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Applications of fuzzy bitopological semi-open sets

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Abstract
The aim of this paper is to introduce and study the notions of fuzzy upper and lower (i,j)-semilimit sets. Properties and basic relationships among fuzzy upper (i,j)-semilimit set, fuzzy lower (i,j)-semilimit set and fuzzy (i,j)-semicontinuity are investigated.

Keywords
Fuzzy (i,j)-semiopen sets, fuzzy lower (i,j)-semilimit set, fuzzy upper (i,j)-semilimit set.

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Contents

1. Introduction

The fundamental concept of a fuzzy set introduced by Zadeh [6] in 1965, provides a natural foundation for building new branches of fuzzy mathematics. In 1968 Chang [2] introduced the concept of fuzzy topological spaces as a generalization of topological spaces. Since then many topologists have contributed to the theory of fuzzy topological spaces. Today fuzzy topology has been firmly established as one of the basic disciplines of fuzzy mathematics. In 1987 Kandil [3] introduced the concept of fuzzy bitopological spaces as an extension of fuzzy topological spaces and a generalization of bitopological spaces. Recently the authors of the paper [5] extended the concepts of fuzzy semiopen sets, fuzzy semicontinuous and fuzzy semiopen mappings due to Azad [1] to fuzzy bitopological spaces. In this paper is to introduce and study the notions of fuzzy upper and lower (i,j)-semilimit sets. Properties and basic relationships among fuzzy upper (i,j)-semilimit set, fuzzy lower (i,j)-semilimit set and fuzzy (i,j)-semicontinuity are investigated.

2. Preliminaries

In this paper, the symbol I will denote the interval [0, 1]. A fuzzy set in X is a function with domain X and values in I, that is, an element of I^X. The subset of X in which $\mu \in I^X$ assumes nonzero values, known as the support of μ [6]. A member μ of I^X is contained in a member ρ of I^X denoted by $\mu \leq \rho$ if, and only if $\mu(x) \leq \rho(x)$ for every $x \in X$ [4]. By $\mu \times \rho$ we denote the fuzzy set in $X \times X$ for which $(\mu \times \rho)\ (x,y)$ = min $\{\mu(x), \rho(y)$, for every $(x,y) \in X \times X\}$. For $\mu \in I^X$, the fuzzy set μ^c is denoted by $\mu^c(x) = 1 - \mu(x)$ for every $x \in X$. A fuzzy set in X is called a fuzzy point if, and only if it takes the value 0 for all $y \in X$ except one, say, $x \in X$. If its value at x is α $(0 < \alpha \leq 1)$ we denote the fuzzy point by x_α, where the point x is called its support. The class of all fuzzy points in X is denoted by χ[2]. The fuzzy point x_α is said to be contained in a fuzzy set μ or to belong to μ, denoted by $x_\alpha \in \mu$ if, and only if $\alpha \leq \mu(x)$. A fuzzy set μ in a topological space (X, τ_1, τ_2) is called a fuzzy neighbourhood of a fuzzy point x_α if, and only if there exists $\beta \in \tau$ such that $x_\alpha \in \beta \leq \mu$ [4]. A fuzzy point x_α is said to be quasi-coincident with μ denoted by x_α q μ if, and only if $\alpha > \mu^c(x)$ or $\alpha + \mu(x) > 1$ [4]. A fuzzy set μ is said to be quasi-coincident with a fuzzy set ρ, denoted by μ q ρ, if, and only if there exists $x \in X$ such that $\mu(x) > \rho^c(x)$ or $\mu(x) + \rho(x) > 1$ [4]. If μ does not quasi-coincident with ρ, then we write $\mu \bar{q} \rho$. A fuzzy set μ of a fuzzy bitopological space (X, τ_1, τ_2) is called fuzzy (i,j)-semiopen [5] if $\mu \leq j\mathrm{Cl}(i\mathrm{Int}(\mu))$, where $j\mathrm{Cl}(\mu) = \wedge\{\rho \in I^X : \mu \leq \rho$ and $1 - \rho \in \tau_j\}$ and $i\mathrm{Int}(\mu) = \vee\{\rho \in I^X : \rho \leq \mu$ and $\rho \in \tau_i\}$. The complement of a fuzzy (i,j)-semiopen set is called a fuzzy (i,j)-semiclosed set [5].

The family of all fuzzy (i,j)-semiopen (resp. fuzzy (i,j)-semiclosed) subsets of (X,τ_1,τ_2) is denoted by (i,j)-$SO(X)$ (resp. (i,j)-$SC(X,Y)$). The fuzzy (i,j)-semiclosure [5] of $\mu \in I^X$ is denoted by (i,j)-$s\mathrm{Cl}(\mu)$ where (i,j)-$s\mathrm{Cl}(\mu) = \wedge\{\rho \in (i,j)\text{-}SC(X,Y) : \mu \leq \rho\}$. Also, the fuzzy (i,j)-semiinterior [5] of μ is denoted by (i,j)-$s\mathrm{Int}(\mu)$ where (i,j)-$s\mathrm{Int}(\mu) = \vee\{\rho \in (i,j)\text{-}SO(X) : \rho \leq \mu\}$. In addition to that, a fuzzy set μ is fuzzy (i,j)-semiopen (resp. fuzzy (i,j)-semiclosed) if, and only if $\mu = (i,j)$-$s\mathrm{Int}(\mu)$ (resp. $\mu = (i,j)$-$s\mathrm{Cl}(\mu)$). Also, for a fuzzy set $\mu \in I^X$, (i,j)-$s\mathrm{Cl}(1-\mu) = 1-(i,j)$-$s\mathrm{Int}(\mu)$ and (i,j)-$s\mathrm{Int}(1-\mu) = 1-(i,j)$-$s\mathrm{Cl}(\mu)$ [5]. A fuzzy set μ in a fuzzy bitopological space (X,τ_1,τ_2) is called a fuzzy open-Q-neighbourhood [4] (resp. (i,j)-semi-open Q-neighbourhood [5]) of a fuzzy point x_α if, and only if there exists $\rho \in \tau$ (resp. $\rho \in (i,j)$-$SO(X)$) such that $x_\alpha q \rho$ and $\rho \leq \mu$. A fuzzy point $x_\alpha \in \mathrm{Cl}(\mu)$ if, and only if each open-Q-neighbourhood x_α is quasi-coincident with μ. A fuzzy set μ in a fuzzy bitopological space (X,τ_1,τ_2) is called fuzzy (i,j)-semineighbourhood of a fuzzy point x_α if there exists $\rho \in (i,j)$-$SO(X)$ such that $x_\alpha \in \rho \leq \mu$. The family of all fuzzy open-Q-neighbourhood (resp. fuzzy (i,j)-semi-open Q-neighbourhood [5]) of the fuzzy point x_α in X is $N(x_\alpha)$ (resp. $N_s(x_\alpha)$). The set $N_s(x_\alpha)$ with the relation $\leq^*$ (that is, $\mu_1 \leq^* \mu_2$ if, and only if $\mu_2 \leq \mu_1$) forms a directed set.

3. Fuzzy (i,j)-semicontinuously converge

Definition 3.1. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is said to be fuzzy (i,j)-semicontinuous if for every fuzzy point x_α in X and every fuzzy (i,j)-semi-open Q-neighbourhood μ of $f(x_\alpha)$, there exists a fuzzy (i,j)-semi-open Q-neighbourhood ρ of x_α such that $f(\rho) \leq \mu$. The family of all fuzzy (i,j)-semicontinuous functions from (X,τ_1,τ_2) into (Y,σ_1,σ_2) is denoted by $SC(X,Y)$.*

Definition 3.2. *Let (X,τ_1,τ_2) be a fuzzy bitopological space and let $\{p_i, i \in I\}$ be a net of fuzzy points in X. We say that the fuzzy net $\{p_i : i \in I\}$ fuzzy (i,j)-semiconverges to a fuzzy point p of X if for every fuzzy (i,j)-semi-open Q-neighbourhood μ of p in X there exists $i_0 \in I$ such that $p_i\ q\ \mu$ for every $i \in I$ and $i \geq i_0$.*

Theorem 3.3. *Let μ be a fuzzy subset of a fuzzy bitopological space (X,τ_1,τ_2). Then, a fuzzy point $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu)$ if, and only if for every $\rho \in (i,j)$-$SO(X)$ for which $x_\alpha\ q\ \rho$ we have $\rho\ q\ \mu$.*

Proof. The fuzzy point $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu)$ if, and only if $x_\alpha \in \gamma$ for every fuzzy (i,j)-semiclosed set γ of X for which $\mu \leq \rho$. Equivalently, $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu)$ if, and only if $\alpha \leq 1-\rho(x)$ for every fuzzy (i,j)-semiopen set ρ for which $\mu \leq 1-\rho$. Thus, $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu)$ if, and only if $\rho(x) \leq 1-\alpha$, for every fuzzy (i,j)-semiopen set ρ for which $\rho \leq 1-\mu$. So, $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu)$ if, and only if for every fuzzy (i,j)-semiopen set ρ of X such that $\rho(x) > 1-\alpha$ we have ρ not less than $1-\mu$. Therefore, $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu)$ if, and only if for every fuzzy (i,j)-semiopen set ρ of X such that $\rho(x)+\alpha > 1$ we have $\rho q \mu$. Thus, $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu)$ if, and only if for every fuzzy (i,j)-semiopen set ρ of X such that $x_\alpha q \rho$ we have $\rho q \mu$. □

Theorem 3.4. *Let $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ be a fuzzy (i,j)-semicontinuous function, x_α be a fuzzy point in X and μ, ρ be a fuzzy (i,j)-semi-open Q-neighbourhood of x_α and $f(x_\alpha)$, respectively such that $f(\mu) \leq \rho$. Then there exists a fuzzy point x_θ in X such that $x_\alpha q \mu$ and $f(x_\alpha)\overline{q}\rho$.*

Proof. Since $f(\mu) \nleq \rho$, $\mu \nleq f^{-1}(\rho)$. Then there exists $x \in Y$ such that $\mu(x) > f^{-1}(\rho(x))$ or $\mu(x) - f^{-1}(\rho(x)) > 0$ and then $\mu(x)+1-f^{-1}(\rho(x)) > 1$ or $\mu(x)+(f^{-1}(\rho))^c(x)) > 1$. Let $(f^{-1}(\rho))^c(x)) = r$. Clearly, for the fuzzy point x_r we have $x_r q \mu$ and $x_r \in (f^{-1}(\rho))^c$. Hence for the fuzzy point $x_\alpha = x_r$ we have $x_\alpha q \mu$ and $f(x_\alpha)\overline{q}\rho$. □

Theorem 3.5. *A function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$ is fuzzy (i,j)-semicontinuous if, and only if for every fuzzy point x_α of X and for every net $\{p_i : i \in I\}$ which fuzzy (i,j)-semiconverges to x_α the net $\{f(p_i) : i \in I\}$ of Y fuzzy (i,j)-semiconverges to $f(x_\alpha)$.*

Proof. Straightforward. □

Definition 3.6. *A net $\{f_i : j \in J\}$ in (i,j)-$FC(X,Y)$ fuzzy (i,j)-semi-continuously converges to $f \in (i,j)$-$FC(X,Y)$ if for every nrt $\{p_i : i \in I\}$ in X which fuzzy (i,j)-semi-converges to a fuzzy point p in X we have that the fuzzy net $\{f_j(p_i), (i,j) \in I \times J\}$ fuzzy fuzzy (i,j)-semi-converges to a fuzzy point $f(p)$ in Y.*

Theorem 3.7. *For a function $f : (X,\tau_1,\tau_2) \to (Y,\sigma_1,\sigma_2)$, the following properties hold:*

1. *If $\{f_i : i \in I\}$ is a net in (i,j)-$SC(X,Y)$ with $f_i = f$ for every $i \in I$, then $\{f_i : i \in I\}$ is fuzzy (i,j)-semicontinuously converges to $f \in (i,j)$-$SC(X,Y)$.*
2. *If $\{f_i : i \in I\}$ is a net in (i,j)-$SC(X,Y)$, which fuzzy (i,j)-semicontinuously converges to $f \in (i,j)$-$SC(X,Y)$ and $\{g_j : j \in J\}$ be a subnet of $\{f_i : i \in I\}$, then the net $\{g_j : j \in J\}$ fuzzy (i,j)-semicontinuously converges to f.*
3. *If $\{f_i : i \in I\}$ is a net in (i,j)-$SC(X,Y)$, which does not fuzzy (i,j)-semicontinuously converges to $f \in (i,j)$-$SC(X,y)$, then there exists no subnet of $\{f_i : i \in I\}$, which fuzzy continuously converges to f.*

Proof. Clear. □

Theorem 3.8. *A net $\{f_i : i \in I\}$ in (i,j)-$SC(X,Y)$ fuzzy (i,j)-semicontinuously converges to $f \in (i,j)$-$SC(X,Y)$ if, and only if for every fuzzy point x_α in Y and every fuzzy (i,j)-semiopen Q-neighbourhood ρ of $f(x_\alpha)$ in Z there exists an element $i_0 \in I$ and a fuzzy (i,j)-semiopen Q-neighbourhood μ of x_α in Y such that $f_i(\mu) \leq \rho$ for every $i \geq i_0$, $i \in I$.*

Proof. Let x_α be a fuzzy point in Y and ρ an (i,j)-semiopen Q-neighbourhood of $f(x_\alpha)$ in Y such that for every $i \in I$ and for every fuzzy (i,j)-semi-open Q-neighbourhood μ of x_α in Y we can choose a fuzzy point x_μ in Y such that $x_\mu q \mu$ and $f_i(x_\mu)\overline{q}\rho$. Clearly, the fuzzy net $\{x_\mu, \mu \in N(x_\alpha)\}$ fuzzy (i,j)-semiconverges to x_α but the fuzzy net $\{f_i(p_\mu), (\mu,i) \in N(x_\alpha)\times I\}$ does not fuzzy (i,j)-semiconverges to $f(x_\alpha)$ in Y. Conversely, let $\{f_i : i \in I\}$ be a fuzzy net in (i,j)-$SC(X,Y)$ which fuzzy (i,j)-semiconverges to the fuzzy point x_α in Y and ρ an arbitrary fuzzy (i,j)-semi-open Q-neighbourhood of $f(x_\alpha)$ in Y. By assumption there exists a fuzzy (i,j)-semi-open Q-neighbourhood ρ of x_α in Y and an element $i_0 \in I$ such that $f_i(\mu) \le \rho$ for every $i \ge i_0$, $i \in I$. Since the fuzzy net $\{x_i : i \in I\}$ (i,j)-semiconverges to x_α in Y. There exists $i_0 \in I$ such that $x_i q \mu$ for every $i \ge i_0$, $i \in I$. Let $(i_0, j_o) \in I \times J$. Then for every $(i,j) \in I \times J$, $(i,j) \ge (i_0, j_o)$ we have $f_j(x_i) q f_j(\mu)$ and $f_j(\mu) \le \rho$, that is, $f_j(x_i) q \rho$. Thus, the net $\{f_j(x_i) : (i,j) \in I \times J\}$ fuzzy (i,j)-semiconverges to $f(x_\alpha)$ and the net $\{f_j : j \in J\}$ fuzzy (i,j)-semicontinuously converges to f. □

Definition 3.9. *A fuzzy set μ of a fuzzy bitopological space (X,τ_1,τ_2) is said to be generalized (i,j)-semiclosed if (i,j)-$s\mathrm{Cl}(\mu) \le \rho$ whenever $\mu \le \rho$ and ρ is (i,j)-semiopen in X.*

Definition 3.10. *A fuzzy bitopological space (X,τ_1,τ_2) is said to be (i,j)-semi-T_1 if every fuzzy point in X is fuzzy (i,j)-semiclosed.*

Theorem 3.11. *A fuzzy bitopological space (X,τ_1,τ_2) is (i,j)-semi-T_1 if, and only if for each $x \in X$ and each $\alpha \in [0,1]$ there exists a fuzzy (i,j)-semiopen set μ such that $\mu(x) = 1-\alpha$ and $\mu(y) = 1$ for $y \ne x$.*

Proof. Let $\alpha = 0$. We set $\mu = X$. Then μ is fuzzy (i,j)-semiopen set such that $\mu(x) = 1-0$ and $\mu(y) = 1$ for $y \ne x$. Now, let $\alpha \in (0,1]$ and $x \in X$. We set $\mu = (x_\alpha)^c$. The set μ is fuzzy (i,j)-semiopen such that $\mu(x) = 1-\alpha$ and $\mu(y) = 1$ for $y \ne x$. Conversely, let x_α be an arbitrary fuzzy point of X. We prove that the fuzzy point x_α is fuzzy (i,j)-semiclosed. By assumption there exists a fuzzy (i,j)-semiopen set μ such that $\mu(x) = 1-\alpha$ and $\mu(y) = 1$ for $y \ne x$. Clearly, $\mu^c = x_\alpha$. Thus, the fuzzy point x_α is fuzzy (i,j)-semiclosed and therefore the fuzzy bitopological space X is (i,j)-semi-T_1. □

Definition 3.12. *A fuzzy bitopological space (X,τ_1,τ_2) is called quasi-(i,j)-semi-T_1 if for any fuzzy points x_α and y_β for which $supp(x_\alpha) \ne x \ne supp(y_\beta) = y$, there exists a fuzzy (i,j)-semiopen set μ such that $x_\alpha \in \mu$ and $y_\beta \notin \mu$ and another (i,j)-semiopen set ρ such that $x_\alpha \notin \rho$ and $y_\beta \in \varphi$.*

Definition 3.13. *A fuzzy bitopological space (X,τ_1,τ_2) is called an (i,j)-semi-T_2 space if for any fuzzy points x_α and y_β for which $supp(x_\alpha) \ne supp(y_\beta)$, there exists two fuzzy (i,j)-semiopen Q-neighbourhoods ρ and μ of x_α and y_β, respectively such that $\rho \wedge \mu = 0$.*

Definition 3.14. *A fuzzy point x_α is called weak (resp. strong) if $\alpha \le \frac{1}{2}$ (resp. $\alpha > \frac{1}{2}$) [4].*

Theorem 3.15. *If (X,τ_1,τ_2) is a quasi-(i,j)-semi-T_1 fuzzy bitopological space and x_α a weak fuzzy point in X, then $(x_\alpha)^c$ is a fuzzy (i,j)-semineighbourhood of each fuzzy point y_β with $x \ne y$.*

Proof. Let $x \ne y$ and y_β be a fuzzy point of X. Since X is quasi-(i,j)-semi-T_1, there exists a fuzzy (i,j)-semiopen set μ of X such that $y_\beta \in \mu$ and $x_\alpha \notin \mu$. Then $\alpha \succ \mu(x)$. Also, $\alpha \le \frac{1}{2}$. Thus, $\mu(x) = 1-\alpha$. Then $\mu(y) \le 1 = (x_\alpha)^c(y)$ for every $y \in X\backslash\{x\}$. So $\mu \le (x_\alpha)^c$. Hence the fuzzy point x_α is an (i,j)-semineighbourhood of y_β. □

Definition 3.16. *A fuzzy bitopological space (X,τ_1,τ_2) is called an (i,j)-semiregular if for any fuzzy points x_α and a fuzzy (i,j)-semiclosed set ρ not containing x_α, there exist μ, $\eta \in (i,j)$-$SO(X)$ such that $x_\alpha \in \mu$, $\rho \le \eta$ and $\mu \wedge \eta = 0$.*

Theorem 3.17. *If (X,τ_1,τ_2) is an (i,j)-semiregular space, then for any strong fuzzy point x_α and any fuzzy (i,j)-semiopen set μ containing x_α, there exists a fuzzy (i,j)-semiopen set ρ containing x_α such that $\beta(\rho) \le \mu$.*

Proof. Suppose that x_α is any strong fuzzy point contained in $\mu \in (i,j)$-$SO(X)$. Then $\frac{1}{2} < \alpha \le \rho(x)$. Thus, the complement of μ, that is, the set μ^c, is a fuzzy (i,j)-semiclosed set to which does not belong to the fuzzy point x_α. Then there exist $\rho, \eta \in (i,j)$-$SO(X)$ such that $x_\alpha \in \rho$ and $\mu^c \le \eta$ with $\rho \wedge \mu = 0$. Hence $\rho \le \eta^c$ and (i,j)-$s\mathrm{Cl}(\rho) \le (i,j)$-$s\mathrm{Cl}(\eta^c) = \eta^c$. Now $\mu^c \le \eta$ implies $\eta^c \le \mu$. Then (i,j)-$s\mathrm{Cl}(\rho) \le \mu$. □

Theorem 3.18. *If (X,τ_1,τ_2) is a fuzzy (i,j)-semiregular space, then the strong fuzzy points in Xare fuzzy generalized (i,j)-semiclosed.*

Proof. Let x_α be any strong fuzzy point in X and μ be a fuzzy (i,j)-semiopen set such that $x_\alpha \in \mu$. Then there exists $\rho \in (i,j)$-$SO(X)$ such that $x_\alpha \in \rho$ and (i,j)-$s\mathrm{Cl}(\rho) \le \mu$. We have (i,j)-$s\mathrm{Cl}(x_\alpha) \le (i,j)$-$s\mathrm{Cl}(\rho) \le \mu$. Hence the fuzzy point x_α is fuzzy generalized (i,j)-semiclosed. □

Definition 3.19. *A fuzzy bitopological space (X,τ_1,τ_2) is called a fuzzy (i,j)-semiregular if for any weak fuzzy point x_α and a fuzzy (i,j)-semiclosed set ρ not containing x_α, there exist $\mu, \eta \in (i,j)$-$SO(X)$ such that $x_\alpha \in \mu$, $\rho \le \eta$ and $\mu \wedge \eta = \phi$.*

Definition 3.20. *A fuzzy set μ in a fuzzy bitopological space (X,τ_1,τ_2) is said to be fuzzy (i,j)-seminearly crisp if (i,j)-$s\mathrm{Cl}(\mu) \wedge ((i,j)\text{-}s\mathrm{Cl}(\mu))^c = \phi$.*

Theorem 3.21. *Let (X,τ_1,τ_2) be a fuzzy bitopological space. If for any weak fuzzy point x_α and $\mu \in (i,j)$-$SO(X)$ containing x_α, there exists a fuzzy (i,j)-semiopen and (i,j)-seminearly crisp fuzzy set ρ containing x_α such that (i,j)-$s\mathrm{Cl}(\rho) \le \mu$, then X is fuzzy (i,j)-semiregular.*

Proof. Assume that η is a fuzzy (i,j)-semiclosed set not containing the weak fuzzy point x_α. Then η^c is a fuzzy (i,j)-semiopen set containing x_α. By hypothesis, there exists a fuzzy (i,j)-semiopen and (i,j)-seminearly crisp fuzzy set ρ such that $x_\alpha \in \rho$ and (i,j)-$s\mathrm{Cl}(\rho) \leq \eta^c$. We set $\gamma = (i,j)$-$s\mathrm{Int}((i,j)$-$s\mathrm{Cl}(\rho))$ and $\mu = 1-(i,j)$-$s\mathrm{Cl}(\rho)$. Then η is fuzzy (i,j)-semiopen, $x_\alpha \in \gamma$ and $\eta \leq \gamma$. We are going to prove that $\mu \wedge \gamma = \phi$. Now assume that there exists $y \in X$ such that $(\gamma \wedge \mu)(y) = \alpha \neq \phi$. Then $y_\alpha \in \gamma \wedge \mu$. Hence $y_\alpha \in (i,j)$-$s\mathrm{Cl}(\rho)$ and $y_\alpha \in (((i,j)$-$s\mathrm{Cl}(\rho)))^c$. This is a contradiction since ρ is (i,j)-seminearly crisp. Hence (X,τ_1,τ_2) is (i,j)-semiregular. □

Definition 3.22. *Let μ be a fuzzy set of a fuzzy bitopological space (X,τ_1,τ_2). A fuzzy point x_α is called an (i,j)-semiboundary point of a fuzzy set μ if, and only if $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu) \wedge (1-(i,j)$-$s\mathrm{Cl}(\mu))$. By (i,j)-$sd(\mu)$, we denote the fuzzy set (i,j)-$s\mathrm{Cl}(\mu) \wedge (1-(i,j)$-$s\mathrm{Cl}(\mu))$.*

Theorem 3.23. *Let X be a fuzzy bitopological space. Suppose that x_α and y_β be weak and strong fuzzy points, respectively. If x_α is generalized (i,j)-semiclosed, then $y_\beta \in (i,j)$-$s\mathrm{Cl}(x_\alpha) \Rightarrow x_\alpha \in (i,j)$-$s\mathrm{Cl}(y_\beta)$.*

Proof. Suppose $y_\beta \in (i,j)$-$s\mathrm{Cl}(x_\alpha)$ and $x_\alpha \notin (i,j)$-$s\mathrm{Cl}(y_\beta)$. Then (i,j)-$s\mathrm{Cl}(y_\beta) < \alpha$. Also $\alpha \leq \frac{1}{2}$. So (i,j)-$s\mathrm{Cl}(y_\beta)(x) \leq 1-\alpha$ and therefore $\alpha \leq 1-(i,j)$-$s\mathrm{Cl}(y_\beta)(x)$. So $x_\alpha \in (((i,j)$-$s\mathrm{Cl}(y_\beta)))^c$. But x_α is generalized (i,j)-semiclosed and $(((i,j)$-$s\mathrm{Cl}(y_\beta)))^c$ is fuzzy (i,j)-semiopen. Hence (i,j)-$s\mathrm{Cl}(x_\alpha) \leq ((i,j)$-$s\mathrm{Cl}(y_\beta))^c$. By assumption, we have $y_\beta \in (i,j)$-$s\mathrm{Cl}(x_\alpha)$. Thus, $y_\beta \in ((i,j)$-$s\mathrm{Cl}(y_\beta))^c$. We prove that this is a contradiction. Indeed, we have $\beta \leq 1-(i,j)$-$s\mathrm{Cl}(y_\beta)(y)$ or (i,j)-$s\mathrm{Cl}(y_\beta)(y) \leq 1-\beta$. Also, $y_\beta \in (i,j)$-$s\mathrm{Cl}(y_\beta)$. Thus $\beta \leq 1-\beta$. But y_β is a strongly fuzzy point, that is, $\beta > \frac{1}{2}$. So the above relation $\beta \leq 1-\beta$ is a contradiction. Hence $x_\alpha \in (i,j)$-$s\mathrm{Cl}(y_\beta)$. □

Theorem 3.24. *Let μ be a fuzzy set of a fuzzy bitopological space (X,τ_1,τ_2). Then $\mu \vee (i,j)$-$sd(\mu) \leq (i,j)$-$s\mathrm{Cl}(\mu)$.*

Proof. Let $x_\alpha \in \mu \vee (i,j)$-$sd(\mu)$. Then $x_\alpha \in \mu$ or $x_\alpha \in (i,j)$-$sd(\mu)$. If $x_\alpha \in (i,j)$-$sd(\mu)$, then $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu)$. Let us suppose that $x_\alpha \in \mu$. We have (i,j)-$s\mathrm{Cl}(\mu) = \wedge\{\rho : \rho \in I^X$, ρ is (i,j)-semiclosed and $\mu \leq \rho\}$. So if $x_\alpha \in \mu$, then $x_\alpha \in \rho$ for and fuzzy (i,j)-semiclosed set ρ of X for which $\mu \leq \rho$ and therefore $x_\alpha \in (i,j)$-$s\mathrm{Cl}(\mu)$. □

Definition 3.25. *A fuzzy point x_α in a fuzzy bitopological space (X,τ_1,τ_2) is said to be :*

1. *well-(i,j)-semiclosed if there exists $y_\beta \in (i,j)$-$s\mathrm{Cl}(x_\alpha)$ such that $supp(x_\alpha) \neq supp(y_\beta)$;*
2. *just-(i,j)-semiclosed if the fuzzy set $pcl(x_\alpha)$ is again fuzzy point.*

Remark 3.26. *Clearly, in a fuzzy (i,j)-semi-T_1 space every fuzzy point is just-(i,j)-semiclosed.*

Theorem 3.27. *If (X,τ_1,τ_2) is a fuzzy bitopological space and x_α is a fuzzy generalized (i,j)-semiclosed but well-(i,j)-semiclosed fuzzy point, then (X,τ_1,τ_2) is not fuzzy (i,j)-semi-T_1.*

Proof. Let X be a fuzzy (i,j)-semi-T_1. By the fact that x_α is fuzzy well-(i,j)-semiclosed, there exists a fuzzy point y_β with $supp(x_\alpha) \neq supp(y_\beta)$ such that $y_\beta \in (i,j)$-$s\mathrm{Cl}(x_\alpha)$. Then there exists $\mu \in (i,j)$-$SO(X)$ such that $x_\alpha \in \mu$ and $y_\beta \notin \mu$. Then (i,j)-$s\mathrm{Cl}(x_\alpha) \leq \mu$ and $y_\beta \in \mu$. But this is a contradiction and hence X cannot be (i,j)-semi-T_1 space. □

Theorem 3.28. *Let (X,τ_1,τ_2) be a fuzzy bitopological space. If x_α and x_β are two fuzzy points such that $\alpha \leq \beta$ and x_β is fuzzy (i,j)-semiopen, then x_α is just-(i,j)-semiclosed if it is fuzzy generalized (i,j)-semiclosed.*

Proof. We prove that the fuzzy set (i,j)-$s\mathrm{Cl}(x_\alpha)$ is again a fuzzy point. We have $x_\alpha \in x_\beta$ and the fuzzy set x_β is fuzzy (i,j)-semiopen. Since x_α is fuzzy generalized (i,j)-semiclosed, (i,j)-$s\mathrm{Cl}(x_\alpha) \leq x_\beta$. Thus, (i,j)-$s\mathrm{Cl}(x_\alpha)(x) \leq \beta$ and (i,j)-$s\mathrm{Cl}(x_\alpha)(z) \leq 0$ for every $z \in X\backslash\{x\}$. So the fuzzy set (i,j)-$s\mathrm{Cl}(x_\alpha)$ is a fuzzy point. □

4. Fuzzy Upper And Lower (i,j)-semilimit sets

Definition 4.1. *Let I be a directed set. Let χ be the collection of all fuzzy points of an ordinary set X. The function $S : I \to \chi$ is called a fuzzy net in X. For every $i \in I$, $S(I)$ is often denoted by s_i and hence a net S is often denoted by $\{s_i : i \in I\}$.*

Definition 4.2. *Let $\{\mu_i : i \in I\}$ be a net of fuzzy sets in a fuzzy bitopological space (X,τ_1,τ_2). Then by (i,j)-s-$\overline{\lim}_I(\mu_i)$, we denote fuzzy upper (i,j)-semilimit of the net $\{\mu_i : i \in I\}$ in I^X, that is, the fuzzy set which is the union of all fuzzy points x_α in X such that for every $i_0 \in I$ and for every (i,j)-semiopen Q-neighbourhood μ of x_r in X there exists an element $i \in I$ for which $i \geq i_0$ and $\mu_i q \mu$. In other cases we get (i,j)-s-$\overline{\lim}_I(\mu_i) = \phi$.*

Definition 4.3. *Let $S = \{s_i : i \in I\}$ be a fuzzy net in X. Then S is said to be quasi-coincident with μ if for each $i \in I$, s_i is quasi-coincident with μ. A fuzzy net $\{g_j : j \in J\}$ in X is called a fuzzy subnet of a fuzzy net $\{s_i : i \in I\}$ in X if there is a function $N : J \to I$ such that (i) $g_i = S_{N_{(j)}}$ and (ii) for the element $i_0 \in I$, there is $j_0 \in J$ such that if $j \geq j_0$, $j \in J$, then $N(j) \geq i_0$.*

Theorem 4.4. *Let $\{\mu_i : i \in I\}$ and $\{\rho_i : i \in I\}$ be two fuzzy nets of fuzzy sets in X. Then the following properties hold:*

1. *The fuzzy upper (i,j)-semilimit is (i,j)-semiclosed.*
2. (i,j)-s-$\overline{\lim}_I(\mu_i) = (i,j)$-$s$-$\overline{\lim}_I((i,j)$-$s\mathrm{Cl}(\mu_i))$
3. *If $\mu_i = \mu$ for every $i \in I$, then (i,j)-s-$\overline{\lim}_I(\mu_i) = (i,j)$-$s\mathrm{Cl}(\mu)$.*
4. *The fuzzy upper (i,j)-semilimit is not affected by changing a finite number of the μ_i.*
5. *If $\mu_i \leq \rho_i$ for every $i \in I$, then (i,j)-s-$\overline{\lim}_I(\mu_i) \leq (i,j)$-$s$-$\overline{\lim}_I(\rho_i)$.*

6. (i,j)-s-$\overline{\lim}_I(\mu_i) \leq (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\})$.
7. (i,j)-s-$\overline{\lim}_I(\mu_i \vee \rho_i) = (i,j)$-$s$-$\overline{\lim}_I(\mu_i) \vee$ (i,j)-s-$\overline{\lim}_I(\rho_i)$.
8. (i,j)-s-$\overline{\lim}_I(\mu_i \wedge \rho_i) \leq (i,j)$-$s$-$\overline{\lim}_I(\mu_i) \wedge$ (i,j)-s-$\overline{\lim}_I(\rho_i)$.

Proof. (1). It is sufficient to prove that (i,j)-$s\mathrm{Cl}((i,j)$-s-$\overline{\lim}_I(\mu_i)) \leq (i,j)$-s-$\overline{\lim}_I(\mu_i)$. Suppose $x_r \in (i,j)$-$s\mathrm{Cl}((i,j)$-s-$\overline{\lim}_I(\mu_i))$ and let μ be an arbitrary fuzzy (i,j)-semiopen Q-neighbourhood of x_α. Then $\mu q(i,j)$-s-$\overline{\lim}_I(\mu_i))$. Hence there exists an element $x^1 \in X$ with $\mu(x^1) + (i,j)$-s-$\overline{\lim}_I(\mu_i)(x_1) \geq 1$. (i,j)-s-$\overline{\lim}_I(\mu_i)(x_1) = \alpha$. Then for the fuzzy point x^1_α in X we have $x^1_\alpha q\mu$ and $x^1_\alpha \in (i,j)$-s-$\overline{\lim}_I(\mu_i))$. Thus, for every element $i_0 \in I$, there exists $i \geq i_0$, $i \in I$ such that $\mu_i q\mu$. This means that $x_\alpha \in (i,j)$-s-$\overline{\lim}_I(\mu_i))$.
(2). Clearly, it is sufficient to prove that for every (i,j)-semiopen set μ the condition $\mu q\mu_i$ is equivalent to $\mu q(i,j)$-$s\mathrm{Cl}\,\mu_i$. Let $\mu q\mu_i$. Then there exists an element $x \in X$ such that $\mu(x) + (\mu_i)(x) \geq 1$. Since $\mu_i \leq (i,j)$-$s\mathrm{Cl}(\mu_i)$, $\mu(x) + (i,j)$-$s\mathrm{Cl}(\mu_i)(x) \geq 1$ and $\mu q(i,j)$-$s\mathrm{Cl}\,\mu_i$. Conversely, let $\mu q(i,j)$-$s\mathrm{Cl}\,\mu_i$. Then there exists an element $x \in X$ such that $\mu(x) + (i,j)$-$s\mathrm{Cl}(\mu_i)(x) \geq 1$. Let (i,j)-$s\mathrm{Cl}(\mu_i(x)) = r$. Then $x_r \in (i,j)$-$s\mathrm{Cl}(\mu_i)$ and the fuzzy (i,j)-semiopen set μ is a fuzzy (i,j)-semiopen Q-neighbourhood of x_r. Hence, $\mu q\mu_i$.
(3), (4). Obvious.
(5). Let $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i))$ and let μ be a fuzzy (i,j)-semiopen Q-neighbourhood of x_r in Y. Then for every $i_0 \in I$ there exists $i \in I$, $i \geq i_0$ such that $\mu_i q\mu$ and therefore $\vee\{\mu_i : i \in I\}$. Thus, $x_r \in (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\})$.
(6). Clear.
(7). Clearly, $\mu_i \leq \mu_i \vee \rho_i$ and $\rho_i \leq \mu_i \vee \rho_i$ for every $i \in I$. Hence by (6), (i,j)-s-$\overline{\lim}_I(\mu_i) \leq (i,j)$-s-$\overline{\lim}_I(\mu_i \vee \rho_i)$ and (i,j)-s-$\overline{\lim}_I(\rho_i) \leq (i,j)$-s-$\overline{\lim}_I(\mu_i \vee \rho_i)$. Thus (i,j)-s-$\overline{\lim}_I(\mu_i) \vee (i,j)$-s-$\overline{\lim}_I(\rho_i) \leq (i,j)$-s-$\overline{\lim}_I(\mu_i \vee \rho_i)$. Conversely, let $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i \vee \rho_i))$. We prove that $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i)) \vee (i,j)$-s-$\overline{\lim}_I(\rho_i))$. Let us suppose that $x_r \notin (i,j)$-s-$\overline{\lim}_I(\mu_i)) \vee (i,j)$-s-$\overline{\lim}_I(\rho_i))$. Then $x_r \notin (i,j)$-s-$\overline{\lim}_I(\mu_i)$ and $x_r \notin (i,j)$-s-$\overline{\lim}_I(\rho_i))$. Hence there exists a fuzzy (i,j)-semiopen Q-neighbourhoods μ_1 of x_r and an element $i_1 \in I$ such that $\mu_i \bar{q}\mu_1$ for every $i \in I$, $i \geq i_1$. Also, there exists a fuzzy (i,j)-semiopen Q-neighbourhoods μ_2 of x_r and an element $i_2 \in I$ such that $\rho_i \bar{q}\mu_2$ for every $i \in I$, $i \geq i_2$. Let $\mu = \mu_1 \vee \mu_2$ and let $i_0 \in I$ such that $i_0 \geq i_1$ and $i_0 \geq i_2$. Then the fuzzy set μ is a fuzzy (i,j)-semiopen Q-neighbourhood of x_r and $\mu_i \vee \rho_i \bar{q}\mu$ for every $i \in I$, $i \geq i_0$. Since $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i \vee \rho_i)$, this is a contradiction. Thus, $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i) \vee (i,j)$-s-$\overline{\lim}_I(\rho_i))$.
(8). Straightforward. □

Definition 4.5. *Let $\{\mu_i : i \in I\}$ be a net of fuzzy sets in a fuzzy bitopological space X. Then by (i,j)-s-$\underline{\lim}_I(\mu_i)$, we denote the fuzzy lower (i,j)-semilimit of the net $\{\mu_i : i \in I\}$ in I^X, that is, the fuzzy set which is the union of all fuzzy points x_r in X such that for every (i,j)-semiopen Q-neighbourhood μ of x_r in X there exists an element $i_0 \in I$ such that $\mu_i q\mu$ for every $i \in I$ and $i \geq i_0$. In other case, we get (i,j)-s-$\underline{\lim}_I(\mu_i) = \phi$.*

Theorem 4.6. *For the fuzzy upper and lower (i,j)-semilimit, we have (i,j)-s-$\underline{\lim}_I(\mu_i) \leq (i,j)$-s-$\overline{\lim}_I(\mu_i)$.*

Proof. The proof follows from the respetive definitions. □

Theorem 4.7. *Let $\{\mu_i : i \in I\}$ be a net of fuzzy sets in X such that $\mu_{i_1} \leq \mu_{i_2}$ if, and only if $i_1 \leq i_2$. Then (i,j)-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\}) = (i,j)$-s-$\underline{\lim}_I(\mu_i)$.*

Proof. Let $x_r \in (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\})$ and μ be a fuzzy (i,j)-semiopen Q-neighbourhood of x_r in X. Then $\mu q \vee \{\mu_i : i \in I\})$. Hence there exists an element $i_0 \in I$ such that $\mu_q\mu_o$. By assumption we have $\mu q\mu_i$ for every $i \in I$, $i \geq i_0$. Thus $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i)$. Conversely, let $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i)$ and let μ be an arbitrary fuzzy (i,j)-semiopen Q-neighbourhood of x_r in X. Then there exists an element $i_0 \in I$ such that $\mu_q\mu_i$ for every $i \in I$, $i \geq i_0$. Hence $\mu q \vee \{\mu_i : i \in I\})$ and $x_r \in (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\})$. □

Theorem 4.8. *Let $\{\mu_i : i \in I\}$ be a net of fuzzy sets in X. Then (i,j)-s-$\overline{\lim}_I(\mu_i)) = \bigwedge\{(i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \geq i_0\}) : i_0 \in I\}$.*

Proof. Let $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i)$ and let $i_0 \in I$. We prove that $x_r \in (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \geq i_0\})$. Let μ be an arbitrary fuzzy (i,j)-semiopen Q-neighbourhood of x_r in X. Then there exists $i \geq i_0$, $i \in I$ such that $\mu_q\mu_i$. Thus, $\mu q \vee \{\mu_i : i \geq i_0\})$ and $x_r \in (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\})$. Conversely, let $x_r \in \bigwedge\{(i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in i_0\}) : i_0 \in I\}$. Then $x_r \in (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \geq i_0\})$ for every $i_0 \in I$. We prove that $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i)$. Let μ be an arbitrary fuzzy (i,j)-semiopen Q-neighbourhood of x_r in X and let $i_0 \in I$. Then $\mu q \vee \{\mu_i : i \geq i_0\}$. We prove that there exists $i \in I$, $i \geq i_0$ such that $\mu_i q\mu$. Let us suppose that $\mu \bar{q}\mu_i$ for every $i \in I$, $i \geq i_0$. Then for every $i \in I$, $i \geq i_0$ and for every $x \in X$ we have $\mu(x) + \mu_i(x) \leq 1$ and $\mu(x) + (\bigvee\{\mu_i : i \geq i_0\})(x) \leq 1$, which is a contradiction. Hence $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i)$. □

Theorem 4.9. *Let $\{\mu_i : i \in I\}$ be a net of fuzzy (i,j)-semiclosed sets in X such that $\mu_{i_1} \leq \mu_{i_2}$ if, and only if $i_2 \leq i_1$. Then (i,j)-s-$\overline{\lim}_I(\mu_i) = \wedge\{\mu_i : i \in I\}$.*

Proof. Let $x_r \in \wedge\{\mu_i : i \in I\}$. Then $x_r \in \mu_i$ or $r \leq \mu_i(x)$ for every $i \in I$. Let $i_0 \in I$ and μ be a fuzzy (i,j)-semiopen Q-neighbourhood of x_r, that is, $r + \mu(x) \succ 1$. Then there exists $i \in I$, $i \geq i_0$ such that $\mu_i(x) + \mu(x) \geq r + \mu(x) \succ 1$. Hence $\mu_i q\mu$ and $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i)$. Conversely, let $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i)$ and let $x_r \notin \wedge\{\mu_i : i \in I\}$. Then there exists $i_0 \in I$ such that $x_r \notin \mu_{i_0}$, that is, $r \succ \mu_{i_0}(x)$. Let $\mu = (\mu^c_{i_0})$. Then μ is fuzzy (i,j)-semiopen Q-neighbourhood of x_r in X and $i \geq i_0$, $\mu \bar{q}\mu_i$, which is a contradiction. □

Theorem 4.10. *Let $\{\mu_i : i \in I\}$ and $\{\rho_i : i \in I\}$ be two fuzzy nets of fuzzy sets in X. Then the following properties hold:*

1. *If $\mu_i \leq \rho_i$ for every $i \in I$, then (i,j)-s-$\underline{\lim}_I(\mu_i) \leq (i,j)$-s-$\underline{\lim}_I(\rho_i)$.*
2. *(i,j)-s-$\underline{\lim}_I(\mu_i \vee \rho_i) \geq (i,j)$-s-$\underline{\lim}_I(\mu_i) \vee$ (i,j)-s-$\underline{\lim}_I(\rho_i)$.*

3. (i,j)-*s*-$\underline{\lim}_I(\mu_i \wedge \rho_i) \leq (i,j)$-*s*-$\underline{\lim}_I(\mu_i) \wedge (i,j)$-*s*-$\underline{\lim}_I(\rho_i)$.
4. *The fuzzy lower* (i,j)*-semilimit is* (i,j)*-semiclosed.*
5. (i,j)-*s*-$\underline{\lim}_I(\mu_i) = (i,j)$-*s*-$\underline{\lim}_I((i,j)\text{-}s\mathrm{Cl}(\mu_i))$.
6. *If* $\mu_i = \mu$ *for every* $i \in I$*, then* (i,j)-*s*-$\underline{\lim}_I(\mu_i) = \beta(\mu)$.
7. *The fuzzy lower* (i,j)*-semilimit is not affected by changing a finite number of the* μ_i.
8. $\wedge\{\mu_i : i \in I\} \leq (i,j)$-*s*-$\underline{\lim}_I(\mu_i)$.
9. (i,j)-*s*-$\underline{\lim}_I(\mu_i) \leq (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\}$.
10. $\vee\{\wedge\{\mu_i : i \geq i_0\} i_0 \in I\} \leq (i,j)$-*s*-$\underline{\lim}_I(\mu_i)$.

Proof. (2). Let $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i) \vee (i,j)$-s-$\underline{\lim}_I(\rho_i)$. Then either $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i)$ or $x_r \in (i,j)$-s-$\underline{\lim}_I(\rho_i)$. Let $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i)$. Then for every fuzzy (i,j)-semiopen Q-neighbourhood μ of x_r, in X there exists an element $i_0 \in I$ such that $\mu_i q \mu$ for every $i \in I$, $i \geq i_0$ and $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i \vee \rho_i)$.
(8). Let $x_r \in \wedge\{\mu_i : i \in I\}$. We prove that $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i)$. Let us suppose that $x_r \notin (i,j)$-s-$\underline{\lim}_I(\mu_i)$. Then there exists a fuzzy (i,j)-semiopen Q-neighbourhood μ of x_r such that for every $i \in I$ there exists $i_0 \geq i$ for which $\mu_{i_0} \overline{q} \mu$. This means that $\mu_{i_0}(x) + \mu(x) \leq 1$ for every $x \in X$. Now, since $x_r \in \wedge\{\mu_i : i \in I\}$ and μ is a fuzzy (i,j)-semiopen Q-neighbourhood of x_r we have that $r \leq \mu_i(x)$ for every $i \in I$ and $r + \mu(x) \succ 1$. Thus, $\mu_i(x) + \mu(x) \succ 1$ for every $i \in I$. By the above this is a contradiction. Hence $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i)$.
(10). Let $x_r \in \{\wedge\{\mu_i : i \geq i_0\} 1_0 \in I\}$. Then there exists $i_0 \in I$ such that $x_r \in \wedge\{\mu_i : i \geq i_0\}$. Hence $x_r \in \mu_i$ for every i, $i \geq i_0$ and $r \leq \mu_i(x)$ for every $i \in I$, $i \geq i_0$. We prove that $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i)$. Let μ be an arbitrary fuzzy (i,j)-semiopen Q-neighbourhood of x_r in X. Then we have $x_r q \mu$ or equivalently $r + \mu(x) \succ 1$. Since $r \leq \mu_i(x)$ for every $i \in I$, $i \geq i_0$ we have $\mu_i(x) + \mu(x) \succ 1$ for every $i \in I$, $i \geq i_0$. Thus $\mu_i q \mu$ for every $i \in I$, $i \geq i_0$ and $x_r \in (i,j)$-s-$\underline{\lim}_I(\mu_i)$. The other proofs can be obtained by using the definitions. □

Definition 4.11. *A net* $\{\mu_i : i \in I\}$ *of fuzzy sets in a fuzzy bitopological space* X *is said to be fuzzy* (i,j)*-semiconvergent to the fuzzy set* μ *if* (i,j)-*s*-$\underline{lim}_I(\mu_i) = (i,j)$-*s*-$\overline{\lim}_I(\mu_i) = \mu$. *We write* (i,j)-*s*-$lim_I(\mu_i) = \mu$.

Theorem 4.12. *Let* $\{\mu_i : i \in I\}$ *be an* (i,j)*-semiconvergent net of fuzzy sets in* X.

1. *If* $\mu_{i_1} \geq \mu_{i_2}$ *for* $i_1 \leq i_2$*, then* (i,j)-*s*-$\lim_I(\mu_i) = \wedge\{(i,j)$-$s\mathrm{Cl}(\mu_i) : i \in I\}$.
2. *If* $\mu_{i_1} \leq \mu_{i_2}$ *for* $i_1 \leq i_2$*, then* (i,j)-*s*-$\lim_I(\mu_i) = (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\})$.

Proof. (1). Clearly $\wedge\{(i,j)$-$s\mathrm{Cl}(\mu_i)\} \leq (i,j)$-s-$\underline{lim}_I((i,j)$-$s\mathrm{Cl}(\mu_i)) = (i,j)$-s-$\underline{lim}_I(\mu_i) \leq (i,j)$-s-$\overline{\lim}_I(\mu_i) =$ (i,j)-s-$\overline{\lim}_I((i,j)$-$s\mathrm{Cl}(\mu_i)) = \wedge\{(i,j)$-$s\mathrm{Cl}(\mu_i) : i \in I\}$. Thus, (i,j)-s-$\lim_I(\mu_i) = \wedge\{(i,j)$-$s\mathrm{Cl}(\mu_i) : i \in I\}$.
(2). Clearly (i,j)-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\}) \leq (i,j)$-s-$\underline{lim}_I(\mu_i) \leq (i,j)$-s-$\overline{\lim}_I(\mu_i) \leq (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\})$. Thus, (i,j)-s-$\lim_I(\mu_i) = (i,j)$-$s\mathrm{Cl}(\vee\{\mu_i : i \in I\})$. □

Theorem 4.13. *Let* $\{\mu_i : i \in I\}$ *and* $\{\rho_i : i \in I\}$ *be two p-convergent nets of fuzzy sets in X. Then the following properties hold:*

1. *If* $\mu_i \leq \rho_i$ *for every* $i \in I$*, then* (i,j)-*s*-$\lim_I(\mu_i) \leq (i,j)$-*s*-$\lim_I(\rho_i)$.
2. (i,j)-*s*-$\lim_I(\mu_i \vee \rho_i) = (i,j)$-*s*-$\lim_I(\mu_i) \vee (i,j)$-*s*-$\lim_I(\rho_i)$.
3. (i,j)-$s\mathrm{Cl}((i,j)$-*s*-$\lim_I(\mu_i)) = (i,j)$-*s*-$\lim_I(\mu_i) = (i,j)$-*s*-$\lim_I((i,j)$-$s\mathrm{Cl}(\mu_i))$.
4. *If* $\mu_i = \mu$ *for every* $i \in I$*, then* (i,j)-*s*-$\lim_I(\mu_i) = (i,j)$-$s\mathrm{Cl}(\mu)$.

Proof. (1). Follows by Theorems 4.4 and 4.10.
(2). By Theorems 4.4 and 4.10, we have (i,j)-s-$\overline{\lim}_I(\mu_i \vee \rho_i) = (i,j)$-s-$\overline{\lim}_I(\mu_i) \vee (i,j)$-s-$\overline{\lim}_I(\rho_i) \leq (i,j)$-s-$\underline{\lim}_I(\mu_i) \vee (i,j)$-s-$\underline{\lim}_I(\rho_i) \leq (i,j)$-s-$\underline{\lim}_I(\mu_i \vee \rho_i)$. So (i,j)-s-$\lim_I(\mu_i \vee \rho_i) = (i,j)$-s-$\lim_I(\mu_i) \vee (i,j)$-s-$\lim_I(\rho_i)$.
(3),(4) Follows by Theorems 4.4 and 4.10. □

Theorem 4.14. *Let* μ_1 *and* μ_2 *be fuzzy* (i,j)*-semiopen Q-neighbourhoods of* x_r *and* y_r *in* X *and* Y*, respectively. Then the fuzzy set* $\mu_1 \times \mu_2$ *is a fuzzy* (i,j)*-semiopen Q-neighbourhood of* $(x,y)_r$ *in* $X \times Y$.

Proof. Clear. □

Theorem 4.15. *Let* $\{\mu_i : i \in I\}$ *and* $\{\rho_i : i \in I\}$ *be two nets of fuzzy sets in X. Then the following properties hold:*

1. (i,j)-*s*-$\overline{\lim}_I(\mu_i \times \rho_i) \leq (i,j)$-*s*-$\overline{\lim}_I(\mu_i) \times (i,j)$-*s*-$\overline{\lim}_I(\rho_i)$.
2. (i,j)-*s*-$\underline{\lim}_I(\mu_i \times \rho_i) \leq (i,j)$-*s*-$\underline{\lim}_I(\mu_i) \times (i,j)$-*s*-$\underline{\lim}_I(\rho_i)$.
3. *If* $\{\mu_i : i \in I\}$ *and* $\{\rho_i : i \in I\}$ *are* (i,j)*-semiconvergent nets, then* (i,j)-*s*-$\lim_I(\mu_i \times \rho_i) \leq (i,j)$-*s*-$\lim_I(\mu_i) \times (i,j)$-*s*-$\lim_I(\rho_i)$.

Proof. (1). Let $(x,y)_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i \times \rho_i)$. We must prove $(x,y)_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i) \times (i,j)$-s-$\overline{\lim}_I(\rho_i)$ or equivalently $r \leq ((i,j)$-s-$\overline{\lim}_I(\mu_i)) \times (i,j)$-s-$\overline{\lim}_I(\rho_i))(x,y)$. Let $i_0 \in I$, μ_1 be an arbitrary fuzzy (i,j)-semiopen Q-neighbourhood of x_r in X and μ_2 be a constant fuzzy (i,j)-semiopen Q-neighbourhood of y_r in Y. Then the fuzzy set $\mu_1 \times \mu_2$ is a fuzzy (i,j)-semiopen Q-neighbourhood of $(x,y)_r$ in $X \times Y$. Hence there exists $i \in I$, $i \geq i_0$ such that $(\mu_1 \times \mu_2) q (\mu_i \times \rho_i)$. We have $\mu_1 q \mu_i$ and $\mu_2 q \rho_i$. Thus, $x_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i)$. Similarly, we can prove that $y_r \in (i,j)$-s-$\overline{\lim}_I(\rho_i)$. Hence $(x,y)_r \in (i,j)$-s-$\overline{\lim}_I(\mu_i) \times (i,j)$-s-$\overline{\lim}_I(\rho_i)$. The proof of (2) and (3) are similar. □

Theorem 4.16. *A net* $\{f_i : i \in I\}$ *in* (i,j)*-SC*(X,Y) *fuzzy* (i,j)*-semicontinuously converges to* $f \in (i,j)$*-SC*(X,Y) *if, and only if* (i,j)-*s*-$\overline{\lim}_I(f_i^{-1}(\mu)) \leq f^{-1}(\mu)$ *for every fuzzy* (i,j)*-semiclosed subset* μ *of* Y.

Proof. Let $\{f_i : i \in I\}$ be a net in (i, j)-$SC(X,Y)$, which fuzzy (i, j)-semicontinuously converges to f and let μ be an arbitrary fuzzy (i, j)-semiclosed subset of Y. Let $x_r \in (i, j)$-s-$\overline{\lim}_I({f_i}^{-1}(\mu))$ and μ be an arbitrary fuzzy (i, j)-semiopen Q-neighbourhood of $f(x_r)$ in Y^1. Since the net $\{f_i : i \in I\}$ fuzzy (i, j)-semicontinuously converges to f, there exists a fuzzy (i, j)-semiopen Q-neighbourhood ρ of x_r in X and an element $i_0 \in I$ such that $f_i(\rho) \leq \mu_1$ for every $i \in I$, $i \geq i_0$ by Theorem 3.7. On the other hand, there exists an element $i \geq i_0$ such that $\rho q {f_i}^{-1}(\mu)$. Hence $f_i(\rho)q\mu$ and therefore $\mu_1 q\mu$. This means that $f(x_r) \in (i, j)$-$s\,\mathrm{Cl}(\mu) = \mu$. Thus $x_r \in f^{-1}(\mu)$. Conversely, let $\{f_i : i \in I\}$ be a net in (i, j)-$SC(X,Y)$ and $f \in (i, j)$-$SC(X,Y)$ such that (i, j)-s-$\overline{\lim}_I({f_i}^{-1}(\mu)) \leq f^{-1}(\mu)$ for every fuzzy (i, j)-semiclosed subset μ of Y. We prove that the net $\{f_i : i \in I\}$ fuzzy (i, j)-semicontinuously converges to f. Let x_r be a fuzzy point of X and μ be a fuzzy (i, j)-semiopen Q-neighbourhood of x_r in X. Since $x_r \notin f^{-1}(\mu)$, where ${\mu_1}^c = \mu$. We have $x_r \notin (i, j)$-s-$\overline{\lim}_I({f_i}^{-1}(\mu))$. Then there exists an element $i_0 \in I$ and a fuzzy (i, j)-semiopen Q-neighbourhood ρ of x_r in X such that ${f_i}^{-1}(\mu)\overline{q}\rho$ for every $i \in I$, $i \geq i_0$. Then $\rho \leq (f^{-1}(\mu))^c = {f_i}^{-1}(\mu^c) = {f_i}^{-1}(\mu_1)$ and $f_i(\rho) \leq \mu_1$ for every $i \in I$, $i \geq i_0$, that is, the net $\{f_i : i \in I\}$ fuzzy (i, j)-semicontinuously converges to f. □

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