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On approximation of fixed point in Busemann space via generalized Picard normal s-iteration process

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Abstract
This paper deals with strong as well as $\Delta-$convergence results for SKC map in Busemann space via generalized Picard normal s-iteration process. We design an example for the Suzuki-Karapinar conditioned mapping in this paper. Also we discuss generalized Picard normal s-iteration process is faster than some famous iteration processes. An numerical example is presented in this paper to support our result.

Keywords
Busemann Space, $\Delta-$convergence, strong convergence, SKC map, iteration method.

AMS Subject Classification
47H09, 47H10.

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1. Introduction

Busemann [3] developed the theory of non-positive curvature for pathmetric space. Then the class of geodesic for curvature non-positivity in convex space introduced by Gromov [10]. The term "Busemann" first time used by Bowditch [2]. Foertsch et al. [5] proved that if CAT(0) spaces satisfies Ptolemy condition, then CAT(0) is Busemann space. The examples of Busemann space are uniformly convex Banach space such as L_p, l_p and W_p^m for $1 < p < \infty$ and Minkowski space and strictly convex Banach spaces and simply connected Riemannian manifolds of nonpositive sectional curvature are examples of the Busemann space. The hyperbolic space appeared for denoting Busemann convex space in [23].

On the other hand, many researchers designed iterative process for approximating fixed point of nonexpansive map and for some class of nonexpansive map (see [1, 12, 16, 19, 20, 25]) in last fifty years and compare which iteration process is faster.

Let Y be a normed space and M be a nonempty subset of Y. The mapping $S : M \to M$ is said to be
(1) **nonexpansive**, if $\|St - Su\| \leq \|t - u\|$ for all $t, u \in M$.
(2) **quasi-nonexpansive**, if $\|St - p\| \leq \|t - p\|$ for all $t \in M$ and $p \in F(S)$, where $F(S)$ denotes the set of all fixed point of S, i.e.,

$$F(S) = \{t : St = t, t \in M\}.$$

Many nonlinear problem can be written as fixed point problem

$$t = St,$$

where fixed point map S may be nonlinear. The solution of such problem is called a fixed point of the map S.

The fixed point iteration can be defined by

$$t_{n+1} = St_n \quad \forall n \in N. \tag{1.1}$$

The iterative process (1.1) is known as Picard iteration, method of successive substitution or Richardson iteration. Picard iteration for Banach contraction mapping converges to unique fixed point of S, but this iteration fails for approximating fixed point for nonexpansive map. On the other hand, this iteration method enable us to identify existence of fixed point of S.

In 1953, Mann [19] introduced an iteration process : Assume that $t_0 \in M$ is arbitrary. The $\{t_n\}$ as follows

$$t_{n+1} = \alpha_n t_n + (1-\alpha_n)St_n, \quad n \geq 0, \tag{1.2}$$

where $\{\alpha_n\}$ is real sequence in the interval $(0,1)$. The iteration $\{t_n\}$ is known as Mann iteration process.

Assume that M is convex subset of normed nonlinear space Y and $S: M \to M$ is a map. Assume that $t_0 \in M$ is arbitrary. The following sequence $\{t_n\}$ in M defined as follows

$$\begin{cases} t_{n+1} = Su_n, \\ u_n = (1-\alpha_n)t_n + \alpha_n St_n, \quad n \in N \ . \end{cases} \tag{1.3}$$

$$\begin{cases} t_{n+1} = Su_n, \\ u_n = (1-\alpha_n)v_n + \alpha_n Sv_n, \\ v_n = (1-\beta_n)t_n + \beta_n St_n, \quad n \in N \ . \end{cases} \tag{1.4}$$

$$\begin{cases} t_{n+1} = Tu_n, \\ u_n = (1-\alpha_n)v_n + \alpha_n Sv_n, \\ v_n = (1-\beta_n)w_n + \beta_n Sw_n, \\ w_n = (1-\gamma_n)t_n + \gamma_n St_n, \quad n \in N \ . \end{cases} \tag{1.5}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequence in $(0,1)$.

The iteration process (1.3), (1.4) and (1.5) are known as normal s-iteration, Picard normal s-iteration and generalized Picard normal s-iteration process, respectively and given by Sahu [24], Kadioglu and Yildirim [13] and Dashputre et al. [6] respectively.

In this paper, we will show generalized Picard normal s-iteration process is faster than Picard normal s-iteration process, normal s-iteration process, Mann iteration process and Picard iteration process in Busemann space. Also we will establish strong as well as $\Delta-$convergence theorems for generalized Picard normal s-iteration process generated by SKC map in uniformly convex Busemann space. We will also discuss the example to support our result with different initial values which ensures the fastness of generalized Picard normal s-iteration process.

2. Preliminaries

Let $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ represent set of all natural numbers, set of nonnegative numbers and set of real numbers, respectively.

Assume that (Y,d) is a metric space. A geodesic path joining $t \in Y$ to $u \in Y$ or a geodesic from t to u is a mapping c from closed interval $[0,l] \subset \mathbb{R}$ to Y such that $c(0) = t, c(l) = u$, and $d(c(a), c(a')) = |a - a'|$ for all $a, a' \in [0,l]$. A map c is isometry if $d(t,u) = l$. The image of c is said to be geodesic (or metric) segment joining the t and u. When it is unique, then this geodesic segment is represented by $[t,u]$. A space (Y,d) is called geodesic space, if every two points of Y can be joined by a geodesic and Y is called uniquely geodesic, if there is only one geodesic joining t and u for each $t,u \in Y$. A subset $X \subseteq Y$ is called convex, if X includes every geodesic segment joining any two of its points. The geodesic triangle $\Delta(t_1,t_2,t_3)$ in the geodesic metric space (Y,d) consists of three points t_1,t_2,t_3 in Y (vertices of Δ) and a geodesic segment between each pair of vertices (the edges of Δ).

Assume that $c : [t,u] \longrightarrow Y$ is path in a metric space Y. The path c is called an affinely reparametrized geodesic, if either c is the constant path or there is a geodesic path $c' : [x,y] \longrightarrow Y$ such that $c = c' o \psi$, where $\psi : [t,u] \longrightarrow [x,y]$ is unique affine homeomorphism between the intervals $[t,u]$ and $[x,y]$ which preserves ratio of distances and collinearity.

Assume that Y is uniquely geodesic space. If $c([t,u])$ is geodesic segment joining a and b and $\eta \in [a,b]$, then $z := c((1-\eta)t + \eta u)$ is unique point in $c([t,u])$ satisfying $d(z,a) = \eta d(a,b)$ and $d(z,b) = (1-\eta)d(a,b)$. Here $[u,v]$ represents geodesic segment $c[t,u]$ and z represents $(1-\eta)t \oplus \eta v$. The subset $M \subseteq Y$ is called geodesically convex, if M contains every geodesic segment joining any two of its point.

Assume that Y is geodesic metric space and $f : Y \longrightarrow \mathbb{R}$ is mapping. If f is convex (strictly convex), if for every geodesic path $c : [t,u] \longrightarrow Y$, $foc : [a,b] \longrightarrow \mathbb{R}$ is convex (strictly convex). If $g : f(X) \longrightarrow \mathbb{R}$ is an increasing convex (strictly convex) and $f : X \longrightarrow \mathbb{R}$ is a convex function, then $gof : X \longrightarrow \mathbb{R}$ is convex (strictly convex).

Definition 2.1. *A geodesic metric space (Y,d) is Busemann space, if for any two affinely repametrized geodesic $c : [t,u] \longrightarrow Y$ and $c' : [t',u'] \longrightarrow Y$, the mapping $D_{c,c'} : [t,u] \times [t',u'] \longrightarrow \mathbb{R}$ defined by*

$$D_{c,c'} = d(c(a), c'(a'))$$

is convex.

Popadopoulos [22] proved that the followings are equivalent to definition 2.1.
For any $t,u,v,w \in Y$ and $\lambda, \lambda' \in [0,1]$, the following conditions hold :
(a) $d(v, (1-\lambda)t \oplus \lambda u) \leq (1-\lambda)d(v,t) \oplus \lambda d(t,u)$,
(b) $d((1-\lambda)t \oplus \lambda u, (1-\lambda')t \oplus \lambda' u) = |\lambda - \lambda' d(t,u)|$,
(c) $(1-\lambda)t \oplus \lambda u = \lambda y \oplus (1-\lambda)t$,
(d) $d((1-\lambda)t \oplus \lambda v, (1-\lambda)u \oplus \lambda w) \leq (1-\lambda)d(t,u) \oplus \lambda d(v,w)$.

Clarkson [4] introduced the notion of uniform convexity in Banach spaces and the term modulus of convexity was coined by Goebel and Reich [9]. The interpretation of these notions, as can be found below, was given by Gelander et al. [7].

Definition 2.2. *A Busemann space (Y,d) is uniformly convex, if for any $s,t,u \in Y$, $\tau > 0$ and $\varepsilon \in (0,2]$, there is a $\delta \in$*

$(0,1]$ *such that* $d\left(\frac{1}{2}t \oplus \frac{1}{2}u, s\right) \leq (1-\delta)\tau$ *whenever* $d(t,s) \leq \tau, d(u,s) \leq \tau$ *and* $d(t,u) \geq \varepsilon\tau$.

Definition 2.3. *The mapping* η *from* $(0,\infty)\times(0,2]$ *to* $(0,1]$ *is known as modulus of continuity, if it provides such a* $\delta = \eta(\tau,\varepsilon)$ *for given* $\tau > 0$ *and* $\varepsilon \in (0,2]$ *and* η *is monotone, if it decreases with* τ *(for a fixed* ε*).*

The following Lemma is an very important property of uniformly convex Busemann space.

Lemma 2.4. *Assume that* (Y,d) *is complete uniformly convex Busemann space with monotone modulus of convexity* η *and* $t \in Y$. *Assume that* $\{a_n\}$ *is sequence in* $[0,1]$ *satisfying* $0 < \liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n < 1$. *If* $\{t_n\}$ *and* $\{u_n\}$ *are sequences in* Y *such that for some* $\tau \geq 0$, $\limsup_{n\to\infty} d(u_n,\tau) \leq \tau$ *and* $d(1-a_n)t_n \oplus a_n u_n, t) = \tau$ *hold. Then* $\lim_{n\to\infty} d(t_n,u_n) = 0$.

Assume that M is nonempty subset of Busemann space (Y,d) and $\{t_n\}$ is any bounded sequence in Y while $diam(M)$ represents the diameter of M.
Define continuous functional $r_a(.,\{t_n\}) : Y \to \mathbb{R}^+$ by

$$r_a(t,\{t_n\}) = \limsup_{n\to\infty} d(t_n,t), \quad t \in Y.$$

The asymptotic center $r(\{t_n\})$ of $\{t_n\}$ is defined by

$$r(\{t_n\}) = \inf_{t\in Y} r(t,\{t_n\}).$$

The asymptotic radius $r_M(\{t_n\})$ of $\{t_n\}$ with respect to M is defined by

$$r_M(\{t_n\}) = \inf_{t\in M} r(t,\{t_n\}).$$

The asymptotic center $A(\{t_n\})$ of $\{t_n\}$ is defined by

$$A(\{t_n\}) = \{t \in Y : r(t,\{t_n\}) = r(\{t_n\})\}.$$

The asymptotic center $AC(\{t_n\})$ of $\{t_n\}$ with respect to M is defined by

$$AC(M,\{t_n\}) = \{t \in Y : r(t,\{t_n\} = r_M(\{t_n\}))\}.$$

The set $AC(M,\{t_n\})$ can be empty, singleton or have infinitely many points. If the asymptotic radius and asymptotic center are taken with respect to X, then they are denoted by $r_a(Y,\{t_n\}) = r_a(\{t_n\})$ and $AC(Y,\{t_n\}) = AC(\{t_n\})$ respectively.

If $\{t_n\}$ converges $t \in M$, then $AC(\{t_n\}) = \{t\}$ and if $\{t_n\}$ converges $t \notin M$, then $r_M(\{t_n\}) = d(t,M)$, where $d(t,M) = \inf_{t\in M} = \inf_{t\in M} d(t,m)$ is the distance of point t from set M and $AC(\{t_n\}) = \{c \in M : d(t,c) = d(t,M) = P_M(t),\}$ where $\{P_M(t)\}$ is set of all closest points to t from set M. It is well known that $\{P_M(t)\}$ is singleton, if M is nonempty closed and convex subset in complete Busemann space [22].

If $\{t_n\}$ is bounded sequence, then map $\varphi : Y \longrightarrow \mathbb{R}$ given by $\varphi(t) = r(t,\{t_n\})$ is a continuous convex function [22, Example 8.4.7]. Furthermore, if M is closed convex subset of a complete uniformly convex Busemann space Y and $\{t_n\}$ is bounded sequence in Y, then there exists a unique point t_0 in M such that $\varphi(t_0) = \inf_{t\in M} \varphi(t)$, i.e., every bounded sequence $\{t_n\}$ in completely convex Busemann space Y has a unique asymptotic center with respect to any nonempty convex closed subset M of Y (see [17, 21]).

Now we consider the following definitions given in [15].

Definition 2.5. *A sequence* $\{t_n\}$ *in* Y *is called to* $\Delta-$*converge to* $t \in Y$, *if for every subsequence* $\{u_n\}$ *of* $\{t_n\}$, t *is unique asymptotic center of* $\{t_n\}$. *We write* $\Delta - \lim_n t_n = t$ *and say* t *is the* $\Delta-$ *limit of* $\{t_n\}$. *Such type of bounded sequence is called regular.*

If $\{t_n\}$ is bounded sequence in Y. Then $\{t_n\}$ is called regular with respect to subset M of Y, if the asymptotic radii of all subsequences of $\{t_n\}$ with respect to M are same.

The following Lemma has been proved in [8] for Banach space and proved for Busemann space in [15].

Lemma 2.6. *Assume that* Y *is a Busemann space. If* M *is a subset of* Y *and* $\{t_n\}$ *is a bounded sequence in* Y. *Then* $\{t_n\}$ *has a subsequence which is regular with respect to* M.

Thus, by Lemma 2.6, it is clear that any bounded sequence has a $\Delta-$convergent subsequence and any convergent sequence is Δ-convergent.

3. Suzuki-Karapinar Conditioned Mappings (SKC Mappings) in Busemann Space

The Suzuki-Karapinar Conditioned Mappings (SKC Mappings) has been introduced in [14].

Definition 3.1. *Assume that* M *is nonempty subset of a Busemann space* (Y,d). *Then a map* $S : M \longrightarrow M$ *is called SKC mapping (Suzuki-Karapinar Conditioned Mapping), if for all* $t,u \in M$

$$\frac{1}{2}d(t,St) \leq d(t,u) \qquad \textit{implies that}$$

$$d(St,Su) \leq \max\left\{d(t,u), \frac{d(t,St)+d(u,Su)}{2}, \frac{d(u,St)+d(t,Su)}{2}\right\}.$$

The following Propositions of [15] are properties of SKC mappings in the Busemann space.

Proposition 3.2. *Assume that* M *is nonempty subset of Busemann space* Y *and* $S : M \longrightarrow M$ *is SKC mapping. Then*

$$d(t,Su) \leq 5d(St,t) + d(t,u)$$

holds for all $t,u \in M$.

Proposition 3.3. *Assume that* M *is nonempty subset of Busemann space* Y. *Then SKC map* $S : M \longrightarrow M$ *is quasi nonexpansive, if set of fixed point of* S *is nonempty.*

Proposition 3.4. *Assume that M is nonempty closed subset of Busemann space Y and $S : M \longrightarrow M$ is SKC map, then the set $F(S)$ of fixed point of S is closed.*

Proposition 3.5. *Assume that M is closed convex subset of complete uniformly convex Busemann space Y and $S : M \longrightarrow M$ is SKC map. If sequence $\{t_n\}$ is Δ-convergent to t and $d(t_n, St_n) \to \infty$ as $n \to \infty$, then $t \in M$ and $St = t$.*

Theorem 3.6. *Assume that M is closed convex subset of complete uniformly convex Busemann space Y and $S : M \longrightarrow M$ be SKC map with $F(S) \neq \phi$. Then $F(S)$ is convex Δ-closed.*

4. Main Results

Consider generalized Picard normal s-iteration process in Busemann space Y as follows :

$$\begin{cases} t_{n+1} = Su_n, \\ u_n = (1-\alpha_n)v_n + \alpha_n Sv_n, \\ v_n = (1-\beta_n)w_n + \beta_n Sw_n, \\ w_n = (1-\gamma_n)t_n + \gamma_n Sx_n, \qquad n \in N \ . \end{cases} \tag{4.1}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in (0,1)$.

Lemma 4.1. *Assume that M is closed convex subset of a complete uniformly convex Busemann space Y and $S : M \longrightarrow M$ is SKC map with $F(S) \neq \phi$. Assume that $t_1 \in M$ and $\{t_n\}$ is defined by (4.1). Then $\lim_{n\to\infty} d(t_n, t)$ exists for all $t \in F(S)$.*

Proof. Let $t \in F(S)$. From Proposition 3.2, we have

$$\begin{aligned} d(t_{n+1}, t) &= d(Su_n, t) \\ &\leq 5d(St, t) + d(u_n, t) \\ &= d(u_n, t) \end{aligned}$$

$$\begin{aligned} d(u_n, t) &= d((1-\alpha_n)v_n + \alpha_n Sv_n), t) \\ &\leq (1-\alpha_n)d(v_n, t) + \alpha_n d(v_n, t) \\ &\leq (1-\alpha_n)d(v_n, t) + \alpha_n(5d(St, t) + d(v_n, t)) \\ &= d(v_n, t) \end{aligned} \tag{4.2}$$

$$\begin{aligned} d(v_n, t) &= d((1-\beta_n)w_n + \beta_n Sw_n), t) \\ &\leq (1-\beta_n)d(w_n, t) + \beta_n d(w_n, t) \\ &\leq (1-\beta_n)d(w_n, t) + \beta_n(5d(St, t) + d(w_n, t)) \\ &= d(w_n, t) \end{aligned} \tag{4.3}$$

Similarly,

$$\begin{aligned} d(w_n, t) &= d((1-\gamma_n)t_n + \gamma_n St_n), t) \\ &\leq (1-\gamma_n)d(t_n, t) + \gamma_n d(St_n, t) \\ &\leq (1-\gamma_n)d(t_n, t) + \gamma_n(5d(St, t) + d(t_n, t)) \\ &= d(t_n, t) \end{aligned} \tag{4.4}$$

Thus, we have

$$d(t_{n+1}, t) \leq d(t_n, t).$$

Therefore $\{d(t_n, t)\}$ is decreasing sequence of positive real numbers and bounded below. Thus $\lim_{n\to\infty} d(t_n, t)$ exists for all $t \in F(S)$. □

The following result is necessary and sufficient condition for existence of fixed point of SKC map in Busemann space.

Theorem 4.2. *Assume that M is closed convex subset of complete uniformly convex Busemann space Y with monotone modulus of uniform convexity and $S : M \longrightarrow M$ is SKC map with $F(S) \neq \phi$. Let $t_1 \in M$ and $\{t_n\}$ is defined by (4.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Then $F(S)$ is nonempty if and only if $\{t_n\}$ is bounded and $\lim_{n\to\infty} d(St_n, t_n) = 0$.*

Proof. Assume that $t \in F(S)$. Then, by Lemma 4.1, $\lim_{n\to\infty} d(t_n, t)$ exists and hence $\{t_n\}$ is bounded.
Suppose that $\lim_{n\to\infty} d(x_n, x) = l$. This implies

$$\lim_{n\to\infty} d(Su_n, t) = l.$$

Now

$$\begin{aligned} d(St_n, t) &\leq 5d(t, St) + d(t_n, t) \\ &= d(t_n, t) = l \\ \lim_{n\to\infty} d(St_n, t) &\leq l \end{aligned}$$

Since, from (4.4), we have $d(w_n, t) \leq d(t_n, t)$. Therefore

$$\lim_{n\to\infty} d(w_n, t) \leq l. \tag{4.5}$$

Now, from(4.2) and (4.3), we have

$$\begin{aligned} d(Su_n, t) &\leq 5d(t, St) + d(u_n, t) \\ &= d(u_n, t) \\ &\leq d(w_n, t) \\ \lim_{n\to\infty} d(Su_n, t) &\leq d(w_n, t) \\ l &\leq d(w_n, t) \end{aligned} \tag{4.6}$$

Thus, from (4.5) and (4.6), we have

$$\lim_{n\to\infty} d(w_n, t) = l.$$

That is, $\lim_{n\to\infty} d(w_n, t) = \lim_{n\to\infty} d((1-\gamma_n)t_n + \gamma_n St_n) = l$.
From Lemma 2.4, we have $\lim_{n\to\infty} d(t_n, St_n) = 0$.

Conversely, assume that the sequence $\{t_n\}$ and $\lim_{n\to\infty} d(t_n, St_n) = 0$. By Lemma 2.6, the sequence $\{t_n\}$ has a subsequence $\{a_n\}$(say) which is regular with respect to M. Let $AC(\{a_n\}) = \{t\}$. Therefore,

$$\limsup_{n\to\infty} d(a_n, St) \leq \limsup_{n\to\infty} [5d(a_n, Sa_n) + d(a_n, t)] = \limsup_{n\to\infty} d(a_n, t).$$

By uniqueness of asymptotic centre, t is fixed point of S. □

Lemma 4.3. *Assume that $\{t_n\}$ is bounded sequence in complete uniformly convex Busemann space Y with $A(\{t_n\})$, $\{b_n\}$ is subsequence of $\{t_n\}$ with $A(\{b_n\}) = \{b_n\}$ and sequence $\{d(t_n, b)\}$ converges. Then $t = b$.*

Proof. Let $t \neq b$, then by uniqueness of asymptotic center,

$$\begin{aligned}
\limsup_{n\to\infty} d(b_n, b) &\leq \limsup_{n\to\infty} d(b_n, t)\\
&\leq \limsup_{n\to\infty} d(t_n, t)\\
&< \limsup_{n\to\infty} d(t_n, b)\\
&\leq \limsup_{n\to\infty} d(b_n, b)
\end{aligned}$$

Thus, we get a contradiction. Hence $t = b$. □

Theorem 4.4. *Assume that M is closed convex subset of a complete uniformly convex Busemann space Y with a monotone modulus of uniform convexity and $S : M \longrightarrow M$ is SKC map with $F(S) \neq \phi$. Assume that $\{t_n\}$ be defined by (4.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Then $\{t_n\}$ $\Delta-$convergence to a fixed point of S.*

Proof. From Theorem 4.2, the sequence $\{t_n\}$ is bounded and $\lim_{n\to\infty} d(t_n, St_n) = 0$.
Assume that $Z(\{t_n\}) = \bigcup A(\{b_n\})$, where $\{b_n\}$ are subsequences of $\{t_n\}$. First we prove that $Z(\{t_n\}) \subset F(S)$. Assume that $b \in Z(\{t_n\})$, then there is subsequence $\{b_n\}$ of $\{t_n\}$ such that $A(\{b_n\}) = \{b\}$. Using Lemma 2.6, there is subsequence $\{a_n\}$ of $\{b_n\}$ such that $\Delta - \lim_{n\to\infty} a_n = a$. Since $\lim_{n\to\infty} d(a_n, Sa_n) = 0$. Thus by Proposition 3.5, we have $Sa = a$ and $a \in M$. Also by Lemma 4.1, $\lim_{n\to\infty} d(t_n, a)$ exists and so $\lim_{n\to\infty} d(b_n, a)$ also exists. By Lemma 4.3, $b = a \in F(S)$.
Now, we will prove sequence $\{t_n\}$ $\Delta-$ converges to fixed point of S. For this, we will prove that $Z(\{t_n\})$ contains exactly one point. Assume that $A(\{t_n\}) = \{t\}$ and $\{b_n\}$ is subsequence of $\{t_n\}$ such that $\Delta - \lim_{n\to\infty} b_n = b$. Using Lemma 4.1, $\{d(t_n, b)\}$ converges. By Lemma 4.3, $t = b$ and this completes the proof. □

Theorem 4.5. *Assume that M is convex closed and boundedly compact subset of complete uniformly convex Busemann space Y with a monotone modulus of uniform convexity and $S : M \longrightarrow M$ be SKC map with $F(S) \neq \phi$. Assume that $\{t_n\}$ is defined by (4.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Then $\{t_n\}$ converges strongly to a fixed point of S.*

Proof. By Theorem 4.2 and Theorem 4.4, we have the sequence $\{t_n\}$ is bounded and $\Delta-$converges to fixed point t of S, i.e., $\{t_n\}$ converges to $t \in F(S)$.
On contrary, suppose that the sequence $\{t_n\}$ does not converges strongly to t. Since M is boundedly compact, therefore we assume that there is $t_1 \in M$ with $t_1 = t$ such that $\{tx_n\}$ converges strongly to t_1. Thus

$$\lim_{n\to\infty} d(t_n, t_1) = 0 \leq \lim_{n\to\infty} d(t_n, t).$$

By uniqueness of asymptotic center of $\{t_n\}$, we have $t_1 = t$. Therefore we get contradiction. □

Corollary 4.6. *Assume that M is convex closed subset of a complete uniformly convex Busemann space Y with monotone modulus of uniform convexity and $S : M \longrightarrow M$ be SKC map with $F(S) \neq \phi$. Assume that $\{t_n\}$ is defined by (4.1), where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0,1)$. Then $\{t_n\}$ converges strongly to fixed point of S.*

5. Example

Consider the mapping $S : [0,2] \longrightarrow [0,2]$ given by

$$S(t,u) = \begin{cases} \left(1+\frac{t}{2}, 1+\frac{u}{2}\right) & \text{(t,u)} \in [0,2) \times [0,2];\\ \left(\frac{1}{2}, \frac{1}{4}\right) & \text{(t,u)} \in \{2\} \times \{2\}\ . \end{cases} \tag{5.1}$$

Put $t = \frac{31}{16}$ and $u = \frac{31}{16}$, then

$$\begin{aligned}
\|St - Su\|_2 &= \left\|\left(1+\frac{31}{32}, 1+\frac{31}{32}\right) - \left(\frac{1}{2}, \frac{1}{4}\right)\right\|_2\\
&= \sqrt{\left(\frac{1}{2}+\frac{31}{32}\right)^2 + \left(\frac{3}{4}+\frac{31}{32}\right)^2}\\
&= \frac{1}{32}\sqrt{5234}
\end{aligned}$$

and

$$\begin{aligned}
\|t - u\|_2 &= \sqrt{\left(\frac{31}{16}-2\right)^2 + \left(\frac{31}{16}-2\right)^2}\\
&= \frac{1}{16}\sqrt{2}.
\end{aligned}$$

Thus we have $\|St - Su\| \geq \|t - u\|$. Therefore S is not nonexpansive mapping. Now we will check the mapping S is SKC map.
Consider the following cases:
Case I : If $t_1, u_1, t_2, u_2 < 2$

$$\frac{1}{2}\|t - St\|_2 = \frac{1}{2}\sqrt{\left(\frac{t_1-2}{2}\right)^2 + \left(\frac{u_1-2}{2}\right)^2} \quad \text{and}$$
$$\|t-u\|_2 = \sqrt{(t_1-t_2)^2 + (u_1-u_2)^2}.$$

Now $\frac{1}{2}\|t - St\|_2 \leq \|t - u\|_2$ holds, if

$$\begin{aligned}
\frac{1}{2}\sqrt{\left(\frac{t_1-2}{2}\right)^2 + \left(\frac{u_1-2}{2}\right)^2} &\leq \sqrt{(t_1-t_2)^2 + (u_1-u_2)^2}\\
\text{which implies}\quad \frac{1}{2}\left(\frac{t_1-2}{2}\right)^2 &\leq t_1 - t_2 \quad \text{and}\\
\text{and}\quad \frac{1}{2}\left(\frac{u_1-2}{2}\right)^2 &\leq u_1 - u_2.
\end{aligned}$$

which is true for all $t_1, t_2, u_1, u_2 < 2$. Thus $\frac{1}{2}\|t - St\|_2 \leq \|t - u\|_2$.
Now

$$\begin{aligned}
\|St - Su\|_2 &= \|(1+\frac{t_1}{2}, 1+\frac{u_1}{2}) - (1+\frac{t_2}{2}, 1+\frac{u_2}{2})\|_2\\
&= \sqrt{(\frac{t_1-t_2}{2})^2 + (\frac{u_1-u_2}{2})^2}\\
&\leq \|t - u\|_2.
\end{aligned}$$

Therefore

$$\|St-Su\|_2 \leq \max\left\{\|t-u\|_2, \frac{1}{2}\Big(\|t-St\|_2+\|u-Su\|_2\Big), \frac{1}{2}\Big(\|t-Su\|_2+\|u-St\|_2\Big)\right\}.$$

Case II : If $t_1,t_2,u_1,u_2 = 2$. Then

$$\begin{aligned}\|St-Su\|_2 &= \left\|\left(\frac{1}{2},\frac{1}{4}\right)-\left(\frac{1}{2},\frac{1}{4}\right)\right\| \\ &= 0 \\ &\leq \max\left\{\|t-u\|_2, \frac{1}{2}\Big(\|t-St\|_2+\|u-Su\|_2\Big), \frac{1}{2}\Big(\|t-Su\|_2+\|u-St\|_2\Big)\right\}.\end{aligned}$$

Case III : $t_1,u_1 < 2$ and $t_2,u_2 = 2$. Then

$$\begin{aligned}\frac{1}{2}\Big\|t-St\Big\|_2 &= \frac{1}{2}\Big\|(t_1,s_1)-\Big(1+\frac{t_1}{2},1+\frac{s_1}{2}\Big)\Big\|_2 \\ &= \frac{1}{2}\sqrt{\Big(\frac{t_1-2}{2}\Big)^2+\Big(\frac{u_1-2}{2}\Big)^2}\end{aligned}$$

$$\text{and} \quad \|s-t\|_2 = \sqrt{(t_1-2)^2+(u_1-2)^2}.$$

Now $\frac{1}{2}\|t-St\|_2 \leq \|t-u\|_2$ holds, if

$$\begin{aligned}\frac{1}{2}\sqrt{\Big(\frac{t_1-2}{2}\Big)^2+\Big(\frac{u_1-2}{2}\Big)^2} &\leq \sqrt{(t_1-2)^2+(u_1-2)^2} \\ \text{which implies} \quad \frac{1}{2}\Big(\frac{t_1-2}{2}\Big)^2 &\leq t_1-2 \quad \text{and} \\ \text{and} \quad \frac{1}{2}\Big(\frac{u_1-2}{2}\Big)^2 &\leq u_1-2\end{aligned}$$

which is true for all $t_1 \geq 2$ and $u_1 \geq 2$.
Thus $\frac{1}{2}\|t-St\|_2 \leq \|t-u\|_2$.
Now for $t_1 \geq 2$ and $u_1 \geq 2$, we have

$$\begin{aligned}\|St-Su\|_2 &= \Big\|\Big(\frac{1}{2},\frac{1}{4}\Big)-\Big(\frac{1}{2},\frac{1}{4}\Big)\Big\|_2 \\ &\leq 0 \\ &= \sqrt{(t_1-t_2)^2+(u_1-u_2)^2} \\ &\leq \|s-t\|_2.\end{aligned}$$

Therefore

$$\|St-Su\|_2 \leq \max\left\{\|t-u\|_2, \frac{1}{2}\Big(\|t-St\|_2+\|u-Su\|_2\Big), \frac{1}{2}\Big(\|t-Su\|_2+\|u-St\|_2\Big)\right\}.$$

Case IV : If $t_1 = u_1 = 2$ and $t_2,u_2 < 2$. Then

$$\begin{aligned}\frac{1}{2}\|t-St\|_2 &= \frac{1}{2}\Big\|(2,2)-\Big(\frac{1}{2},\frac{1}{4}\Big)\Big\|_2 \\ &= \frac{1}{2}\sqrt{\Big(\frac{3}{2}\Big)^2+\Big(\frac{7}{4}\Big)^2}\end{aligned}$$

$$\text{and} \quad \|t-u\|_2 = \sqrt{(2-t_2)^2+(2-u_2)^2}.$$

Now $\frac{1}{2}\|t-St\|_2 \leq \|t-u\|_2$ holds, if

$$\begin{aligned}\frac{1}{2}\sqrt{\Big(\frac{3}{2}\Big)^2+\Big(\frac{7}{4}\Big)^2} &\leq \sqrt{(2-t_2)^2+(2-u_2)^2} \\ \text{which implies} \quad \frac{3}{4} &\leq 2-t_2 \quad \text{and} \\ \text{and} \quad \frac{7}{8} &\leq 2-u_2\end{aligned}$$

which is true for all $t_2 \leq \frac{5}{4}$ and $u_2 \leq \frac{9}{8}$. Thus $\frac{1}{2}\|t-St\|_2 \leq \|t-u\|_2$.

Now for $x_2 \leq \frac{5}{4}$ and $y_2 \leq \frac{9}{8}$, we have

$$\begin{aligned}\|St-Su\|_2 &= \Big\|\Big(\frac{1}{2},\frac{1}{4}\Big)-(t_2,u_2)\Big\|_2 \\ &\leq \sqrt{\Big(\frac{1}{2}-t_2\Big)^2+\Big(\frac{1}{4}-u_2\Big)^2} \\ &= \sqrt{(2-t_2)^2+(2-u_2)^2} \\ &\leq \|t-u\|_2.\end{aligned}$$

Therefore

$$\|St-Su\|_2 \leq \max\left\{\|t-u\|_2, \frac{1}{2}\Big(\|t-St\|_2+\|u-Su\|_2\Big), \frac{1}{2}\Big(\|t-Su\|_2+\|u-St\|_2\Big)\right\}.$$

Thus the mapping S is SKC map.

6. Numerical Result

We compare the new iteration process, i.e., generalized Picard normal s-iteration process with Picard normal s-iteration process, normal s-iteration process, Mann iteration process and Picard iteration process. We set the stop parameter to $d(t_n,t) \leq 10^{-12}$. For example defined in Section 5, consider $\alpha_n = \frac{1}{(n+11)^{\frac{1}{8}}}, \beta_n = \frac{n}{(2n+7)^{\frac{1}{2}}}, \gamma_n = \frac{1}{(3n+5)^{\frac{1}{2}}}$ and we take initial value $(t_0,u_0) = (1.2,1.5)$.

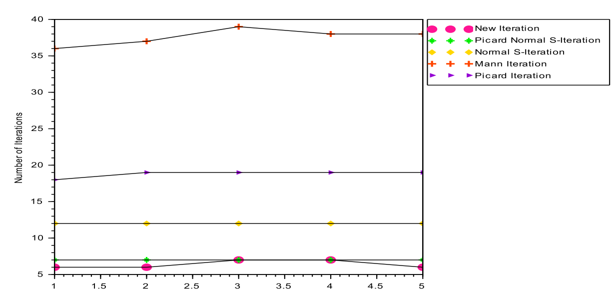

Figure 1. Comparison among number of iterations of different iterative scheme

Table 1. Comparison of iteration of different iteration scheme

Number of Iterations For $\alpha_n = \frac{1}{(n+11)^{\frac{1}{8}}}, \beta_n = \frac{n}{(2n+7)^{\frac{1}{2}}}, \gamma_n = \frac{1}{(3n+5)^{\frac{1}{2}}}$					
Initial Value	**Generalized Picard normal s-iteration**	**Picard normal s-iteration**	**Normal s-iteration**	**Mann iteration**	**Picard iteration**
(1.2, 1.5)	6	7	12	36	18
(1.8, 1.25)	6	7	12	37	19
(0.2, 1.75)	7	7	12	39	19
(0.9, 0.5)	7	7	12	38	19
(1.9, 1.1)	6	7	12	36	19

Table 2. Numerical result for generalized Picard normal s-iteration process

n	x_n	w_n	z_n	y_n	x_{n+1}
0	(1.2,1.5)	(1.32871119482, 1.33911100647)	(1.32986672945, 1.33766658819)	(1.33177248778, 1.33528439028)	(1.33294312194, 1.33382109757)
1	(1.33294312194, 1.33382109757)	(1.33304659236, 1.33369175954)	(1.33317627585, 1.33352965519)	(1.3332617583, 1.33342280212)	(1.33331543958, 1.33335570053)
2	(1.33331543958, 1.33335570053)	(1.33331948595, 1.33335064256)	(1.33332812724, 1.33333633168)	(1.33333093466, 1.33333633168)	(1.33333273366, 1.33333408292)
3	(1.33333273366, 1.33333408292)	(1.33333285387, 1.33333393267)	(1.33333322526, 1.33333346843)	(1.33333328304, 1.3333333962)	(1.33333332076, 1.33333334905)
4	(1.33333332076, 1.33333334905)	(1.33333332305, 1.33333334619)	(1.3333333324, 1.3333333345)	(1.3333333329, 1.33333333388)	(1.33333333322, 1.33333333347)
5	(1.33333333322, 1.33333333347)	(1.33333333324, 1.33333333345)	(1.33333333334, 1.33333333333)	(1.33333333333, 1.33333333333)	(1.33333333333, 1.33333333333)

From Table 1, Figure 1 and Table 2, it is clear that $\{t_n\}$ converges strongly to $(\frac{4}{3}, \frac{4}{3})$ of $F(S)$ and generalized Picard normal s-iteration process converges to fixed point faster than Picard normal s-iteration process, normal s-iteration process, Mann iteration process and Picard iteration process.

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