Malaya Journal of Matematik, Vol. 8, No. 3, 1085-1091, 2020
https://doi.org/10.26637/MJM0803/0061

On certain subclass of univalent functions with finitely many fixed coefficients defined by Bessel function

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Abstract
In this present investigation, we study a new class of functions that are analytic and Univalent with finitely many fixed coefficients defined by modified Hadamard product involving Bessel function. Further, we also establish coefficient condition, radii of starlikeness and convexity, extreme points and integral operators applied to functions in this class.

Keywords
Analytic; Starlike; Convex; Bessel Function.

AMS Subject Classification
30C45, 30C80.

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Article History: Received **18** February **2020**; Accepted **14** June **2020**

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1. Introduction

Let $\mathscr{A}$ denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic and univalent in the open disc

$$\mathbb{D}=\{z:|z|<1\}.$$

Let $\mathscr{S}$ denote the subclass of $\mathscr{A}$ where in addition the functions in $\mathscr{S}$ are also univalent in $\mathbb{D}$. The class SD(α) was introduced in [8] and was recently considered in [12] that consists functions of the form (1.1) satisfying the criteria

$$\Re\left\{\frac{f(z)}{z}\right\}\geq\alpha\left|f'(z)-\frac{f(z)}{z}\right|,\alpha\geq 0. \tag{1.2}$$

New subclasses of $\mathscr{S}$ by fixing a finite number of coefficients of functions has been considered earlier by many authors (see [4, 5] for details). We also denote by $\mathscr{T}$ a subclass of $\mathscr{A}$ introduced and studied by Silverman [9],consisting of functions of the form

$$f(z)=z-\sum_{n=2}^{\infty}a_nz^n,\ a_n>0;z\in\mathbb{D}. \tag{1.3}$$

Robertson [7] introduced the subclasses of $\mathscr{A}$, given by $\mathscr{S}^*(\beta)$ and $\mathscr{C}(\beta)$ respectively called as starlike functions of order β and convex functions of order β consisting of functions which satisfy the following inequalities:

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\beta \text{ and } \Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\beta.$$

The generalized Bessel function $\omega_{u,b,c}(z)$ of the first kind of order u in terms of Euler gamma function is given by the representation

$$\omega_{u,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-c)^n}{n!\Gamma(u+n+\frac{b+1}{2})}\left(\frac{z}{2}\right)^{2n+u},z\in\mathbb{C}. \tag{1.4}$$

The function $\varphi_{u,b,c}(z)$ defined by the transformation

$$\varphi_{u,b,c}(z)=2^u\Gamma(u+\frac{b+1}{2})z^{1-\frac{u}{2}}\omega_{u,b,c}(\sqrt{z}), \tag{1.5}$$

using the generalised Bessel function $\omega_{u,b,c}(z)$ is studied by many researchers [2, 3]. Ramachandran et al.[6] obtained the following series representation for the function $\varphi_{u,b,c}(z)$ given by (1.5)

$$\varphi_{u,b,c}(z) = z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (\kappa)_n n!} z^{n+1}, z \in \mathbb{C} \tag{1.6}$$

where $\kappa = u + \frac{b+1}{2} \notin \mathbb{Z}_0^-, N = \{1, 2, \cdots\}$,
$\mathbb{Z}_0^- = \{0, -1, -2, \cdots\}$. and $(\kappa)_n$ is the Pochhammer symbol given by

$$(\kappa)_n = \begin{cases} 1, & n = 0 \\ \kappa(\kappa+1)(\kappa+2)\cdots(\kappa+n-1), & n \in \mathbb{N} \end{cases}$$
$$= \frac{\Gamma(\kappa+n)}{\Gamma(\kappa)}. \tag{1.7}$$

The Hadamard product or convolution of two functions f given by (1.1) and g defined as $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.8}$$

For convenience, $\varphi_{u,b,c}(z)$ is replaced by $\varphi_{\kappa,c}(z)$.
Ramachandran et al. [6] introduced a operator B_κ^c: $\mathscr{S} \to \mathscr{S}$ which is defined by the convolution

$$\begin{aligned} B_\kappa^c f(z) &= \varphi_{\kappa,c}(z) * f(z) \\ &= z + \sum_{n=1}^{\infty} \frac{(-c)^n}{4^n (\kappa)_n n!} z^{n+1} \\ &= z + \sum_{n=2}^{\infty} \frac{(-c)^{n-1}}{4^{n-1} (\kappa)_{n-1} (n-1)!} z^n \\ &= z + \sum_{n=2}^{\infty} \mathscr{E}(c, \kappa, n) a_n z^n, \end{aligned} \tag{1.9}$$

where $\mathscr{E}(c, \kappa, n) = \frac{(-c)^{n-1}}{4^{n-1} (\kappa)_{n-1} (n-1)!}$
The function B_κ^c of the form (1.9) which is nothing but a transformation of the generalized hypergeometric function.
Also $B_\kappa^c f(z) = z_0 F_1 \left(\kappa, -\frac{cz}{4} \right) * f(z)$ and
$\varphi(\kappa, c) \left(\frac{-cz}{4} \right) = z_0 F_1(\kappa, z)$
A class $UB(\lambda, \eta, k, c)$ has been recently studied by Shanmugam et al. [10] and [11]. A function f of the form (1.3) is said to be in the class $UB(\lambda, \eta, k, c)$ if

$$\Re \left\{ \frac{zG''(z)}{G'(z)} \right\} > k \left| \frac{zG''(z)}{G'(z)} - 1 \right| + \eta, \tag{1.10}$$

where $c > 1, 0 \le \lambda < 1, k \ge 0, 0 \le \eta < 1, z \in \mathbb{D}$ and

$$G(z) = (1-\lambda)(B_\kappa^c f(z)) + \lambda z (B_\kappa^c f(z))'.$$

Motivated by the work of Ramachandran et al. [6], here we consider and study the subclass $BSD(\alpha, \lambda, c)$ with fixed finitely many coefficients defined by modified Hadamard product with Bessel function.
We begin with the definition of the class $BSD(\alpha, \lambda, c)$.

Definition 1.1. *Let $c > 1, 0 \le \lambda < 1, \alpha \ge 0$. A function $f \in \mathscr{S}$ is in $BSD(\alpha, \lambda, c)$, if it satisfies the following inequality*

$$\Re \left\{ \frac{G(z)}{z} \right\} \ge \alpha \left| G'(z) - \frac{G(z)}{z} \right|, \tag{1.11}$$

where $G(z) = (1-\lambda)(B_\kappa^c f(z)) + \lambda z (B_\kappa^c f(z))'$.

Further, let $TBSD(\alpha, \lambda, c) = \mathscr{T} \cap BSD(\alpha, \lambda, c)$.

Theorem 1.2. *Let the function f be of the form (1.3). Then, $f \in TBSD(\alpha, \lambda, c)$ if and only if for $\alpha \ge 0$*

$$\sum_{n=2}^{\infty} (1+(n-1)\lambda)(1+(n-1)\alpha)\mathscr{E}(c, \kappa, n)|a_n| \le 1. \tag{1.12}$$

Proof. Let f of the form (1.3) satisfies (1.12). Then we have

$$\begin{aligned} &\Re \left\{ \frac{G(z)}{z} \right\} - \alpha \left| G'(z) - \frac{G(z)}{z} \right| \\ &= \left\{ \Re \left\{ \frac{G(z)}{z} - 1 \right\} + 1 \right\} - \alpha \left| G'(z) - \frac{G(z)}{z} \right| \\ &\ge 1 - \left| \frac{G(z)}{z} - 1 \right| - \alpha \left| G'(z) - \frac{G(z)}{z} \right| \\ &= 1 - \left| \sum_{n=2}^{\infty} (1+(n-1)\lambda)\mathscr{E}(c, \kappa, n) a_n z^{n-1} \right| \\ &\quad - \alpha \left| \sum_{n=2}^{\infty} (n-1)(1+(n-1)\lambda)\mathscr{E}(c, \kappa, n) a_n z^{n-1} \right| \\ &= 1 - \sum_{n=2}^{\infty} (1+(n-1)\lambda)\mathscr{E}(c, \kappa, n)|a_n| \\ &\qquad - \alpha \sum_{n=2}^{\infty} (n-1)(1+(n-1)\lambda) D(c, \kappa, n)|a_n| \\ &\ge 0. \end{aligned}$$

Therefore, $f \in TBSD(\alpha, \lambda, c)$. Conversely, let

$$\Re \left\{ \frac{G(z)}{z} \right\} - \alpha \left| G'(z) - \frac{G(z)}{z} \right| > 0.$$

This implies,

$$\Re \left\{ 1 - \sum_{n=2}^{\infty} (1+(n-1)\lambda)\mathscr{E}(c, \kappa, n)|a_n| z^{n-1} \right\}$$
$$-\alpha \left| \sum_{n=2}^{\infty} (n-1)(1+(n-1)\lambda)\mathscr{E}(c, \kappa, n) a_n z^{n-1} \right| > 0.$$

If we allow z to take real values and as $z \to 1$, we get

$$1 - \sum_{n=2}^{\infty} (1+(n-1)\lambda)\mathscr{E}(c, \kappa, n)|a_n|$$
$$-\alpha \sum_{n=2}^{\infty} (n-1)(1+(n-1)\lambda)\mathscr{E}(c, \kappa, n) a_n \ge 0$$

or

$$\sum_{n=2}^{\infty}(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)|a_n|\leq 1.$$

□

Corollary 1.3. *Let $f\in TBSD(\alpha,\lambda,c)$. Then,*

$$a_n\leq \frac{1}{(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)}, n\geq 2. \quad (1.13)$$

The subclass $TBSD(\alpha,\lambda,c,p_k)$ of $TBSD(\alpha,\lambda,c)$ consists of functions

$$f(z)=z-\sum_{i=2}^{k}\frac{p_i}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,n)}z^i-\sum_{n=k+1}^{\infty}a_nz^n \quad (1.14)$$

where $\alpha\geq 0, 0\leq p_i\leq 1$ and $0\leq\sum_{i=2}^{k}p_i\leq 1$.

2. Main Results

We start with obtaining the coefficient bounds for the class $TBSD(\alpha,\lambda,c,p_k)$ for functions f of the form (1.14).

Theorem 2.1. *A function of the form (1.14) belongs to the class $TBSD(\alpha,\lambda,c,p_k)$ if and only if*

$$\sum_{n=k+1}^{\infty}(1+(n-1)\lambda)(1+(n-1)\alpha)\mathscr{E}(c,\kappa,n)a_n \leq 1-\sum_{i=2}^{k}p_i \quad (2.1)$$

where $\alpha\geq 0$, $0\leq p_i\leq 1$ and $0\leq\sum_{i=2}^{k}p_i\leq 1$. The result is sharp.

Proof. From (1.14), we have for $i=2,3,\cdots k$,

$$a_i=\frac{p_i}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}, \quad (2.2)$$

$0\leq p_i\leq 1,\ 0\leq\sum_{i=2}^{k}p_i\leq 1.$

By Theorem 1.2,

$$\begin{aligned}
&\sum_{n=2}^{\infty}(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)a_n\\
&=\sum_{i=2}^{k}(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)a_i\\
&+\sum_{n=k+1}^{\infty}(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)a_n\\
&=\sum_{i=2}^{k}p_i\\
&\quad+\sum_{n=k+1}^{\infty}(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,n)a_n\\
&\leq 1.
\end{aligned}$$

Conversely,

$$\begin{aligned}
&\Re\left\{\frac{G(z)}{z}\right\}-\alpha\left|G'(z)-\frac{G(z)}{z}\right|\\
&\geq 1-\left|\frac{G(z)}{z}-1\right|-\alpha\left|G'(z)-\frac{G(z)}{z}\right|\\
&=1-\sum_{n=2}^{\infty}(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)|a_n|\\
&\qquad-\alpha\sum_{n=2}^{\infty}(n-1)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)|a_n|\\
&=1-\sum_{n=2}^{\infty}(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)|a_n|\\
&=1-\sum_{i=2}^{k}(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)|a_i|\\
&\quad-\sum_{n=k+1}^{\infty}(1+(n-1)\lambda)(1+(n-1)\alpha)\mathscr{E}(c,\kappa,n)|a_n|\\
&=1-\sum_{i=2}^{k}p_i\\
&\quad-\sum_{n=k+1}^{\infty}(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)|a_n|\\
&\geq 0.
\end{aligned}$$

Hence $f\in TBSD(\alpha,\lambda,c,p_k)$. □

Finally, it is observed that the inequality (2.1) of Theorem 2.1 is sharp and for $n\geq 1$, the extremal function is given by

$$f(z)=z-\sum_{i=2}^{k}\frac{p_i}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}z^i-\frac{1-\sum_{i=2}^{k}p_i}{(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)}z^n. \quad (2.3)$$

Corollary 2.2. *Let $f\in TBSD(\alpha,\lambda,c,p_k)$. Then, for $n\geq k+1$,*

$$a_n\leq\frac{1-\sum_{i=2}^{k}p_i}{(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)}z^n.$$

The sharpness is obtained for the function f given by (2.3).

Theorem 2.3. *The class $TBSD(\alpha,\lambda,c,p_k)$ is convex.*

Proof. Let $f,g\in TBSD(\alpha,\lambda,c,p_k)$.
Then,

$$f(z)=z-\sum_{i=2}^{k}\frac{p_i}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}z^i-\sum_{n=k+1}^{\infty}a_nz^n$$

and

$$g(z)=z-\sum_{i=2}^{k}\frac{p_i}{(1+(i-1)\lambda)(1+(i-1)\alpha)\mathscr{E}(c,\kappa,i)}z^i-\sum_{n=k+1}^{\infty}b_nz^n,$$

$0\le p_i\le 1$, $0\le\sum_{i=2}^{k}p_i\le 1$.
Let us assume that $h(z)=\mu f(z)+(1-\mu)g(z)$. Hence,

$$h(z)=z-\sum_{i=2}^{k}\frac{p_i}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}z^i-\sum_{n=k+1}^{\infty}(\mu a_n+(1-\mu)b_n)z^n$$

Consider,

$$\begin{aligned}\sum_{n=k+1}^{\infty}&(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)(\mu a_n+(1-\mu)b_n)\\ &=\mu\sum_{n=k+1}^{\infty}(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)a_n\\ &+(1-\mu)\sum_{n=k+1}^{\infty}(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)b_n\\ &\le\mu\left(1-\sum_{i=2}^{k}p_i\right)+(1-\mu)\left(1-\sum_{i=2}^{k}p_i\right)\\ &=1-\sum_{i=2}^{k}p_i.\end{aligned}$$

Therefore, $h(z)\in TBSD(\alpha,\lambda,c,p_k)$. □

Theorem 2.4. *Let*

$$f_k(z)=z-\sum_{i=2}^{k}\frac{p_i}{(1+(i-1)\lambda)(1+(i-1)\alpha)\mathscr{E}(c,\kappa,i)}z^i \quad (2.4)$$

and for $n\ge k+1$, let

$$f_n(z)=z-\sum_{i=2}^{k}\frac{p_i}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}z^i-\frac{\left(1-\sum_{i=2}^{k}p_i\right)}{(1+(n-1)\lambda)(1+(n-1)\alpha)\mathscr{E}(c,\kappa,n)}z^n. \quad (2.5)$$

Then $f\in TBSD(\alpha,\lambda,c,p_k)$ if and only if the function f can be represented in the form

$$f(z)=\sum_{n=k}^{\infty}\lambda_nf_n(z), \quad (2.6)$$

where $\lambda_n\ge 0$, $(n\ge k)$ and $\sum_{n=k}^{\infty}\lambda_n=1$.

Proof. Let $f\in T$ can be expressed in the form (2.6). Then

$$f(z)=z-\sum_{i=2}^{k}\frac{p_i}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}z^i-\sum_{n=k+1}^{\infty}\frac{\lambda_n\left(1-\sum_{i=2}^{k}p_i\right)}{(1+(n-1)\lambda)(1+(n-1)\alpha)\mathscr{E}(c,\kappa,n)}z^n. \quad (2.7)$$

Now,

$$\begin{aligned}\sum_{n=k+1}^{\infty}&(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)\lambda_n\\ &\times\frac{\left(1-\sum_{i=2}^{k}p_i\right)}{(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)}\\ &=\sum_{n=k+1}^{\infty}\lambda_n\left(1-\sum_{i=2}^{k}p_i\right)\\ &=\left(1-\sum_{i=2}^{k}p_i\right)\sum_{n=k+1}^{\infty}\lambda_n\\ &=\left(1-\sum_{i=2}^{k}p_i\right)(1-\lambda_k)\\ &\le 1-\sum_{i=2}^{k}p_i,\end{aligned}$$

which implies $f\in TBSD(\alpha,\lambda,c,p_k)$.
Conversely, for $n\ge k+1$, let

$$\lambda_n=\frac{(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)}{\left(1-\sum_{i=2}^{k}p_i\right)}, \quad (2.8)$$

and $\lambda_k=1-\sum_{n=k+1}^{\infty}\lambda_n$.

Thus f can be expressed as $f(z)=\sum_{n=k}^{\infty}\lambda_nf_n(z)$. □

Corollary 2.5. *The extreme points of the class $TBSD(\alpha,\lambda,c,p_k)$ are the functions $f_n,(n\ge k)$ given by (2.4) and (2.5).*

Next, we prove few theorems by using integral operators. The Alexander operator [1] for the functions in the class $\mathscr{S}$

maps the class of starlike functions onto the class of close to convex functions and is defined as

$$I(f)=\int_0^z \frac{f(t)}{t}dt.$$

Theorem 2.6. *Let f of the form (1.14) be in the class $TBSD(\alpha,\lambda,c,p_k)$. Then $I(f)\in TBSD(\alpha,\lambda,c,p_k)$ where $q_k=\frac{p}{k}$.*

Proof. First of all,

$$\begin{aligned} I(f) &= z-\sum_{i=2}^{k}\frac{q_i}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}z^i \\ &- \sum_{n=k+1}^{\infty}\frac{a_n}{n}z^n. \end{aligned}$$

Now,

$$\begin{aligned} &\sum_{n=k+1}^{\infty}(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)\frac{a_n}{n} \\ &\leq \frac{1}{k+1}\sum_{n=k+1}^{\infty}(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)a_n \\ &\leq \frac{1}{k+1}\left(1-\sum_{i=2}^{k}p_i\right) \\ &= \frac{1}{k+1}-\sum_{i=2}^{k}\frac{p}{k+1} \\ &\leq 1-\sum_{i=2}^{\infty}\frac{p}{i}. \end{aligned}$$

Hence $I(f)\in TBSD(\alpha,\lambda,c,p_k)$. ☐

Next, we obtain the radii results for the function in the class $TBSD(\alpha,\lambda,c,p_k)$ to be starlike or convex of order β. These results are stated in next two theorems.

Theorem 2.7. *Let the function f given by (1.14) belongs to the class $TBSD(\alpha,\lambda,c,p_k)$. Then $f\in\mathscr{S}^*(\beta)$ in the disk $|z|<r_1$, where r_1 is the largest value that satisfies*

$$\sum_{i=2}^{\infty}\left[\frac{(2-i)-\beta}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}p_ir^{i-1}\right] + \frac{((2-n)-\beta)\left(1-\sum_{i=2}^{k}p_i\right)}{(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)}r^{n-1}\leq\beta. \tag{2.9}$$

Proof. To show the theorem, it is enough to establish that

$$\left|\frac{zf'(z)}{f(z)}-1\right|\leq 1-\beta \text{ for } |z|<r_1.$$

Now, Upon simple computations, we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \quad \leq 1-\beta,\text{ for } |z|<r, \text{ if and only if}$$

$$\sum_{i=2}^{k}\left[\frac{((2-i)-\beta)}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}p_ir^{i-1}\right] + \sum_{n=k+1}^{\infty}((2-n)-\beta)a_nr^{n-1}\leq 1-\beta. \tag{2.10}$$

By using Corollary 1.3, we get

$$a_n=\frac{1-\sum_{i=2}^{k}p_i}{(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)}\lambda_n, \tag{2.11}$$

where $\lambda_n\geq 0$, $n\geq k+1$ and $\sum_{n=k+1}^{\infty}\lambda_n\leq 1$.

For each fixed r, choosing an integer $n_0=n_0(r)$, for which

$$\frac{((2-n)-\beta)r^{n-1}}{(1+(n-1)\lambda)(1+(n-1)\alpha)\mathscr{E}(c,\kappa,n)}$$

is a maximum, we obtain

$$\sum_{n=k+1}^{\infty}((2-n)-\beta)a_nr^{n-1} \leq \frac{((2-n_0)-\beta)\left(1-\sum_{i=2}^{k}p_i\right)}{(1+(n_0-1)\lambda)(1+(n_0-1)\alpha)D(c,\kappa,n_0)}r^{n_0-1}. \tag{2.12}$$

Hence f is starlike of order β in $|z|\leq r_1$, provided

$$\sum_{i=2}^{k}\left[\frac{((2-i)-\beta)}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}p_ir^{i-1}\right] + \frac{((2-n_0)-\beta)\left(1-\sum_{i=2}^{k}p_i\right)}{(1+(n_0-1)\lambda)(1+(n_0-1)\alpha)D(c,\kappa,n_0)}r^{n_0-1} \tag{2.13}$$

$$\leq 1-\beta.$$

We find the value of r_0 and corresponding $n_0(r_0)$, so that

$$\sum_{i=2}^{k}\left[\frac{((2-i)-\beta)}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)}p_ir_0^{i-1}\right] + \frac{((2-n_0)-\beta)\left(1-\sum_{i=2}^{k}p_i\right)}{(1+(n_0-1)\alpha)(1+(n_0-1)\lambda)\mathscr{E}(c,\kappa,n_0)}r_0^{n_0-1} \tag{2.14}$$

$$=1-\beta.$$

This is the radius of starlikeness of order β for functions in the class $TBSD(\alpha,\lambda,c,p_k)$. ☐

The radius of convexity for functions in the class $TBSD(\alpha,\lambda,c,p_k)$ is given in the next theorem.

Theorem 2.8. *Let the function f given by (1.14) belong to the class $TBSD(\alpha,\lambda,c,p_k)$. Then $f \in \mathscr{C}(\beta)$ in $|z| < r_2$, where r_2 is the largest value that satisfies*

$$\sum_{i=2}^{\infty} \frac{i(i-\beta)p_i r^{i-1}}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)} + \frac{n(n-\beta)\left(1-\sum_{i=2}^{k} p_i\right) r^{n-1}}{(1+(n-1)\alpha)(1+(n-1)\lambda)\mathscr{E}(c,\kappa,n)} \le 1-\beta. \tag{2.15}$$

Proof. Upon simple computations,for $|z| < r$,

$$\left|\frac{zf''(z)}{f(z)}\right| \le 1-\beta$$

if and only if

$$\sum_{i=2}^{\infty} \frac{i(i-\beta)r^{i-1}}{(1+(i-1)\alpha)(1+(i-1)\lambda)} + \sum_{n=k+1}^{\infty} n(n-\beta)a_n r^{n-1} \le 1-\beta. \tag{2.16}$$

By virtue of Corollary 1.3 and for each fixed r, choosing an integer $n_0 = n_0(r)$ for which

$$\frac{n_0(n_0-\beta)r^{n_0-1}}{(1+(n_0-1)\alpha)(1+(n_0-1)\lambda)\mathscr{E}(c,\kappa,n_0)}$$

is maximum, we get

$$\sum_{n=k+1}^{\infty} n(n-\beta)a_n r^{n-1} \le \frac{n_0(n_0-\beta)\left(1-\sum_{i=2}^{k} p_i\right) r_0^{n_0-1}}{(1+(n_0-1)\alpha)(1+(n_0-1)\lambda)\mathscr{E}(c,\kappa,n_0)}. \tag{2.17}$$

Therefore, f is convex of order β in $|z| < r_2$, provided

$$\sum_{i=2}^{\infty} \frac{i(i-\beta)r^{i-1}}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)} + \frac{n_0(n_0-\beta)\left(1-\sum_{i=2}^{k} p_i\right) r^{n_0-1}}{(1+(n_0-1)\alpha)(1+(n_0-1)\lambda)\mathscr{E}(c,\kappa,n_0)} \tag{2.18}$$

$$\le 1-\beta.$$

We find the value of r_0 and corresponding $n_0(r_0)$, so that

$$\sum_{i=2}^{\infty} \frac{i(i-\beta)r_0^{i-1}}{(1+(i-1)\alpha)(1+(i-1)\lambda)\mathscr{E}(c,\kappa,i)} + \frac{n_0(n_0-\beta)\left(1-\sum_{i=2}^{k} p_i\right) r_0^{n_0-1}}{(1+(n_0-1)\alpha)(1+(n_0-1)\lambda)\mathscr{E}(c,\kappa,n_0)} = 1-\beta.$$

This gives the radius of convexity of order β for the functions f in $TBSD(\alpha,\lambda,c,p_k)$.

□

Acknowledgements

The work of third-named author is supported by a grant from the Science and Engineering Research Board, Government of India under Mathematical Research Impact Centric Support of Department of Science and Technology (DST)(vide ref: MTR/2017/000607).

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ISSN(P):2319 − 3786
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