\Theta))$.

In [24], Huban et al. obtained the following result:

$$\operatorname{ber}(S) \leq \frac{1}{2} \left(\|S\|_{\text{ber}} + \left\| S^2 \right\|_{\text{ber}}^{1/2} \right). \tag{1.2}$$

After that, in [22], and [9], respectively, the same authors proved for $S \in \mathbb{L}(\mathcal{H}(\Theta))$

$$\frac{1}{4} \left\| |S|^2 + |S^*|^2 \right\|_{\text{ber}} \leq \operatorname{ber}^2(S) \leq \frac{1}{2} \left\| |S|^2 + |S^*|^2 \right\|_{\text{ber}} \tag{1.3}$$

where $|S| = (S^*S)^{1/2}$ is the acsolute value of S, and

$$\operatorname{ber}^{2\alpha}(S) \leq \frac{1}{2} \left\| |S|^{2\alpha} + |S^*|^{2\alpha} \right\|_{\text{ber}} \tag{1.4}$$

where $\alpha \geq 1$.

Huban et al. demonstrated the following Berezin radius estimate for the product of two functional Hilbert space operators

$$\operatorname{ber}^{\alpha}(T^*S) \leq \frac{1}{2} \left\| |S|^{2\alpha} + |T|^{2\alpha} \right\|, \ \alpha \geq 1, \tag{1.5}$$

in [22, Theorem 3.11].

On Bergman and Hardy spaces, the Berezin symbol (or transform) has been thoroughly investigated for Hankel and Toeplitz operators. Several mathematical works have examined the Berezin symbol and Berezin radius throughout the years; a few of them are [6, 7, 12, 19, 20, 25, 26, 32]. In order to functional Hilbert space (reproducing kernel Hilbert space) operators, this study establishes numerous improvements of the aforementioned Berezin radius inequalities. In specifically, it is demonstrated that

$$\mathrm{ber}^2(S) \le \frac{1}{6}\left\| |S|^2 + |S^*|^2 \right\|_{\mathrm{ber}} + \frac{1}{3}\mathrm{ber}(S)\, \| |S| + |S^*| \|_{\mathrm{ber}} \tag{1.6}$$

for the arbitrary bounded linear operator $S \in \mathbb{L}(\mathcal{H}(\Theta))$. Furthermore covered are a few additional connected issues. The related results are obtained in [4].

2. Known Lemmas

The following series of corollaries are necessary for us to succeed in our mission.

According to the Cauchy-Schwarz inequality,

$$|\langle u, v\rangle| \le \|u\| \, \|v\| \tag{2.1}$$

holds true for every vectors u and v in an inner product space.

Contrarily, the traditional Schwarz inequality for positive operators states that for any $u, v \in \mathcal{H}$, if $S \in \mathbb{L}(\mathcal{H})$ is a positive operators, then

$$|\langle Su, v\rangle|^2 \le \langle Su, u\rangle \langle Sv, v\rangle \, . \tag{2.2}$$

A companion of Schwarz inequality (2.2) known as the Kato's inequality or the so called mixed Cauchy Schwarz inequality was first proposed by Kato [27] in 1952. It states:

$$|\langle Su, v\rangle|^2 \le \left\langle |S|^{2r} u, u\right\rangle \left\langle |S^*|^{2(1-r)} v, v\right\rangle, 0 \le r \le 1 \tag{2.3}$$

for any operators $S \in \mathcal{B}(\mathcal{H})$ and any vectors $u, v \in \mathcal{H}$.

$$|\langle Su, u\rangle| \le \sqrt{\langle |S| \, u, u\rangle \langle |S^*| \, u, u\rangle}. \tag{2.4}$$

in particular is present.

$$\begin{aligned} |\langle Su, u\rangle|^2 &\le \frac{1}{3}\langle |S| \, u, u\rangle \langle |S^*| \, u, u\rangle + \frac{2}{3}|\langle Su, u\rangle| \sqrt{\langle |S| \, u, u\rangle \langle |S^*| \, u, u\rangle} \\ &\le \langle |S| \, u, u\rangle \langle |S^*| \, u, u\rangle \end{aligned} \tag{2.5}$$

was proven to be the refinement of (2.4) in [30].

The following well-known lemmas will make it necessary to demonstrate our findings. The Power-Mean (PM) inequality comes first.

Lemma 2.1. *([31]) According to the PM inequality,*

$$x^r y^{1-r} \le rx + (1-r)\, y \le (rx^\alpha + (1-r)\, y^\alpha)^{\frac{1}{\alpha}} \tag{2.6}$$

holds for every $0 \le r \le 1$, $x, y \ge 0$ *and* $\alpha \ge 1$.

The McCarty inequality for positive operators is the following lemma.

Lemma 2.2. *([15]) If* $S \in \mathbb{L}(\mathcal{H})$ *is a positive operator and* $u \in \mathcal{H}$ *is an unit vector, then we have*

$$\langle Su, u\rangle^\alpha \le (\ge) \langle S^\alpha u, u\rangle \, , \, \alpha \ge 1 \;\; (0 \le \alpha \le 1) \, . \tag{2.7}$$

Lemma 2.3. *([5]) If $S,T\in\mathbb{L}(\mathcal{H})$ and f is a non-negative convex function on $[0,\infty)$, then we have*

$$\left\|f\left(\frac{S+T}{2}\right)\right\|\leq\left\|\frac{f(S)+f(T)}{2}\right\|. \tag{2.8}$$

Lemma 2.4. *If $u,v\in\mathcal{H}$ and $0\leq\xi\leq1$, then we have*

$$|\langle u,v\rangle|^2\leq(1-\xi)|\langle u,v\rangle|\,\|u\|\,\|v\|+\xi\|u\|^2\|v\|^2\leq\|u\|^2\|v\|^2. \tag{2.9}$$

Lemma 2.5. *Let $u,v\in\mathcal{H}$. Then*

$$|\langle u,v\rangle|\leq(1-\xi)\sqrt{|\langle u,v\rangle|\,\|u\|\,\|v\|}+\xi\|u\|\,\|v\|\leq\|u\|\,\|v\|. \tag{2.10}$$

The next finding expands and clarifies Kato's inequality (2.3), which in turn expands and clarifies (2.5).

Lemma 2.6. *([4]) If $S\in\mathbb{L}(\mathcal{H}(\Theta))$, $0\leq\xi,r\leq1$ and $\alpha\geq1$, then we have*

$$\begin{aligned}\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^{2\alpha}&\leq\xi\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle\\&+(1-\xi)\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^{\alpha}\sqrt{\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle}\\&\leq\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle.\end{aligned} \tag{2.11}$$

Proof. Let $\tau,\upsilon\in\Theta$ be an arbitrary. By using (2.7), we get

$$\begin{aligned}&\xi\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle\\&+(1-\xi)\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^{\alpha}\sqrt{\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle}\\&\geq\xi\left\langle|S|^{2r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle^{\alpha}\left\langle|S^*|^{2(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle^{\alpha}\\&+(1-\xi)\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^{\alpha}\sqrt{\left\langle|S|^{2r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle^{\alpha}\left\langle|S^*|^{2(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle^{\alpha}}\\&=\xi\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^{2\alpha}+(1-\xi)\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^{\alpha}\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^{\alpha}\\&=\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^{2\alpha}\end{aligned} \tag{2.12}$$

for every $0\leq\xi\leq1$ and $\alpha\geq1$. As opposed to that, we get

$$\begin{aligned}&\xi\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle\\&+(1-\xi)\left|\left\langle S\widehat{k}_\tau,\widehat{k}_\upsilon\right\rangle\right|^{\alpha}\sqrt{\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle}\\&\leq\xi\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle\\&+(1-\xi)\sqrt{\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle}\sqrt{\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle}\\&=\xi\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle+(1-\xi)\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle\\&=\left\langle|S|^{2\alpha r}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\left\langle|S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon,\widehat{k}_\upsilon\right\rangle.\end{aligned} \tag{2.13}$$

Combining (2.12) and (2.13), we deduce that

$$\begin{aligned}\left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^{2\alpha} &\leq \xi\left\langle |S|^{2\alpha r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\left\langle |S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon, \widehat{k}_\upsilon\right\rangle \\ &\quad + (1-\xi)\left|\left\langle S\widehat{k}_\tau, \widehat{k}_\upsilon\right\rangle\right|^{\alpha}\sqrt{\left\langle |S|^{2\alpha r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\left\langle |S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon, \widehat{k}_\upsilon\right\rangle} \\ &\leq \left\langle |S|^{2\alpha r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\left\langle |S^*|^{2\alpha(1-r)}\widehat{k}_\upsilon, \widehat{k}_\upsilon\right\rangle.\end{aligned}$$

■

3. Main Results

Now, our refined Berezin radius inequality could be presented like this:

Theorem 3.1. *If $X, Y \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \leq \xi \leq 1$ and $\alpha \geq 1$, then we have*

$$\begin{aligned}\mathrm{ber}^{2\alpha}(Y^*X) &\leq (1-\xi)\,\mathrm{ber}^{\alpha}(Y^*X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}} + \frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}} \\ &\leq \frac{1}{2}\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}.\end{aligned} \tag{3.1}$$

Proof. Assume that $\widehat{k}_\tau \in \mathcal{H}$ is a normalized reproducing kernel. If we take $u = X\widehat{k}_\tau$ and $v = Y\widehat{k}_\tau$ in the inequality in (2.9), then we have

$$\begin{aligned}\left|\left\langle X\widehat{k}_\tau, Y\widehat{k}_\tau\right\rangle\right|^2 &\leq \left|\left\langle Y^*X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^2 \\ &\leq (1-\xi)\left|\left\langle X\widehat{k}_\tau, Y\widehat{k}_\tau\right\rangle\right|\left\|X\widehat{k}_\tau\right\|\left\|Y\widehat{k}_\tau\right\| + \xi\left\|X\widehat{k}_\tau\right\|^2\left\|Y\widehat{k}_\tau\right\|^2 \\ &= (1-\xi)\left|\left\langle Y^*X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|\left\langle |X|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}}\left\langle |Y|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \\ &\quad + \xi\left\langle |X|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\left\langle |Y|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle.\end{aligned}$$

Employing the PM inequality (2.6), we get

$$\begin{aligned}\left|\left\langle Y^*X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^2 \leq \Bigg((1-\xi)&\left|\left\langle Y^*X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\alpha}\left\langle |X|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{\alpha}{2}}\left\langle |Y|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{\alpha}{2}} \\ &+\xi\left\langle |X|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\alpha}\left\langle |Y|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\alpha}\Bigg)^{\frac{1}{\alpha}},\end{aligned}$$

which implies that

$$\begin{aligned}&\left|\left\langle Y^*X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2\alpha} \\ &\leq (1-\xi)\left|\left\langle Y^*X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\alpha}\left\langle |X|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{\alpha}{2}}\left\langle |Y|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{\alpha}{2}} + \xi\left\langle |X|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\alpha}\left\langle |Y|^2\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\alpha} \\ &\leq (1-\xi)\left|\left\langle Y^*X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\alpha}\left\langle |X|^{2\alpha}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}}\left\langle |Y|^{2\alpha}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} + \xi\left\langle |X|^{2\alpha}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\left\langle |Y|^{2\alpha}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\end{aligned}$$

(by the inequality (2.7))

$$\le \frac{1}{2}(1-\xi)\left|\left\langle Y^*X\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^{\alpha}\left(\left\langle |X|^{2\alpha}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\left\langle |Y|^{2\alpha}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right)$$
$$+\frac{1}{2}\xi\left(\left\langle |X|^{4\alpha}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\left\langle |Y|^{4\alpha}\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right)$$
(by the inequality (2.6))
$$=\frac{1}{2}(1-\xi)\left|\left\langle Y^*X\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^{\alpha}\left\langle\left(|X|^{2\alpha}+|Y|^{2\alpha}\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle+\frac{1}{2}\xi\left\langle\left(|X|^{4\alpha}+|Y|^{4\alpha}\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle,$$

and

$$\sup_{\tau\in\Theta}\left|\left\langle Y^*X\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^{2\alpha}\le\frac{1}{2}(1-\xi)\sup_{\tau\in\Theta}\left\{\left|\left\langle Y^*X\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right|^{\alpha}\left\langle\left(|X|^{2\alpha}+|Y|^{2\alpha}\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle\right\}$$
$$+\frac{1}{2}\xi\sup_{\tau\in\Theta}\left\langle\left(|X|^{4\alpha}+|Y|^{4\alpha}\right)\widehat{k}_\tau,\widehat{k}_\tau\right\rangle.$$

Therefore, we have

$$\mathrm{ber}^{2\alpha}(Y^*X)\le(1-\xi)\,\mathrm{ber}^{\alpha}(Y^*X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}.$$

The desired first inequality is therefore obtained in (3.1). Nonetheless, from the inequalities (1.5) and (2.8), we get

$$\begin{aligned}\mathrm{ber}^{2\alpha}(Y^*X)&\le\frac{1}{2}(1-\xi)\,\mathrm{ber}^{\alpha}(Y^*X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&\le\frac{1}{2}(1-\xi)\left(\frac{1}{2}\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}\right)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}\\
&+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&=\frac{1}{4}(1-\xi)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}^2+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&\le\frac{1}{4}(1-\xi)\left\|\left(\frac{|X|^{2\alpha}+|Y|^{2\alpha}}{2}\right)^2\right\|_{\mathrm{ber}}+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&\le\frac{1}{4}(1-\xi)\left\|\left(\frac{\left(2|X|^{2\alpha}\right)^2+\left(2|Y|^{2\alpha}\right)^2}{2}\right)\right\|_{\mathrm{ber}}+\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}\\
&\le\frac{1}{2}\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}},\end{aligned}$$

which demonstrates the second inequality in (3.1). ■

The next outcome is much better than the inequalities (3.1).

Theorem 3.2. *If $X,Y\in\mathbb{L}(\mathcal{H}(\Theta))$, $\alpha\ge1$ and $\xi\in[0,1]$, then we get*

$$\begin{aligned}\mathrm{ber}^{2r}(Y^*X)&\le\frac{1}{4}\xi\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}^2+\frac{1}{2}(1-\xi)\,\mathrm{ber}^{\alpha}(X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}\\
&\le\frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}^2+\frac{1}{2}(1-\xi)\,\mathrm{ber}^{\alpha}(X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\mathrm{ber}}\\
&\le\frac{1}{2}\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\mathrm{ber}}.\end{aligned}\tag{3.2}$$

Proof. Let $\tau \in \Theta$ be an arbitrary. Then for all $\xi \in [0,1]$, we have

$$\begin{aligned}\operatorname{ber}^{2\alpha}(Y^*X) &\le \xi \operatorname{ber}^{2\alpha}(Y^*X) + (1-\xi)\operatorname{ber}^{2\alpha}(Y^*X)\\ &= \xi \operatorname{ber}^{2\alpha}(Y^*X) + (1-\xi)\operatorname{ber}^{\alpha}(Y^*X)\operatorname{ber}^{\alpha}(Y^*X)\\ &\le \frac{1}{4}\xi\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|^2 + \frac{1}{2}(1-\xi)\operatorname{ber}^{\alpha}(X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|\\ &\quad \text{(by the inequalities (1.5)),}\end{aligned}$$

which proves the first inequality in (3.2). From the inequalities (2.8),

$$\begin{aligned}\operatorname{ber}^{2\alpha}(X) &\le \frac{1}{4}\xi\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\text{ber}}^2 + \frac{1}{2}(1-\xi)\operatorname{ber}^{\alpha}(X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\text{ber}}\\ &= \frac{1}{4}\xi\left\|\left(\frac{\left(2|X|^{2\alpha}\right)+\left(2|Y|^{2\alpha}\right)}{2}\right)^2\right\|_{\text{ber}} + \frac{1}{2}(1-\xi)\operatorname{ber}^{\alpha}(X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\text{ber}}\\ &\le \frac{1}{4}\xi\left\|\frac{\left(2|X|^{2\alpha}\right)^2+\left(2|Y|^{2\alpha}\right)^2}{2}\right\|_{\text{ber}} + \frac{1}{2}(1-\xi)\operatorname{ber}^{\alpha}(X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\text{ber}}\\ &= \frac{1}{2}\xi\left\||X|^{4\alpha}+|Y|^{4\alpha}\right\|_{\text{ber}} + \frac{1}{2}(1-\xi)\operatorname{ber}^{\alpha}(X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\text{ber}}\end{aligned}$$

provides the second inequality in (3.2). The third disparity in comes as a result of (3.1). ■

By taking $\alpha = 1$ and $\xi = \frac{1}{3}$ in (3.2), the outcome is as follows.

Corollary 3.3. *If $X, Y \in \mathbb{L}(\mathcal{H}(\Theta))$, then we have*

$$\begin{aligned}\operatorname{ber}^{2}(Y^*X) &\le \frac{1}{12}\left\||X|^{2}+|Y|^{2}\right\|_{\text{ber}}^2 + \frac{1}{3}\operatorname{ber}(X)\left\||X|^{2}+|Y|^{2}\right\|_{\text{ber}}\\ &\le \frac{1}{6}\left\||X|^{4}+|Y|^{4}\right\|_{\text{ber}}^2 + \frac{1}{3}\operatorname{ber}(X)\left\||X|^{2}+|Y|^{2}\right\|_{\text{ber}}\\ &\le \frac{1}{2}\left\||X|^{4}+|Y|^{4}\right\|_{\text{ber}}.\end{aligned}$$

Theorem 3.4. *If $X, Y \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \le \xi \le 1$ and $\alpha \ge 1$, then we have*

$$\begin{aligned}\operatorname{ber}^{\alpha}(Y^*X) &\le \frac{1}{\sqrt{2}}(1-\xi)\operatorname{ber}^{\frac{\alpha}{2}}(Y^*X)\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\text{ber}}^{\frac{1}{2}} + \frac{1}{2}\xi\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\text{ber}}\\ &\le \frac{1}{2}\left\||X|^{2\alpha}+|Y|^{2\alpha}\right\|_{\text{ber}}.\end{aligned}$$

Proof. Assume that $\widehat{k}_\tau \in \mathcal{H}$ is a normalized reproducing kernel. We determine the desired inequality by entering $u = X\widehat{k}_\tau$ and $v = Y\widehat{k}_\tau$ in (2.10) and continuing as in the argument of Theorem 3.1. ■

Theorem 3.5. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \le r, \xi \le 1$ and $\varsigma \ge 1$, then we have*

$$\begin{aligned}\operatorname{ber}^{2\varsigma}(X) &\le \xi\left\|r|X|^{2\varsigma}+(1-r)|X^*|^{2\varsigma}\right\|_{\text{ber}}\\ &\quad + \frac{1}{2}(1-\xi)\operatorname{ber}^{\varsigma}(X)\left\||X|^{2r\varsigma}+|X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}}. \qquad (3.3)\end{aligned}$$

Proof. Assume.that $\tau \in \Theta$ is an arbitrary. If we take $\tau = \upsilon$ in the inequality (2.11), then we get

$$\begin{aligned}
\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|^{2\varsigma} &\le \xi \left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \\
&+ (1-\xi) \left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|^{\varsigma} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle} \\
&\le \xi \left\langle |X|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle^{r} \left\langle |X^*|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle^{(1-r)} \\
&\text{(by the inequality (2.7))} \\
&+ (1-\xi) \left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|^{\varsigma} . \left(\frac{1}{2} \left\langle |X|^{2\varsigma r} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle + \left\langle |X^*|^{2\varsigma(1-r)} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right) \\
&\text{(bythe inequality (2.7))} \\
&\le \xi \left[r \left\langle |X|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle + (1-r) \left\langle |X^*|^{2\varsigma} \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \right] \\
&\text{(by the inequality (2.6))} \\
&+ \frac{1}{2}(1-\xi) \left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|^{\varsigma} . \left(\left\langle \left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right) \\
&\le \xi \left\langle \left(r\,|X|^{2\varsigma} + (1-r)\,|X^*|^{2\varsigma} \right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \\
&+ \frac{1}{2}(1-\xi) \left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|^{\varsigma} . \left(\left\langle \left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right),
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\tau\in\Theta} \left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|^{2\varsigma} &\le \xi \sup_{\tau\in\Theta} \left\langle \left(r\,|X|^{2\varsigma} + (1-r)\,|X^*|^{2\varsigma} \right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle \\
&+ \frac{1}{2}(1-\xi) \sup_{\tau\in\Theta} \left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau \right\rangle\right|^{\varsigma} \left\langle \left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right) \widehat{k}_\tau, \widehat{k}_\tau \right\rangle .
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\mathrm{ber}^{2\varsigma}(X) &\le \xi \left\| r\,|X|^{2\varsigma} + (1-r)\,|X^*|^{2\varsigma} \right\|_{\mathrm{ber}} \\
&+ \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\varsigma}(X) \left\| |X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma} \right\|_{\mathrm{ber}}
\end{aligned}$$

which the required result. ■

In [24, Th. 3.3], it is proved that

$$\mathrm{ber}^{2\varsigma}(X) \le \frac{1}{2} \left\| \xi\,|X|^{2\xi\varsigma} + (1-\xi)\,|X^*|^{2\varsigma} \right\|_{\mathrm{ber}}, \; 0 < \xi < 1, \varsigma \ge 1. \tag{3.4}$$

The next finding is stronger than the disparity (3.4).

Theorem 3.6. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \le r, \xi \le 1$ and $\varsigma \ge 1$, then we have*

$$\begin{aligned}
\mathrm{ber}^{2\varsigma}(X) &\le \xi \left\| r\,|X|^{2\varsigma} + (1-r)\,|X^*|^{2\varsigma} \right\|_{\mathrm{ber}} \\
&+ (1-\xi)\,\mathrm{ber}^{\varsigma}(X) \sqrt{\left\| r\,|X|^{2\varsigma} + (1-r)\,|X^*|^{2\varsigma} \right\|_{\mathrm{ber}}} \qquad (3.5) \\
&\le \left\| r\,|X|^{2\varsigma} + (1-r)\,|X^*|^{2\varsigma} \right\|_{\mathrm{ber}} .
\end{aligned}$$

Proof. Assume.that $\tau \in \Theta$ is an arbitrary. If we take $\tau = \upsilon$ in the inequality (2.11), then we get

$$\begin{aligned}\left|\left\langle X\widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle\right|^{2\varsigma} &\leq \xi \left\langle |X|^{2\varsigma r} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle \left\langle |X^{*}|^{2\varsigma(1-r)} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle \\ &\quad + (1-\xi) \left|\left\langle X\widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle\right|^{\varsigma} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle \left\langle |X^{*}|^{2\varsigma(1-r)} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle} \\ &\leq \xi \left\langle |X|^{2\varsigma} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle^{r} \left\langle |X^{*}|^{2\varsigma} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle^{(1-r)} \\ &\quad \text{(by the inequality (2.7))} \\ &\quad + (1-\xi) \left|\left\langle X\widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle\right|^{\varsigma} . \frac{1}{2} \left(\left\langle |X|^{2\varsigma r} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle + \left\langle |X^{*}|^{2\varsigma(1-r)} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle\right) \\ &\leq \xi \left\langle \left(r\,|X|^{2\varsigma} + (1-r)\,|X^{*}|^{2\varsigma}\right) \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle \\ &\quad + (1-\xi) \left|\left\langle X\widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle\right|^{\varsigma} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle \left\langle |X^{*}|^{2\varsigma(1-r)} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle} \\ &\quad \text{(by the inequality (2.6))}\end{aligned}$$

and

$$\begin{aligned}\sup_{\tau\in\Theta} \left|\left\langle X\widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle\right|^{2\varsigma} &\leq \xi \sup_{\tau\in\Theta} \left\langle \left(r\,|X|^{2\varsigma} + (1-r)\,|X^{*}|^{2\varsigma}\right) \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle \\ &\quad + (1-\xi) \sup_{\tau\in\Theta} \left|\left\langle X\widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle\right|^{\varsigma} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle \left\langle |X^{*}|^{2\varsigma(1-r)} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle}.\end{aligned}$$

So, we deduce

$$\begin{aligned}\mathrm{ber}^{2\varsigma}(X) &\leq \xi \left\| r\,|X|^{2\varsigma} + (1-r)\,|X^{*}|^{2\varsigma}\right\|_{\mathrm{ber}} \\ &\quad + (1-\xi)\,\mathrm{ber}^{\varsigma}(X) \sqrt{\left\| |rX|^{2\varsigma} + (1-r)\,|X^{*}|^{2\varsigma}\right\|_{\mathrm{ber}}}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{ber}^{2\varsigma}(X) &\leq \xi \left\| r\,|X|^{2\varsigma} + (1-r)\,|X^{*}|^{2\varsigma}\right\|_{\mathrm{ber}} \\ &\quad + (1-\xi)\,\mathrm{ber}^{\varsigma}(X) \sqrt{\left\| |rX|^{2\varsigma} + (1-r)\,|X^{*}|^{2\varsigma}\right\|_{\mathrm{ber}}} \\ &\leq \left\| r\,|X|^{2\varsigma} + (1-r)\,|X^{*}|^{2\varsigma}\right\|_{\mathrm{ber}} \\ &\quad \text{(by the inequality (3.4))}\end{aligned}$$

allows us to deduce the second inequality from the first inequality, demonstrating the required result. ∎

Theorem 3.7. *If* $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \leq r, \xi \leq 1$ *and* $\varsigma \geq 1$, *then we have*

$$\mathrm{ber}^{\varsigma}(X) \leq \frac{1}{2}\xi \left\| |X|^{2\varsigma r} + |X^{*}|^{2\varsigma(1-r)}\right\|_{\mathrm{ber}} + \frac{1}{\sqrt{2}} (1-\xi)\,\mathrm{ber}^{\frac{\varsigma}{2}}(X) \left\| |X|^{2\varsigma r} + |X^{*}|^{2\varsigma(1-r)}\right\|_{\mathrm{ber}}^{1/2}. \tag{3.6}$$

Proof. Suppose that $\tau, \upsilon \in \Theta$ is an arbitrary. One may see from the inequality (2.12) and (2.13) that

$$\begin{aligned}\left|\left\langle X\widehat{k}_{\tau}, \widehat{k}_{\upsilon}\right\rangle\right|^{\varsigma} &\leq \xi \left\langle |X|^{2\varsigma r} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle^{\frac{1}{2}} \left\langle |X^{*}|^{2\varsigma(1-r)} \widehat{k}_{\upsilon}, \widehat{k}_{\upsilon}\right\rangle^{\frac{1}{2}} \qquad (3.7) \\ &\quad + (1-\xi) \left|\left\langle X\widehat{k}_{\tau}, \widehat{k}_{\upsilon}\right\rangle\right|^{\frac{\varsigma}{2}} \sqrt{\left\langle |X|^{2\varsigma r} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle^{\frac{1}{2}} \left\langle |X^{*}|^{2\varsigma(1-r)} \widehat{k}_{\upsilon}, \widehat{k}_{\upsilon}\right\rangle^{\frac{1}{2}}} \\ &\leq \left\langle |X|^{2\varsigma r} \widehat{k}_{\tau}, \widehat{k}_{\tau}\right\rangle^{\frac{1}{2}} \left\langle |X^{*}|^{2\varsigma(1-r)} \widehat{k}_{\upsilon}, \widehat{k}_{\upsilon}\right\rangle^{\frac{1}{2}}\end{aligned}$$

for every $\varsigma \geq 1$ and $0 \leq r, \xi \leq 1$. Setting $\tau = \upsilon$ in the above inequality, it follows that

$$\begin{aligned}
\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^\varsigma &\leq \xi \left\langle |X|^{2\varsigma r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \left\langle |X^*|^{2\varsigma(1-r)}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \\
&+ (1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma/2} \sqrt{\left\langle |X|^{2\varsigma r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{\frac{1}{2}} \left\langle |X^*|^{2\varsigma(1-r)}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^{1/2}} \\
&\leq \frac{1}{2}\xi\left(\left\langle |X|^{2\varsigma r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left\langle |X^*|^{2\varsigma(1-r)}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right) \\
&+ \frac{1}{\sqrt{2}}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma/2} \sqrt{\left(\left\langle |X|^{2\varsigma r}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left\langle |X^*|^{2\varsigma(1-r)}\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right)} \\
&\text{(by the inequality (2.6))} \\
&\leq \frac{1}{2}\xi\left\langle\left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&+ \frac{1}{\sqrt{2}}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma/2} \sqrt{\left\langle\left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle}
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\operatorname{ber}^\varsigma(X) &\leq \frac{1}{2}\xi\left\||X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right\|_{\text{ber}} \\
&+ \frac{1}{\sqrt{2}}(1-\xi)\operatorname{ber}^{\frac{\varsigma}{2}}(X)\left\||X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right\|_{\text{ber}}^{1/2}.
\end{aligned}$$

The evidence is now complete. ■

From [24, Th. 3.2], it is evident that

$$\operatorname{ber}^\varsigma(X) \leq \frac{1}{2}\left\||X|^{2\xi\varsigma} + |X^*|^{2(1-\xi)\varsigma}\right\|_{\text{ber}} \tag{3.8}$$

if $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 < \xi < 1$and $\varsigma \geq 1$.

The implication that follows demonstrates that our finding (3.6) is more powerful than the inequality (3.8).

Corollary 3.8. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \leq r, \xi \leq 1$ and $\varsigma \geq 1$, then we have*

$$\begin{aligned}
\operatorname{ber}^\varsigma(X) &\leq \frac{1}{2}\xi\left\||X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}} \\
&+ \frac{1}{\sqrt{2}}(1-\xi)\operatorname{ber}^{\frac{\varsigma}{2}}(X)\left\||X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}}^{1/2} \\
&\leq \frac{1}{2}\left\||X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}}.
\end{aligned}$$

Proof. Assume that $\tau, \upsilon \in \Theta$ is an arbitrary. From (3.6), we get

$$\begin{aligned}
\operatorname{ber}^\varsigma(X) &\leq \frac{1}{2}\xi\left\||X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}} \\
&+ \frac{1}{\sqrt{2}}(1-\xi)\operatorname{ber}^{\frac{\varsigma}{2}}(X)\left\||X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}}^{1/2} \\
&\leq \frac{1}{2}\xi\left\||X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}} + \frac{1}{2}(1-\xi)\left\||rX|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}} \\
&\text{(by the inequality (3.8))} \\
&= \frac{1}{2}\left\||X|^{2r\varsigma} + |X^*|^{2(1-r)\varsigma}\right\|_{\text{ber}},
\end{aligned}$$

as required. ■

Theorem 3.9. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \le r, \xi \le 1$ and $\varsigma \ge 1$, then we have*

$$\mathrm{ber}^{2\varsigma}(X) \le \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\varsigma}(X)\left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\mathrm{ber}} + \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\mathrm{ber}}. \tag{3.9}$$

Proof. Assume that $\widehat{k}_\tau \in \mathcal{H}$ is a normalized reproducing kernel. If we take $\tau = \upsilon$ in the inequality (2.11), then we get

$$\begin{aligned}
\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{2\varsigma} &\le \xi \left\langle |X|^{2\varsigma r}\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle |X^*|^{2\varsigma(1-r)}\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\quad + (1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} \sqrt{\left\langle |X|^{2\varsigma r}\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \left\langle |X^*|^{2\varsigma(1-r)}\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle} \\
&\le \frac{1}{2}\xi \left(\left\langle |X|^{2\varsigma r}\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^2 + \left\langle |X^*|^{2\varsigma(1-r)}\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle^2\right) \\
&\quad \text{(by the inequality (2.6))} \\
&\quad + \frac{1}{2}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} . \left\langle \left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\quad \text{(by the inequality (2.6))} \\
&= \frac{1}{2}\xi \left(\left\langle |X|^{4\varsigma r}\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle + \left\langle |X^*|^{4\varsigma(1-r)}\,\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right) \\
&\quad \text{(by the inequality (2.7))} \\
&\quad + \frac{1}{2}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} . \left\langle \left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&= \frac{1}{2}\xi \left\langle \left(|X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\quad + \frac{1}{2}(1-\xi)\left|\left\langle X\widehat{k}_\tau, \widehat{k}_\tau\right\rangle\right|^{\varsigma} \left\langle \left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
\sup_{\tau\in\Theta}\left|\widetilde{X}(\tau)\right|^{2\varsigma} &\le \frac{1}{2}\xi \sup_{\tau\in\Theta}\left\langle \left(|X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle \\
&\quad + \frac{1}{2}(1-\xi)\sup_{\tau\in\Theta}\left|\widetilde{X}(\tau)\right|^{\varsigma} \left\langle \left(|X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)}\right)\widehat{k}_\tau, \widehat{k}_\tau\right\rangle .
\end{aligned}$$

Hence we get

$$\begin{aligned}
\mathrm{ber}^{2\varsigma}(X) &\le \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\mathrm{ber}} \\
&\quad + \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\varsigma}(X)\left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\mathrm{ber}},
\end{aligned}$$

and the proof is complete. ∎

Corollary 3.10. *If $X \in \mathbb{L}(\mathcal{H}(\Theta))$, $0 \le r, \xi \le 1$ and $\varsigma \ge 1$, then we have*

$$\begin{aligned}
\mathrm{ber}^{2\varsigma}(X) &\le \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\mathrm{ber}} \\
&\quad + \frac{1}{2}(1-\xi)\,\mathrm{ber}^{\varsigma}(X)\left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\mathrm{ber}} \\
&\le \frac{1}{2}\left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\mathrm{ber}} .
\end{aligned} \tag{3.10}$$

Proof. Assume that $\tau \in \Theta$ is arbitrary. From (3.9), we get

$$
\begin{aligned}
\operatorname{ber}^{2\varsigma}(X) &\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{2}(1-\xi)\operatorname{ber}^{\varsigma}(X) \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}} \\
&\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{4}(1-\xi) \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}}^{2}
\end{aligned}
$$

(by the inequality (3.8))

$$
\begin{aligned}
&\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{4}(1-\xi) \left\| \left(\frac{2|X|^{2\varsigma r} + 2|X^*|^{2\varsigma(1-r)}}{2} \right)^2 \right\|_{\text{ber}} \\
&\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{8}(1-\xi) \left\| \left(2|X|^{2\varsigma r}\right)^2 + \left(2|X^*|^{2\varsigma(1-r)}\right)^2 \right\|_{\text{ber}}
\end{aligned}
$$

(by the inequality (2.8))

$$
\begin{aligned}
&\leq \frac{1}{2}\xi \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} + \frac{1}{2}(1-\xi) \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}} \\
&\leq \frac{1}{2} \left\| |X|^{4\varsigma r} + |X^*|^{4\varsigma(1-r)} \right\|_{\text{ber}}.
\end{aligned}
$$

We determine the desired disparity (3.10). ∎

We utilize the inequalities (3.4) and (3.8) for every $X \in \mathcal{B}(\mathcal{H})$, $0 \leq r, \xi \leq 1$ and $\varsigma \geq 1$. In fact, after applying (2.8), we obtain

$$
\begin{aligned}
\operatorname{ber}^{2\varsigma}(X) &= \xi \operatorname{ber}^{2\varsigma}(X) + (1-\xi)\operatorname{ber}^{2\varsigma}(X) \\
&= \xi \operatorname{ber}^{2\varsigma}(X) + (1-\xi)\operatorname{ber}^{\varsigma}(X)\operatorname{ber}^{\varsigma}(X) \\
&= \frac{1}{4}\xi \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}} \\
&+ \frac{1}{2}(1-\xi)\operatorname{ber}^{\varsigma}(X) \left\| |X|^{2\varsigma r} + |X^*|^{2\varsigma(1-r)} \right\|_{\text{ber}},
\end{aligned}
$$

which of course refines (3.9). In instance, we obtain

$$
\operatorname{ber}^{2}(X) \leq \frac{1}{12} \left\| |X| + |X^*| \right\|_{\text{ber}}^{2} + \frac{1}{3}\operatorname{ber}(X) \left\| |X| + |X^*| \right\|_{\text{ber}}
$$

for $\varsigma = 1$, $r = \frac{1}{2}$ and $\xi = \frac{1}{3}$. It follows from Theorem 3.1 in [24] that if $X \in \mathbb{L}(\mathcal{H}(\Theta))$ then we have

$$
\operatorname{ber}(X) \leq \frac{1}{2} \left\| |X| + |X^*| \right\|_{\text{ber}} \leq \frac{1}{2}\left(\|X\|_{\text{ber}} + \left\|X^2\right\|_{\text{ber}}^{1/2} \right). \tag{3.11}
$$

So, from (3.11), we can deduce the inequality

$$
\begin{aligned}
\operatorname{ber}^{2}(X) &\leq \frac{1}{12} \left\| |X| + |X^*| \right\|_{\text{ber}}^{2} + \frac{1}{3}\operatorname{ber}(X) \left\| |X| + |X^*| \right\|_{\text{ber}} \\
&= \frac{1}{12} \left\| |X| + |X^*| \right\|_{\text{ber}}^{2} + \frac{1}{3}\left(\frac{1}{2} \left\| |X| + |X^*| \right\|_{\text{ber}} \right) \left\| |X| + |X^*| \right\|_{\text{ber}} \\
&= \frac{1}{12} \left\| |X| + |X^*| \right\|_{\text{ber}}^{2} + \frac{1}{6} \left\| |X| + |X^*| \right\|_{\text{ber}}^{2} \\
&= \frac{1}{4} \left\| |X| + |X^*| \right\|_{\text{ber}}^{2},
\end{aligned}
$$

which indeed refines (3.11). Thus, we have

$$\begin{aligned}
\mathrm{ber}^2(X) &\leq \frac{1}{12}\left\||X|+|X^*|\right\|_{\mathrm{ber}}^2+\frac{1}{3}\mathrm{ber}(X)\left\||X|+|X^*|\right\|_{\mathrm{ber}} \\
&= \frac{1}{12}\left\|\left(\frac{2|X|+2|X^*|}{2}\right)^2\right\|_{\mathrm{ber}}+\frac{1}{3}\mathrm{ber}(X)\left\||X|+|X^*|\right\|_{\mathrm{ber}} \\
&\leq \frac{1}{24}\left\|(2|X|)^2+(2|X^*|)^2\right\|_{\mathrm{ber}}+\frac{1}{3}\mathrm{ber}(X)\left\||X|+|X^*|\right\|_{\mathrm{ber}} \\
&\text{(by the inequality (2.8))} \\
&= \frac{1}{6}\left\||X|+|X^*|\right\|_{\mathrm{ber}}^2+\frac{1}{3}\mathrm{ber}(X)\left\||X|+|X^*|\right\|_{\mathrm{ber}},
\end{aligned}$$

which the inequality in (1.6), as required.

We recommend [8, 16–19, 22–24] for more recent findings on Berezin radius inequalities for operators and related findings.

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