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Mapping properties of some integral operators associated with generalized Bessel functions

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Abstract

In the present paper, we study the mapping properties of some integral operators on certain classes of harmonic univalent functions associated with generalized Bessel functions of the first kind. To be more precise, we study the mapping properties of Goodman-Rønning-type harmonic univalent functions in the open unit disc $\mathbb{U}$.

Keywords

Harmonic, univalent functions, generalized Bessel functions.

AMS Subject Classification

30C45.

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Contents

1. Introduction

Let A denote the class of functions f of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U}=\{z : z\in\mathbb{C} \text{ and } |z|<1\}$ and satisfy the normalization condition $f(0)=f'(0)-1=0$. Now, we recall that the generalized Bessel function of the first kind $w=w_{p,b,c}$ is defined as the particular solution of the second-order linear homogeneous differential equation

$$z^2\omega''(z)+bz\omega'(z)+\left[cz^2-p^2+(1-b)p\right]\omega(z)=0, \tag{1.2}$$

where $b,p,c\in C$, which is a natural generalization of Bessel's differential equation. This function has the familiar representation

$$\omega(z)=\omega_{p,b,c}(z)=\sum_{n=0}^{\infty}\frac{(-1)^nc^n}{n!\Gamma\left(p+n+\frac{b+1}{2}\right)}\left(\frac{z}{2}\right)^{2n+p}, z\in\mathbb{C}. \tag{1.3}$$

The differential equation (1.2) permits the study of Bessel, modified Bessel, spherical Bessel function and modified spherical Bessel functions all together. Solutions of (1.2) are referred to as the generalized Bessel function of order p. The particular solution given by (1.3) is called the generalized Bessel function of the first kind of order p. Although the series defined above is convergent everywhere, the function $\omega_{p,b,c}$ is generally not univalent in U. It is worth mentioning that, in particular, when $b=c=1$, we reobtain the Bessel function $\omega_{p,1,1}=J_p$ and for $c=-1, b=1$ the function $\omega_{p,1,-1}$ becomes the modified Bessel function I_p. Now, consider the function $u_{p,b,c}$ defined by the transformation

$$u_{p,b,c}(z)=2^p\Gamma\left(p+\frac{b+1}{2}\right)z^{-p/2}\omega_{p,b,c}(z^{1/2}).$$

By using the well-known Pochhammer (or Appell) symbol, defined in terms of the Euler Gamma function for $a\neq 0,-1,-2,\cdots$ by

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=\begin{cases}1 & : \text{if } n=0\\ a(a+1)\cdots(a+n-1) & : \text{if } n=1,2,3,\cdots.\end{cases},$$

We obtain for the function $u_{p,b,c}$ by the following representation

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty} \frac{(-c/4)^n}{\left(p+\frac{(b+1)}{2}\right)_n} \frac{z^n}{n!}, \tag{1.4}$$

where $p+(b+1)/2 \neq 0,-1,-2,\cdots$. This function is analytic on C and satisfies the second-order linear differential equation

$$4z^2u''(z)+2(2p+b+1)zu'(z)+czu(z)=0.$$

For convenience throughout in the sequel, we use the following notations:

$$u_{p,b,c} = u_p, \qquad k = p+\frac{b+1}{2}.$$

Let H be the family of all harmonic functions of the form $f = h+\overline{g}$, where

$$h(z) = z+\sum_{n=2}^{\infty} A_n z^n$$

and

$$g(z) = \sum_{n=1}^{\infty} B_n z^n, \qquad |B_1|<1, \qquad (z\in\mathbb{U}), \tag{1.5}$$

are in the class A. Denote by S_H the subclass of H that are univalent and sense-preserving in U. We also let the subclass S_H^0 of S_H as

$$S_H^0 = \left\{ f = h+\overline{g} \in S_H : g'(0) = B_1 = 0 \right\}.$$

The classes S_H^0 and S_H were first studied in [5]. Also, we let K_H^0, $S_H^{*,0}$ and C_H^0 denote the subclasses of S_H^0 of harmonic functions which are, respectively, convex, starlike and close-to-convex in U. For definitions and properties of these classes, one may refer to [5] or [6].

A function $f(z)$ of the form (1.5) is said to be in the class $N_H(\beta)$, where $0\le\beta<1$, if the the following condition is satisfied

$$\Re\left(\frac{f'(z)}{z'}\right) \ge \beta, \;\; z = re^{i\theta} \in \mathbb{U}.$$

Further we let $TN_H(\beta) \equiv N_H(\beta)\cap T$, where T consists of the functions $f = h+\overline{g}$ in S_H so that h and g are of the form

$$h(z) = z-\sum_{n=2}^{\infty}|A_n|z^n, \qquad g(z) = \sum_{n=1}^{\infty}|B_n|z^n. \tag{1.6}$$

The classes $N_H(\beta)$, $TN_H(\beta)$ were studied by Ahuja and Jahangiri [2].

In 2001, Rosy *et al.* [14] introduced the class $G_H(\gamma)$ consisting of functions of the form (1.5) if it satisfies the following condition

$$\Re\left\{ \left(1+e^{i\alpha}\right)\frac{zf'(z)}{f(z)} - e^{i\alpha} \right\} \ge \gamma, \;\; \alpha\in R, \; z\in\mathbb{U}.$$

Further we define $TG_H(\gamma) \equiv G_H(\gamma)\cap T$. The class $G_H(\gamma)$ is called Goodman-Rønning-type harmonic univalent functions in U.

For complex parameters c_1, k_1, c_2, k_2 $(k_1, k_2 \neq 0,-1,-2,\cdots)$, we introduce the following convolution operator

$$\Omega \equiv \Omega\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} : H \to H$$

defined by

$$\Omega\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f = h(z) * \int_0^z u_{p_1}(t)dt + \overline{g(z) * \int_0^z u_{p_2}(t)dt}$$

for any function $f = h+\overline{g}$ in H.

Letting

$$\Omega\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f(z) = H(z)+\overline{G(z)},$$

where

$$H(z) = z+\sum_{n=2}^{\infty}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}A_n z^n$$

and

$$G(z) = \sum_{n=1}^{\infty}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}B_n z^n. \tag{1.7}$$

Similarly we define the Libera type integral operator

$$L\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f(z) = h(z) * \tfrac{2}{z}\int_0^z zu_{p_1}(t)dt + \overline{g(z) * \tfrac{2}{z}\int_0^z zu_{p_2}(t)dt}, \tag{1.8}$$

or equivalently

$$Ł\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f(z) = H(z)+\overline{G(z)},$$

where

$$H(z) = z+\sum_{n=2}^{\infty}\frac{2(-c_1/4)^{n-1}}{(k_1)_{n-1}(n+1)(n-1)!}A_n z^n,$$

and

$$G(z) = \sum_{n=1}^{\infty}\frac{2(-c_2/4)^{n-1}}{(k_2)_{n-1}(n+1)(n-1)!}B_n z^n. \tag{1.9}$$

Throughout this paper, we will frequently use the notation

$$\Omega(f) = \Omega\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f$$

and

$$L(f) = L\begin{pmatrix} k_1, & c_1 \\ k_2, & c_2 \end{pmatrix} f.$$

The generalized Bessel function is a recent topic of study in the Geometric Function Theory (e.g. see the work of [3], [7]-[11]). Motivated by results on connections between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions (see [1], [4], [12], [13], [15] and [16]), we establish a number of connections between the classes $G_H(\gamma)$, K_H^0, $S_H^{*,0}$, C_H^0 and $N_H(\beta)$ by applying the convolution operators Ω and L.

2. Main Results

In order to establish connections between harmonic convex functions and Goodman-Rønning-type harmonic univalent functions, we shall require the following lemmas.

Lemma 2.1. *([5], [6]). If $f=h+\overline{g}\in K_H^0$ where h and g are given by (1.5) with $B_1=0$, then*

$$|A_n|\le \frac{n+1}{2},\ \ |B_n|\le \frac{n-1}{2}.$$

Lemma 2.2. *([14]). Let $f=h+\overline{g}$ be given by (1.5). If $0\le\gamma<1$ and*

$$\sum_{n=2}^{\infty}(2n-1-\gamma)\,|A_n|+\sum_{n=1}^{\infty}(2n+1+\gamma)\,|B_n|\le 1-\gamma, \quad (2.1)$$

then f is sense-preserving, Goodman-Rønning-type harmonic univalent functions in U and $f\in G_H(\gamma)$.

Remark 2.3. *In [14], it is also shown that $f=h+\overline{g}$ given by (1.6) is in the family $TG_H(\gamma)$, if and only if the coefficient condition (2.1) holds. Moreover, if $f\in TG_H(\gamma)$, then*

$$|A_n|\le \frac{1-\gamma}{2n-1-\gamma}, \qquad n\ge 2,$$

and

$$|B_n|\le \frac{1-\gamma}{2n+1+\gamma}, \qquad n\ge 1.$$

Lemma 2.4. *([3]). If $b,p,c\in C$ and $k\neq 0,-1,-2,\cdots$ then the function u_p satisfies the recursive relation $4ku_p'(z)=-cu_{p+1}(z)$ for all $z\in\mathbb{C}$.*

Lemma 2.5. *If $c<0$ and $k>1$, then*

$$\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(k)_n(n+1)!}=\frac{-4(k-1)}{c}\left[u_{p-1}(1)-1\right].$$

Proof. We can write

$$\sum_{n=0}^{\infty}\frac{(-c/4)^n}{(k)_n(n+1)!}=\frac{(k-1)}{(-c/4)}\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k-1)_{n+1}(n+1)!}$$
$$=\frac{-4(k-1)}{c}\left[u_{p-1}(1)-1\right].$$

□

Theorem 2.6. *Let $c_1,c_2<0$, $k_1,k_2>1$. If for some $\gamma(0\le\gamma<1)$ and the inequality*

$$\begin{aligned}&2u_{p_1}'(1)+(3-\gamma)u_{p_1}(1)-(1+\gamma)\\&\left[\tfrac{-4(k_1-1)}{c_1}\left[u_{p_1-1}(1)-1\right]\right]+2u_{p_2}'(1)+(1+\gamma)u_{p_2}(1)\\&-(1+\gamma)\left[\tfrac{-4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]\right]\le 4(1-\gamma)\end{aligned} \quad (2.2)$$

is satisfied then $\Omega\left(K_H^0\right)\subset G_H(\gamma)$.

Proof. Let $f=h+\overline{g}\in K_H^0$ where h and g are of the form (1.5) with $B_1=0$. We need to show that $\Omega(f)=H+\overline{G}\in G_H(\gamma)$, where H and G defined by (1.7) with $B_1=0$ are analytic functions in U.

In view of Lemma 2.2, we need to prove that

$$P_1\le 1-\gamma,$$

where

$$P_1=\sum_{n=2}^{\infty}(2n-1-\gamma)\left|\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}A_n\right|+\sum_{n=2}^{\infty}(2n+1+\gamma)\left|\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}B_n\right|.$$

In view of Lemma 2.1, we have

$$\begin{aligned}P_1&\le\frac{1}{2}\left[\sum_{n=2}^{\infty}(n+1)(2n-1-\gamma)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}+\sum_{n=2}^{\infty}(n-1)(2n+1+\gamma)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}\right]\\&=\frac{1}{2}\left[\sum_{n=2}^{\infty}\{2n(n-1)+(3-\gamma)n-(1+\gamma)\}\times\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}+\sum_{n=2}^{\infty}\{2n(n-1)+(\gamma+1)n-(\gamma+1)\}\times\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}\right]\\&=\frac{1}{2}\left[2\sum_{n=0}^{\infty}\tfrac{(-c_1/4)^{n+1}}{(k_1)_{n+1}n!}+(3-\gamma)\sum_{n=0}^{\infty}\tfrac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}-(1+\gamma)\sum_{n=0}^{\infty}\tfrac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+2)!}+2\sum_{n=0}^{\infty}\tfrac{(-c_2/4)^{n+1}}{(k_2)_{n+1}n!}\right.\\&\qquad\left.+(\gamma+1)\sum_{n=0}^{\infty}\tfrac{(-c_2/4)^{n+1}}{(k_2)_{n+1}(n+1)!}-(\gamma+1)\sum_{n=0}^{\infty}\tfrac{(-c_2/4)^{n+1}}{(k_2)_{n+1}(n+2)!}\right]\end{aligned}$$

$$= \frac{1}{2}\left[2\frac{(-c_1/4)}{k_1}\sum_{n=0}^{\infty}\frac{(-c_1/4)^n}{(k_1+1)_n n!}+(3-\gamma)\right.$$
$$\left\{u_{p_1}(1)-1\right\}-(1+\gamma)\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+2)!}$$
$$+2\frac{(-c_2/4)}{k_2}\sum_{n=0}^{\infty}\frac{(-c_2/4)^n}{(k_2+1)_n n!}+(1+\gamma)$$
$$\left.\left\{u_{p_2}(1)-1\right\}-(1+\gamma)\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}(n+2)!}\right]$$
$$= \frac{1}{2}\left[2\frac{(-c_1/4)}{k_1}u_{p_1+1}(1)+(3-\gamma)\left\{u_{p_1}(1)-1\right\}\right.$$
$$-(1+\gamma)\left[\frac{-4(k_1-1)}{c_1}\left[u_{p_1-1}(1)-1\right]-1\right]$$
$$+2\frac{(-c_2/4)}{k_2}u_{p_2+1}(1)+(1+\gamma)\left\{u_{p_2}(1)-1\right\}$$
$$\left.-(1+\gamma)\left[\frac{-4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]-1\right]\right]$$
$$= \frac{1}{2}\left[2u'_{p_1}(1)+(3-\gamma)\left(u_{p_1}(1)-1\right)\right.$$
$$-(1+\gamma)\left[\frac{-4(k_1-1)}{c_1}\left[u_{p_1-1}(1)-1\right]-1\right]$$
$$+2u'_{p_2}(1)+(1+\gamma)\left(u_{p_2}(1)-1\right)$$
$$\left.-(1+\gamma)\left[\frac{-4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]-1\right]\right].$$

Now $P_1 \le 1-\gamma$ follows from the given condition

This completes the proof of Theorem 2.6. □

Analogous to Theorem 2.6, we next find conditions of the classes $S_H^{*,0}$, C_H^0 with $G_h(\gamma)$. However we first need the following result which may be found in [5], [6].

Lemma 2.7. *If $f=h+\overline{g}\in S_H^{*,0}$ or C_H^0 where h and g are given by (1.5) with $B_1=0$, then*

$$|A_n| \le \frac{(2n+1)(n+1)}{6},\quad |B_n| \le \frac{(2n-1)(n-1)}{6}.$$

Theorem 2.8. *If $c_1,c_2<0$, $k_1,k_2>1$. If for some $\gamma(0\le\gamma<1)$ and the inequality*

$$4u''_{p_1}(1)+(16-2\gamma)u'_{p_1}(1)+(7-5\gamma)u_{p_1}(1)$$
$$-(1+\gamma)\left[\frac{-4(k_1-1)}{c_1}\left[u_{p_1-1}(1)-1\right]\right]$$
$$+4u''_{p_2}(1)+(8+2\gamma)u'_{p_2}(1)-(1+\gamma)u_{p_2}(1)$$
$$+(1+\gamma)\left[\frac{-4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]\right]\le 12(1-\gamma) \quad (2.3)$$

is satisfied, then

$$\Omega(S_H^{*,0})\subset G_H(\gamma) \text{ and } \Omega(C_H^0)\subset G_H(\gamma).$$

Proof. Let $f=h+\overline{g}\in S_H^{*,0}$ (C_H^0) where h and g are given by (1.5) with $B_1=0$. We need to show that $\Omega(f)=H+\overline{G}\in G_H(\gamma)$, where H and G defined by (1.7) with $B_1=0$ are analytic functions in U. In view of Lemma 2.2, it is enough to show that $P_1\le 1-\gamma$, where

$$P_1=\sum_{n=2}^{\infty}(2n-1-\gamma)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}|A_n|$$
$$+\sum_{n=2}^{\infty}(2n+1+\gamma)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}|B_n|.$$

In view of Lemma 2.7, we have

$$P_1\le\frac{1}{6}\left[\sum_{n=2}^{\infty}(2n+1)(n+1)(2n-1-\gamma)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}\right.$$
$$\left.+\sum_{n=2}^{\infty}(2n-1)(n-1)(2n+1+\gamma)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}\right]$$
$$=\frac{1}{6}\left[\sum_{n=2}^{\infty}\{4n(n-1)(n-2)\right.$$
$$+(16-2\gamma)n(n-1)+(7-5\gamma)n-(\gamma+1)\}$$
$$\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}+\sum_{n=2}^{\infty}\{4n(n-1)(n-2)+(8+2\gamma)$$
$$\left.n(n-1)-(1+\gamma)n+(1+\gamma)\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}\right]$$
$$=\frac{1}{6}\left[\left\{4\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n-1)!}+(16-2\gamma)\right.\right.$$
$$\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}n!}+(7-5\gamma)\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}$$
$$\left.-(1+\gamma)\sum_{n=0}^{\infty}\frac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+2)!}\right\}$$
$$+\left\{4\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}(n-1)!}+(8+2\gamma)\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}n!}\right.$$
$$-(1+\gamma)\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}(n+1)!}$$
$$\left.\left.+(1+\gamma)\sum_{n=0}^{\infty}\frac{(-c_2/4)^{n+1}}{(k_2)_{n+1}(n+2)!}\right\}\right]$$
$$=\frac{1}{6}\left[\left\{4u''_{p_1}(1)+(16-2\gamma)u'_{p_1}(1)+(7-5\gamma)\right.\right.$$
$$\left\{u_{p_1}(1)-1\right\}$$
$$\left.-(1+\gamma)\left[\frac{-4(k_1-1)}{c_1}\left[u_{p_1-1}(1)-1\right]-1\right]\right.$$
$$+\left\{4u''_{p_2}(1)+(8+2\gamma)u'_{p_2}(1)-(1+\gamma)\left\{u_{p_2}(1)-1\right\}\right.$$
$$\left.\left.+(1+\gamma)\left[\frac{-4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]-1\right]\right\}\right].$$

Now $P_1\le 1-\gamma$ follows from the given condition. □

In order to determine connection between $TN_H(\beta)$ and $G_H(\gamma)$, we need the following results in Lemma 2.9.

Lemma 2.9. *([2]). Let $f = h + \overline{g}$ where h and g are given by (1.6) with $B_1 = 0$, and suppose that $0 \leq \beta < 1$. Then*

$$f \in TN_H(\beta) \Leftrightarrow \sum_{n=2}^{\infty} n|A_n| + \sum_{n=2}^{\infty} n|B_n| \leq 1-\beta.$$

Remark 2.10. *If $f \in TN_H(\beta)$, then $|A_n| \leq \frac{1-\beta}{n}$ and $|B_n| \leq \frac{1-\beta}{n}$, $n \geq 2$.*

Theorem 2.11. *If $c_1, c_2 < 0$, $k_1, k_2 > 1$ $(k_1, k_2 \neq 0, -1, -2, \cdots)$. If for some $\beta (0 \leq \beta < 1)$, $\gamma (0 \leq \gamma < 1)$ and the inequality*

$$\begin{aligned}(1-\beta)\Big[2\left\{u_{p_1}(1)-1\right\}+(1+\gamma)\tfrac{4(k_1-1)}{c_1}\\ \left[u_{p_1-1}(1)-1-\tfrac{(-c_1/4)}{k_1-1}\right]\\ +2u_{p_2}(1)-(1+\gamma)\tfrac{4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]\Big]\\ \leq 1-\gamma\end{aligned}$$

is satisfied then

$$\Omega(TN_H(\beta)) \subset G_H(\gamma).$$

Proof. Let $f = h + \overline{g} \in TN_H(\beta)$ where h and g are given by (1.5). In view of Lemma 2.2, it is enough to show that $P_2 \leq 1-\gamma$, where

$$P_2 = \sum_{n=2}^{\infty}(2n-1-\gamma)\tfrac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}|A_n| +$$

$$\sum_{n=1}^{\infty}(2n+1+\gamma)\tfrac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}|B_n|.$$

Using Remark 2.10, we have

$$\begin{aligned}P_2 &\leq (1-\beta)\left[\sum_{n=2}^{\infty}\left\{2-\tfrac{(1+\gamma)}{n}\right\}\tfrac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}\right.\\ &\quad\left.+\sum_{n=1}^{\infty}\left\{2+\tfrac{(1+\gamma)}{n}\right\}\tfrac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}\right]\\ &=(1-\beta)\left[2\sum_{n=0}^{\infty}\tfrac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+1)!}\right.\\ &\quad-(1+\gamma)\sum_{n=0}^{\infty}\tfrac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+2)!}\\ &\quad\left.+2\sum_{n=0}^{\infty}\tfrac{(-c_2/4)^{n}}{(k_2)_{n}n!}+(1+\gamma)\sum_{n=0}^{\infty}\tfrac{(-c_2/4)^{n}}{(k_2)_{n}(n+1)!}\right]\\ &=(1-\beta)\Big[2\left\{u_{p_1}(1)-1\right\}-(1+\gamma)\tfrac{(k_1-1)}{(-c_1/4)}\\ &\quad\sum_{n=0}^{\infty}\tfrac{(-c_1/4)^{n+2}}{(k_1-1)_{n+2}(n+2)!}\\ &\quad+2u_{p_2}(1)+(1+\gamma)\tfrac{(k_2-1)}{(-c_2/4)}\sum_{n=0}^{\infty}\tfrac{(-c_2/4)^{n+1}}{(k_2-1)_{n+1}(n+1)!}\Big]\\ &=(1-\beta)\Big[2\left\{u_{p_1}(1)-1\right\}+(1+\gamma)\tfrac{4(k_1-1)}{c_1}\\ &\quad\left[u_{p_1-1}(1)-1-\tfrac{(-c_1/4)}{k_1-1}\right]\\ &\quad+2u_{p_2}(1)-(1+\gamma)\tfrac{4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]\Big]\\ &\leq 1-\gamma,\end{aligned}$$

by the given hypothesis. □

In next theorem, we establish connections between $TG_H(\gamma)$ and $G_H(\gamma)$.

Theorem 2.12. *Let $c_1, c_2 < 0$, $k_1, k_2 > 1$. If for some $\gamma (0 \leq \gamma < 1)$ the inequality*

$$\tfrac{4(k_1-1)}{c_1}\left[u_{p_1-1}(1)-1\right]+\tfrac{4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right] \geq -2 \quad (2.4)$$

is satisfied, then $\Omega(TG_H(\gamma)) \subset G_H(\gamma)$.

Proof. Let $f = h + \overline{g} \in TG_H(\gamma)$ where h and g are given by (1.6). In view of Lemma 2.2, it is enough to show that $P_2 \leq 1-\gamma$, where

$$P_2 = \sum_{n=2}^{\infty}(2n-1-\gamma)\tfrac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}|A_n| +$$

$$\sum_{n=1}^{\infty}(2n+1+\gamma)\tfrac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}|B_n|.$$

Using Remark 2.3, it follows that

$$\begin{aligned}P_2 &= \sum_{n=2}^{\infty}(2n-1-\gamma)\tfrac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}|A_n| +\\ &\quad\sum_{n=1}^{\infty}(2n+1+\gamma)\tfrac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}|B_n|\\ &\leq(1-\gamma)\left[\sum_{n=2}^{\infty}\tfrac{(-c_1/4)^{n-1}}{(k_1)_{n-1}n!}+\sum_{n=1}^{\infty}\tfrac{(-c_2/4)^{n-1}}{(k_2)_{n-1}n!}\right]\\ &=(1-\gamma)\left[\sum_{n=0}^{\infty}\tfrac{(-c_1/4)^{n+1}}{(k_1)_{n+1}(n+2)!}\right.\\ &\quad\left.+\sum_{n=0}^{\infty}\tfrac{(-c_2/4)^{n}}{(k_2)_{n}(n+1)!}\right]\\ &=(1-\gamma)\Big[-\tfrac{4(k_1-1)}{c_1}\left[u_{p_1-1}(1)-1\right]-1\\ &\quad-\tfrac{4(k_2-1)}{c_2}\left[u_{p_2-1}(1)-1\right]\Big]\\ &\leq(1-\gamma),\end{aligned}$$

by the given condition and this completes the proof. □

In next theorem, we present conditions on the parameters k_1, k_2, c_1, c_2 and obtain a characterization for operator Ω which maps $TG_H(\gamma)$ on to itself.

Theorem 2.13. *If $c_1, c_2 < 0$, $k_1, k_2 > 1$ and $\gamma (0 \le \gamma < 1)$. Then*

$$\Omega(TG_H(\gamma)) \subset TG_H(\gamma),$$

if and only if the inequality (2.4) is satisfied.

Proof. The proof of above theorem is similar to that of Theorem 2.4. Therefore we omits the details involved. □

Theorem 2.14. *Let $c_1, c_2 < 0$, $k_1 > 0, k_2 > 2$. If for some $\gamma (0 \le \gamma < 1)$ and the inequality*

$$2u'_{p_1}(1) + (1-\gamma)(u_{p_1}(1)-1) + 2u'_{p_2}(1) + (\gamma - 1)$$
$$(u_{p_2}(1)-1) + 2(1-\gamma)\left[\frac{-4(k_2-1)}{c_2}[u_{p_2-1}(1)-1] - 1\right]$$
$$-2(1-\gamma)\left[\frac{16(k_2-2)(k_2-1)}{c_2^2}[u_{p_2-2}(1)-1\right.$$
$$\left.\left. + \frac{c_2/4}{k_2-2} - \frac{(c_2/4)^2}{2(k_2-2)(k_2-1)}\right]\right] \le 2(1-\gamma) \tag{2.5}$$

is satisfied then $L\left(K_H^0\right) \subset G_H(\gamma)$.

Proof. Let $f = h + \overline{g} \in K_H^0$ where h and g are of the form (1.5) with $B_1 = 0$. We need to show that $L(f) = H + \overline{G} \in G_H(\gamma)$, where H and G defined by (1.9) with $B_1 = 0$ are analytic functions in U.

In view of Lemma 2.2, we need to prove that

$$P_3 \le 1 - \gamma,$$

where

$$P_3 = \sum_{n=2}^{\infty}(2n-1-\gamma)\left|\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\frac{2A_n}{n+1}\right| +$$
$$\sum_{n=2}^{\infty}(2n+1+\gamma)\left|\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n-1)!}\frac{2B_n}{n+1}\right|.$$

In view of Lemma 2.1, we have

$$P_3 \le \left[\sum_{n=2}^{\infty}(2n-1-\gamma)\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!} + \right.$$
$$\left.\sum_{n=2}^{\infty}(2n+1+\gamma)n(n-1)\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n+1)!}\right]$$
$$= \left[\sum_{n=2}^{\infty}\{2(n-1)+(1-\gamma)\}\frac{(-c_1/4)^{n-1}}{(k_1)_{n-1}(n-1)!}\right.$$
$$+\sum_{n=2}^{\infty}\{2(n+1)n(n-1)+(\gamma-1)(n+1)n+$$
$$\left.2(1-\gamma)(n+1)+2(\gamma-1)\}\frac{(-c_2/4)^{n-1}}{(k_2)_{n-1}(n+1)!}\right]$$

performing the similar calculation as in Theorem 2.6 we obtain the required condition.

This completes the proof. □

Theorem 2.15. *Let $c_1, c_2 < 0$, $k_1, k_2 > 2$, $0 \le \beta < 1$. If for some $\gamma (0 \le \gamma < 1)$ and the inequality*

$$2(1-\beta)\left[2\left\{\frac{-4(k_1-1)}{c_1}(u_{p_1-1}(1)-1)-1\right\}\right.$$
$$-(3+\gamma)\left\{\frac{16(k_1-2)(k_1-1)}{c_1^2}\left(u_{p_1-2}(1)-1+\frac{c_1/4}{k_1-2}\right)-\frac{1}{2}\right\}$$
$$+2\left\{\frac{-4(k_2-1)}{c_2}(u_{p_2-1}(1)-1)\right\}$$
$$\left.-(1-\gamma)\left\{\frac{16(k_2-2)(k_2-1)}{c_2^2}\left(u_{p_2-2}(1)-1+\frac{c_2}{4(k_2-2)}\right)\right\}\right] \tag{2.6}$$
$$\le 1-\gamma$$

is satisfied then $L(TN_H(\beta)) \subset G_H(\gamma)$.

Proof. The proof of above theorem is similar to that of Theorem 2.14, therefore we omit the details involved. □

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