Malaya Journal of Matematik, Vol. 8, No. 3, 1099-1109, 2020
https://doi.org/10.26637/MJM0803/0063

On generalized Caputo fractional differential equations and inclusions with non-local generalized fractional integral boundary conditions

Muthaiah Subramanian[1]*, Sargunam Muthu[2], Murugesan Manigandan[3] and Thangaraj Nandha Gopal[4]

Abstract
In this article, concerning nonlocal generalized fractional integral boundary conditions, we investigate the existence of solutions for new boundary value problems of generalized Caputo-type fractional differential equations and inclusions. In the case of equations, we make use of the Banach fixed point theorem and fixed point theorem due to O'Regan and the nonlinear alternative for contractive maps for inclusions. Examples are given to clarify our main results. Finally, we discuss some variants of the given problem.

Keywords
Fractional differential equations, generalized Caputo fractional derivative, Generalized Riemann-Liouville fractional integral, Non-local, Existence, Inclusions, Fixed point.

AMS Subject Classification
26A33, 34A08, 34A12, 34B10, 34A60.

[1,3,4] *Department of Mathematics, Sri Ramakrishna Mission Vidyalaya College of Arts and Science, Coimbatore-641020, Tamilnadu, India.*
[2] *School of Education Science, DMI-ST.Eugene University, Chipata Campus, Zambia.*
***Corresponding author**: subramanianmcbe@gmail.com;
Article History: Received **24** November **2019**; Accepted **24** January **2020**

Contents

1. Introduction

The technique of dealing with fractional differential equations (FDEs) wasn't much for young researchers until the generalized fractional derivatives defined by various fractional operators were exposed. Generalized fractional differential equations (GCFDEs) of Caputo type resulted in effective numerical solutions to differential equations. It is also well-known that the Caputo derivative plays an essential role in physical memory problems. Katugampola in [15] has also introduced a new fractional integral, which acts as a combined integral for the Riemann-Liouville and Hadamard integrals. Research work in this area has grown significantly worldwide due to extensive applications of FDEs in engineering and science. For examples and details see [1, 3, 6, 7, 9–13, 17, 21, 22, 27–31, 33] and the references cited therein. In recent years non-local boundary value problems (BVPs) for FDEs and inclusions have been studied by many researchers. Ntouyas et.al [23] has studied the existence of solutions with sum and integral boundary conditions for fractional differential inclusions. In [2] the authors analyzed the existence results for the fractional differential inclusion and the integrated boundary conditions of the form of type Erdelyi-Kober:

$$\begin{aligned} D^q x(t) &\in F(t,x(t)), \ t\in[0,T], \\ x(0) &= 0, \ \alpha x(T)=\sum_{i=1}^{m}\beta_i I_{\eta_i}^{\gamma_i,\delta_i}x(\xi_i), \end{aligned}$$

where $1<q\leq 2$, D^q is the Riemann-Liouville fractional derivative (RLFD) of order q, $F:[0,1]\times\mathbb{R}\to P(\mathbb{R})$ is a multivalued map, and $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\alpha,\beta_i\in\mathbb{R}$, $\xi_i\in(0,T)$, $I_{\eta_i}^{\gamma_i,\delta_i}$ is the Erdelyi-Kober

fractional integral of order $\delta_i > 0$, $\eta_i > 0$, $\gamma_i \in \mathbb{R}$ $i = 1,2,...,m$ are given constants. In [4] the authors investigated the FDEs and inclusion with integral boundary conditions of the form of Erdelyi-Kober type:

$$\begin{aligned} {}^cD^q x(t) &= f(t,x(t)), \ t \in [0,T], \\ x(0) &= g(x(t)), \ x(T) = \alpha I^{\gamma,\delta} x(\xi), \end{aligned}$$

and

$$\begin{aligned} {}^cD^q x(t) &\in F(t,x(t)), \ t \in [0,T], \\ x(0) &= g(x), \ x(T) = \alpha I^{\gamma,\delta} x(\xi), \end{aligned}$$

where $1 < q \leq 2$, D^q is the Caputo fractional derivative (CFD) of order q, f: $\mathscr{J} \times \mathbb{R} \to \mathbb{R}$ and g: $\mathscr{C}(\mathscr{J},\mathbb{R}) \to \mathbb{R}$ are given continuous functions, and $F : [0,1] \times \mathbb{R} \to P(\mathbb{R})$ is a multi-valued map, and $P(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\alpha \in \mathbb{R}$, $\xi \in (0,T)$, $I_\eta^{\gamma,\delta}$ is the Erdelyi-Kober fractional integral of order $\delta > 0$, $\eta > 0$ $\gamma \in \mathbb{R}$ are given constants. The GCFDEs have been studied by many researchers in recent years. Zeng et.al [32] discussed FDEs and numerical solutions of the type of Caputo-Katugampola. FDEs with Caputo-Katugampola dependence were obtained by Almeida et.al [5]. Jarad et.al [14] analyzed generalized and Caputo modification of fractions derivatives. In this paper, we begin the investigation of GCFDEs BVPs and inclusions improved with non-local generalized integral boundary conditions. We examine in exact terms the existence and uniqueness of the solutions for the GCFDEs and the form:

$$\begin{aligned} &{}^\rho_{\mathscr{C}}\mathscr{D}^\rho y(\tau) = h(\tau,y(\tau)), \ \tau \in \mathscr{J} := [0,T], \\ &y(0) = 0, \quad \varpi^\rho \mathscr{I}^\varsigma y(\varphi) = g(y), \\ &1 < \rho \leq 2, \ 0 < \varsigma < 1, \ \varphi \in (0,T), \end{aligned} \tag{1.1}$$

and

$$\begin{aligned} &{}^\rho_{\mathscr{C}}\mathscr{D}^\rho y(\tau) \in H(\tau,y(\tau)), \ \tau \in \mathscr{J} := [0,T], \\ &y(0) = 0, \quad \varpi^\rho \mathscr{I}^\varsigma y(\varphi) = g(y), \\ &1 < \rho \leq 2, \ 0 < \varsigma < 1, \ \varphi \in (0,T), \end{aligned} \tag{1.2}$$

where ${}^\rho_{\mathscr{C}}\mathscr{D}^\rho$ denote the generalized Caputo fractional derivative (GCFD) of order $1 < \rho \leq 2$, ${}^\rho\mathscr{I}^\varsigma$ denote the generalized Riemann-Liouville fractional integral (GRLFI) of order $0 < \varsigma < 1$, $\rho > 0$, and h: $\mathscr{J} \times \mathbb{R} \to \mathbb{R}$ and g: $\mathscr{C}(\mathscr{J},\mathbb{R}) \to \mathbb{R}$ are given continuous functions and ϖ is positive real constant and H: $\mathscr{J} \times \mathbb{R} \to \mathscr{E}(\mathbb{R})$ is a multivalued function, $\mathscr{E}(\mathbb{R}$ is the family of all nonempty subsets of $\mathbb{R}$).

We emphasize that $g(y)$ in (1.1) may be interpreted as $g(y) = \sum_{j=1}^{k} \zeta_j y(\tau_j)$, where ζ_j, $j = 1,2,\cdots,k$ are given constants and $\tau_1 < \tau_2 < \cdots < \tau_k \leq 1$. Whatever is left of the paper is sorted out as follows. Section 2 shows the essential founding material identified with our problem and has demonstrated an auxiliary lemma. Section 3 contains the main results for FDEs. Examples in section 4 are used to validate the solutions. Section 5 contains the main results for inclusions. The important observations of the results are made in Section 6.

2. Preliminaries

In section 2 we present some notations, fractional calculus definitions and preliminary results needed later on in our proof [14, 16, 18, 20, 26].

Let us define the space of complex-valued all Lebesgue measurable functions $h : (b,c) \to \mathbb{R} \ni \|h\|_{T_a^p} < \infty$, where $a \in \mathbb{R}$, $1 \leq p \leq \infty$ and

$$\|h\|_{T_a^p} = \left(\int_b^c |\upsilon^a h(\upsilon)|^p \frac{d\upsilon}{\upsilon} \right)^{\frac{1}{p}}, 1 \leq p \leq \infty.$$

$\mathscr{L}^1(b,c)$ refers to the measurable space of all Lebesgue functions ϕ on (b,c) endowed with the norm:

$$\|\phi\|_{\mathscr{L}^1} = \int_c^b |\phi(\upsilon)| d\upsilon < \infty.$$

Definition 2.1. *[16] The left and right-sided GRLFIs of order $\rho > 0$ and $\rho > 0$, of a function $h \in T_a^p(b,c)$, $\forall -\infty < b < \tau < c < \infty$, is defined as*

$$\begin{aligned} ({}^\rho\mathscr{I}^\rho_{b+}h)(\tau) &= \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_b^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}} h(\upsilon) d\upsilon, \\ ({}^\rho\mathscr{I}^\rho_{c-}h)(\tau) &= \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_\tau^c \frac{\upsilon^{\rho-1}}{(\upsilon^\rho - \tau^\rho)^{1-\rho}} h(\upsilon) d\upsilon. \end{aligned}$$

Remark 2.2. *The above definition for GRLFIs reduce to the standard Riemann-Liouville fractional integrals for $\rho \to 1$ (see [16]). and Hadmard fractional integrals $\rho \to 0$ respectively (see [18]).*

Definition 2.3. *[14] For $\rho \geq 0$ and $h \in \mathscr{AC}^n_\delta[b,c]$, the left and right-sided GCFDs of order ρ are described by*

$${}^\rho_{\mathscr{C}}\mathscr{D}^\rho_{b+}h(\varepsilon) = {}^\rho\mathscr{D}^\rho_{b+}\Big[h(\tau) - \sum_{j=0}^{n-1} \frac{\delta^j h(b)}{j!} \Big(\frac{\tau^\rho - b^\rho}{\rho}\Big)^j\Big](\varepsilon),$$

$${}^\rho_{\mathscr{C}}\mathscr{D}^\rho_{c-}h(\varepsilon) = {}^\rho\mathscr{D}^\rho_{c-}\Big[h(\tau) - \sum_{j=0}^{n-1} \frac{(-1)^j\delta^j h(c)}{j!} \Big(\frac{c^\rho - \tau^\rho}{\rho}\Big)^j\Big](\varepsilon),$$

$$\delta = \Big(\varepsilon^{1-\rho} \frac{d}{d\varepsilon}\Big).$$

Remark 2.4. *The above definitions for GCFDs reduces to the standard Caputo derivatives and Hadamard fractional derivatives (see for $\rho \to 1$ (see [14])) and $\rho \to 0$ respectively (see [14]).*

Definition 2.5. *[14] Let $\rho > 0$, $\rho > 0$, $n = [\rho] + 1, 0 < b < \tau < c \leq \infty$. The operators*

$$({}^\rho\mathscr{D}^\rho_{b+}h)(\tau) = \frac{\rho^{\rho-n+1}}{\Gamma(n-\rho)} \Big(\tau^{1-\rho}\frac{d}{d\tau}\Big)^n \int_b^\tau \frac{\upsilon^{\rho-1}h(\upsilon)d\upsilon}{(\tau^\rho - \upsilon^\rho)^{\rho-n+1}},$$

$$({}^\rho\mathscr{D}^\rho_{c-}h)(\tau) = \frac{\rho^{\rho-n+1}}{\Gamma(n-\rho)} \Big(-\tau^{1-\rho}\frac{d}{d\tau}\Big)^n \int_\tau^c \frac{\upsilon^{\rho-1}h(\upsilon)d\upsilon}{(\upsilon^\rho - \tau^\rho)^{\rho-n+1}},$$

for $\tau \in (b,c)$ are called the left and right sided generalized Riemann-Liouville fractional derivatives (GRLFDs) of fractional order ρ, respectively.

Lemma 2.6. *Let $\rho \geq 0$, $n = [\rho]+1$ and $h \in AC^n_\delta[b,c]$, where $0<b<c<\infty$. Then*

(1) for $\rho \notin \mathbb{N}$,

$$ {}^{\rho}_{\mathscr{C}}\mathscr{D}^{\rho}_{b+}h(\tau) = \frac{1}{\Gamma(n-\rho)}\int_b^\tau \frac{\left(\frac{\tau^\rho-\upsilon^\rho}{\rho}\right)^{n-\rho-1}(\delta^n h)(\upsilon)d\upsilon}{\upsilon^{1-\rho}}, = {}^{\rho}\mathscr{I}^{n-\rho}_{b+}(\delta^n h)(\tau), $$

$$ {}^{\rho}_{\mathscr{C}}\mathscr{D}^{\rho}_{c-}h(\tau) = \frac{1}{\Gamma(n-\rho)}\int_\tau^c \frac{\left(\frac{\upsilon^\rho-\tau^\rho}{\rho}\right)^{n-\rho-1}(-1)^n(\delta^n h)(\upsilon)d\upsilon}{\upsilon^{1-\rho}}, = {}^{\rho}\mathscr{I}^{n-\rho}_{c-}(\delta^n h)(\tau). $$

(2) for $\rho \in \mathbb{N}$,

$$ {}^{\rho}_{\mathscr{C}}\mathscr{D}^{\rho}_{b+}h = \delta^n h, \quad {}^{\rho}_{\mathscr{C}}\mathscr{D}^{\rho}_{c-}h = (-1)^n\delta^n h. $$

Lemma 2.7. *Let $\rho \neq 0$, a function $h \in \mathscr{AC}^n_\delta[b,c]$. Then*

$$ {}^{\rho}\mathscr{I}^{\rho}_{b+}{}^{\rho}_{\mathscr{C}}\mathscr{D}^{\rho}_{b+}h(\varepsilon) = \Big[h(\varepsilon) - \sum_{j=0}^{n-1}\frac{(\delta^j h)(b)}{j!}\Big(\frac{\varepsilon^\rho - b^\rho}{\rho}\Big)^j\Big], $$

$$ {}^{\rho}\mathscr{I}^{\rho}_{c-}{}^{\rho}_{\mathscr{C}}\mathscr{D}^{\rho}_{c-}h(\varepsilon) = \Big[h(\varepsilon) - \sum_{j=0}^{n-1}\frac{(-1)^j(\delta^j h)(c)}{j!}\Big(\frac{c^\rho - \varepsilon^\rho}{\rho}\Big)^j\Big]. $$

In particular, if $0<\rho\leq 1$, we have

$$ {}^{\rho}\mathscr{I}^{\rho}_{b+}{}^{\rho}_{\mathscr{C}}\mathscr{D}^{\rho}_{b+}h(\varepsilon) = h(\varepsilon)-h(b), $$
$$ {}^{\rho}\mathscr{I}^{\rho}_{c-}{}^{\rho}_{\mathscr{C}}\mathscr{D}^{\rho}_{c-}h(\varepsilon) = h(\varepsilon)-h(c). $$

Lemma 2.8. *Let $\rho>0$ and $\rho \in \mathbb{R}$. Then*

$$ {}^{\rho}\mathscr{I}^{\rho}\tau^{\iota} = \frac{\Gamma(\frac{\iota}{\rho}+1)}{\Gamma(\frac{\iota}{\rho}+\rho+1)}\frac{\tau^{\rho\rho+\iota}}{\rho^\rho}. $$

Lemma 2.9. *Let $\rho,\varsigma>0$, $1\leq p\leq\infty$, $0<b<c<\infty$. Then for $h \in T^p_a(b,c)$, $\rho>0$.*

$$ {}^{\rho}\mathscr{I}^{\rho}_{b+}{}^{\rho}\mathscr{I}^{\varsigma}_{b+}h = {}^{\rho}\mathscr{I}^{\rho+\varsigma}_{b+}h. $$

We define space $\mathscr{H} = \mathscr{C}(\mathscr{J},\mathbb{R})$ endowed with the norm $\|y\| = \sup\{|y(\tau)|, \tau \in \mathscr{J}\}$. Obviously $(\mathscr{H},\|\cdot\|)$ is a Banach space. Let us introduce $\mathscr{AC}^n_\delta(\mathscr{J})$, which consists of the functions h that have absolutely continuous δ^{n-1} derivative, where $\delta = \tau^{1-\rho}\frac{d}{d\tau}$. We define the space $\mathscr{AC}^n_\delta(\mathscr{J},\mathbb{R}) = \{h : \mathscr{J}\to\mathbb{R} : \delta^{n-1}h \in \mathscr{AC}(\mathscr{J},\mathbb{R}), \delta = \tau^{1-\rho}\frac{d}{d\tau}\}$, which is equipped with the norm $\|h\|_{\mathscr{C}^n_\delta} = \sum_{k=0}^{n-1}\|\delta^k h\|_{\mathscr{C}}$.

Lemma 2.10. *Let $\hat{h} \in \mathscr{C}(0,T)\cap\mathscr{L}^1(0,T)$, $y \in \mathscr{AC}^2_\delta(\mathscr{J})$, the function y is the solution of the problem*

$$ \begin{cases} {}^{\rho}\mathscr{D}^{\rho}y(\tau) = \widehat{h}(\tau), \ \tau \in \mathscr{J}, \\ y(0) = 0, \ \varpi\, {}^{\rho}\mathscr{I}^{\varsigma}y(\varphi) = g(y), \end{cases} \tag{2.1} $$

is equivalent to the integral equation

$$ y(\tau) = \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho-\upsilon^\rho)^{1-\rho}}\widehat{h}(\upsilon)d\upsilon + \frac{\tau^\rho}{\frac{\varpi\rho\varphi^{\rho(\varsigma+1)}}{\rho^{\varsigma+1}\Gamma(\varsigma+2)}} \times\Big[g(y) - \varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi \frac{\upsilon^{\rho-1}}{(\varphi^\rho-\upsilon^\rho)^{1-(\rho+\varsigma)}}\widehat{h}(\upsilon)d\upsilon\Big]. \tag{2.2} $$

The integral solution to the problem (1.1) can be written in the view of Lemma 2.10.

$$ y(\tau) = \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho-\upsilon^\rho)^{1-\rho}}h(\upsilon,y(\upsilon))d\upsilon + \frac{\tau^\rho}{\rho\sigma}\Bigg[g(y) - \varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi \frac{\upsilon^{\rho-1}h(\upsilon,y(\upsilon))d\upsilon}{(\varphi^\rho-\upsilon^\rho)^{1-(\rho+\varsigma)}}\Bigg], \tag{2.3} $$

where

$$ \sigma = \frac{\varpi\varphi^{\rho(\varsigma+1)}}{\rho^{\varsigma+1}\Gamma(\varsigma+2)} \neq 0. \tag{2.4} $$

The operator $\Psi : \mathscr{H}\to\mathscr{H}$ used by

$$ \Psi y(\tau) = \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho-\upsilon^\rho)^{1-\rho}}h(\upsilon,y(\upsilon))d\upsilon + \frac{\tau^\rho}{\rho\sigma}\Bigg[g(y) - \varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi \frac{\upsilon^{\rho-1}h(\upsilon,y(\upsilon))d\upsilon}{(\varphi^\rho-\upsilon^\rho)^{1-(\rho+\varsigma)}}\Bigg]. \tag{2.5} $$

We realize that the system (1.1) does not have a solution until the operator equation $\Psi y = y$ and the existence of an operator Ψ fixed point implies the existence of a problem solution (1.1).

Definition 2.11. *A function $y \in \mathscr{AC}^2_\delta(\mathscr{J},\mathbb{R})$ is a solution of the problem (1.2) if $y(0)=0$, $g(y) = \varpi^\rho\mathscr{I}^\varsigma y(\varphi)$, and $\exists$ a function $h \in \mathscr{L}^1(\mathscr{J},\mathbb{R}) \ni h(\tau) \in H(\tau,y(\tau))$ almost everywhere on $\mathscr{J}$ and*

$$ y(\tau) = \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau \frac{\upsilon^{\rho-1}h(\upsilon)d\upsilon}{(\tau^\rho-\upsilon^\rho)^{1-\rho}} + \frac{\tau^\rho}{\rho\sigma}\Bigg[g(y) - \varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi \frac{\upsilon^{\rho-1}h(\upsilon)d\upsilon}{(\varphi^\rho-\upsilon^\rho)^{1-(\rho+\varsigma)}}\Bigg]. \tag{2.6} $$

Lemma 2.12. *[8] If $\mathscr{M} : \mathscr{W}\to\mathscr{E}_{cl}(\mathscr{T})$ is upper semi-continuous then $Gr(\mathscr{M})$ is a closed subset of $\mathscr{W}\times\mathscr{T}$ i.e., for every sequence $y_{m m\in\mathbb{N}} \subset \mathscr{W}$ and $z_{m m\in\mathbb{N}} \subset \mathscr{T}$ if when $m\to\infty$, $y_m \to y_*$, $z_m \to z_*$ and $z_m \in \mathscr{M}(y_m)$, then $z_* \in \mathscr{M}(y_*)$. Conversely, if $\mathscr{M}$ is completely continuous and has a closed graph, then it is upper semi-continuous.*

Lemma 2.13. *[19] Let $\mathscr{W}$ be a Banach space. Let $H : \mathscr{J}\times\mathbb{R}\to\mathscr{E}_{cp,c}(\mathscr{W})$ be an $\mathscr{L}^1$ Caratheodory multivalued map and let $\mathscr{K}$ be a linear continuous mapping from $\mathscr{L}^1(\mathscr{J},\mathscr{W})\times\mathscr{C}(\mathscr{J},\mathscr{W})$. Then the operator $\mathscr{K}\circ\mathscr{F}_{H,y} : \mathscr{C}(\mathscr{J},\mathscr{W})\to\mathscr{E}_{cp,c}(\mathscr{C}(\mathscr{J},\mathscr{W}))$, $y\mapsto(\mathscr{K}\circ\Psi_{H,y})(y) = \mathscr{K}(\Psi_{H,y})$ is a closed graph operator in $\mathscr{C}(\mathscr{J},\mathscr{W})\times\mathscr{C}(\mathscr{J},\mathscr{W})$.*

Theorem 2.14. *[24] Denote by $\mathscr{U}$ an open set in a closed, convex set $\mathscr{E}$ of a Banach space $\mathscr{W}$. Assume that $0\in\mathscr{U}$. Also assume that $\ominus(\overline{\mathscr{U}})$ is bounded and that $\ominus : \overline{\mathscr{U}}\to\mathscr{E}$ is given by $\ominus = \ominus_1+\ominus_2$, in which $\ominus_1 : \overline{\mathscr{U}}\to\mathscr{E}$ is continuous and completely continuous and $\ominus_2 : \overline{\mathscr{U}}\to\mathscr{E}$ is nonlinear contraction (i.e., $\exists$ a nonnegative nondecreasing function $\kappa : [0,\infty)\to[0,\infty)$ satisfying $\kappa(w)<w$ for $w>0$, $\ni \|\ominus_2 y - \ominus_2 z\| \leq \kappa(\|y-z\|)\forall\, y,z\in\mathscr{U}$). Then, either*

(i) $\ominus$ *has a fixed point* $v \in \overline{\mathscr{U}}$ *or*
(ii) $\exists$ *a point* $v \in \partial\mathscr{U}$ *and* $\beta \in (0,1)$ *with* $v = \beta\ominus(v)$ *where* $\overline{\mathscr{U}}$ *and* $\partial\mathscr{U}$*, respectively, represent the closure and boundary of* $\mathscr{U}$.

Theorem 2.15. *[25] Let* $\mathscr{W}$ *be a Banach space, and* $\mathscr{U}$ *a bounded neighborhood of* $0 \in \mathscr{W}$ *Let* $T_1 : \mathscr{W} \to \mathscr{E}_{cp,c}(\mathscr{W})$ *and* $T_2 : \overline{\mathscr{U}} \to \mathscr{E}_{cp,c}(\mathscr{W})$ *be two multivalued operators satisfying*
(1) T_1 *is contraction, and*
(2) T_2 *is upper semi-continuous and compact.*
Then, if $\mathscr{M} = T_1 + T_2$ *either*
(i) $\mathscr{M}$ *has a fixed point in* $\overline{\mathscr{U}}$ *or*
(ii) there is a point $v \in \partial\mathscr{U}$ *and* $\beta \in (0,1)$ *with* $v \in \beta\mathscr{U}(v)$.

In this section, by using Banach and O'Regan's fixed point theorems, we obtain the existence and uniqueness results.

3. Main Results : Single-valued case

For computational purposes, we represent:

$$\Theta_1 = \frac{T^{\rho\rho}}{\rho^{\rho}\Gamma(\rho+1)} + \frac{T^{\rho}}{\rho|\sigma|}\Big[\frac{|\varpi|\;\varphi^{\rho(\rho+\varsigma)}}{\rho^{\rho+\varsigma}\Gamma(\rho+\varsigma+1)}\Big], \quad (3.1)$$

and

$$\Theta_2 = \frac{T^{\rho}}{\rho|\sigma|}. \quad (3.2)$$

The following hypotheses are necessary to prove the existence and uniqueness results. Let $h : \mathscr{J} \times \mathbb{R} \to \mathbb{R}$ and $g : \mathscr{C}(\mathscr{J},\mathbb{R}) \to \mathbb{R}$ are continuous functions.

$(\mathscr{S}_1)$ $\exists$ a constant $\mathscr{K} > 0 \ni |h(\tau,y) - h(\tau,z)| \le \mathscr{K}|y-z|$, for each $\tau \in \mathscr{J}$ and $y,z \in \mathbb{R}$.

$(\mathscr{S}_2)$ $\exists$ a constant $k > 0 \ni |g(y) - g(z)| \le k|y-z|, k < \dfrac{1}{\Theta_2}$ for each $y,z \in \mathscr{C}(\mathscr{J},\mathbb{R})$.

$(\mathscr{S}_3)$ $g(0) = 0$.

$(\mathscr{S}_4)$ $\exists$ a nonnegative function $\vartheta \in \mathscr{C}(\mathscr{J},\mathbb{R}^+)$ and a nondecreasing function $\xi : [0,\infty] \to (0,\infty)$ $\ni |h(\tau,y)| \le \vartheta(\tau)\xi(\|y\|)$ for any $(\tau,y) \in \mathscr{J}\times\mathbb{R}$.

$(\mathscr{S}_5)$ $\sup\limits_{\nu\in(0,\infty)} \dfrac{\nu}{\Theta_1\|\vartheta\|\xi(\nu)} > \dfrac{1}{1-\Theta_2 k}$.

$(\mathscr{S}_6)$ $H : \mathscr{J}\times\mathbb{R} \to \mathscr{E}_{cp,c}(\mathbb{R})$ is $\mathscr{L}^1$ - Caratheodory multivalued map.

$(\mathscr{S}_7)$ $\exists$ a continuous nondecreasing function $\xi : [0,\infty] \to (0,\infty)$ and a function $\vartheta \in \mathscr{C}(\mathscr{J},\mathbb{R}^+) \ni \|H(\tau,y)\|_{\mathscr{E}} = \sup\{|z| : z \in H(\tau,y)\} \le \vartheta(\tau)\xi(|y|)$ for each $(\tau,y) \in \mathscr{J}\times\mathbb{R}$.

$(\mathscr{S}_8)$ $\exists$ a number $\alpha > 0 \ni$

$$\frac{(1-\Theta_2 k)\alpha}{\Theta_1\|\vartheta\|\xi(\alpha)} > 1, \quad (3.3)$$

where Θ_1 and Θ_2 are given by (3.1) and (3.2) respectively.

Theorem 3.1. *Suppose that* $(\mathscr{S}_1)$ *and* $(\mathscr{S}_2)$ *holds. If*

$$\eta := \mathscr{K}\Theta_1 + k\Theta_2 < 1, \quad (3.4)$$

where Θ_1, Θ_2 *are defined by (3.1) and (3.2). Then the BVP (1.1) has a unique solution on* $\mathscr{J}$.

Proof. For $y,z \in \mathscr{H}$ and each $\tau \in \mathscr{J}$, the operator Ψ equation defined by (2.5), and assumptions $((\mathscr{S}_1),(\mathscr{S}_2))$, we obtain

$$\begin{aligned}
&|(\Psi y)(\tau) - (\Psi z)(\tau)| \\
\le\;& \max_{\tau\in\mathscr{J}}\Bigg\{\frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^{\tau}\frac{\upsilon^{\rho-1}}{(\tau^{\rho}-\upsilon^{\rho})^{1-\rho}} \\
&\times|h(\upsilon,y(\upsilon)) - h(\upsilon,z(\upsilon))|d\upsilon + \frac{\tau^{\rho}}{\rho|\sigma|}\Big[|\varpi|\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \\
&\int_0^{\varphi}\frac{\upsilon^{\rho-1}}{(\varphi^{\rho}-\upsilon^{\rho})^{1-(\rho+\varsigma)}}|h(\upsilon,y(\upsilon)) - h(\upsilon,z(\upsilon))|d\upsilon\Big] \\
&+\frac{\tau^{\rho}}{\rho|\sigma|}|g(y)-g(z)|\Bigg\} \\
\le\;& \max_{\tau\in\mathscr{J}}\Bigg\{\mathscr{K}\|y-z\|\Bigg(\frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^{\tau}\frac{\upsilon^{\rho-1}}{(\tau^{\rho}-\upsilon^{\rho})^{1-\rho}}d\upsilon \\
&+\frac{\tau^{\rho}}{\rho|\sigma|}\Big[|\varpi|\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^{\varphi}\frac{\upsilon^{\rho-1}}{(\varphi^{\rho}-\upsilon^{\rho})^{1-(\rho+\varsigma)}}d\upsilon\Big]\Bigg) \\
&+k\|y-z\|\frac{\tau^{\rho}}{\rho|\sigma|}\Bigg\} \\
\le\;& (\mathscr{K}\Theta_1 + k\Theta_2)\|y-z\|.
\end{aligned}$$

Therefore,

$$\|\Psi y - \Psi z\| \le (\mathscr{K}\Theta_1 + k\Theta_2)\|y-z\|.$$

This follows from the statement (3.4) that Ψ is a contraction in itself from Banach space $\mathscr{H}$. As a consequence, operator Ψ has a fixed point by Banach's fixed point theorem, which corresponds to the unique solution to the problem (1.1). □

Theorem 3.2. *Suppose that* $(\mathscr{S}_2)$*,* $(\mathscr{S}_3)$*,* $(\mathscr{S}_4)$ *and* $(\mathscr{S}_5)$ *holds. Then the BVP (1.1) has at least one solution on* $\mathscr{J}$.

Proof. Consider the operator $\Psi : \mathscr{H} \to \mathscr{H}$ described by (2.5). We break down Ψ into two operators,

$$(\Psi y)(\tau) = (\Psi_1 y)(\tau) + (\Psi_2 y)(\tau),\; \tau \in \mathscr{J}, \quad (3.5)$$

where

$$\begin{aligned}
(\Psi_1 y)(\tau) =\;& \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^{\tau}\frac{\upsilon^{\rho-1}}{(\tau^{\rho}-\upsilon^{\rho})^{1-\rho}}h(\upsilon,y(\upsilon))d\upsilon \\
&-\frac{\tau^{\rho}}{\rho\sigma}\Bigg[\varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^{\varphi}\frac{\upsilon^{\rho-1}}{(\varphi^{\rho}-\upsilon^{\rho})^{1-(\rho+\varsigma)}} \\
&\times h(\upsilon,y(\upsilon))d\upsilon\Bigg],\; \tau\in\mathscr{J}, \quad (3.6)
\end{aligned}$$

and

$$(\Psi_2 y)(\tau) = \frac{\tau^{\rho}g(y)}{\rho\sigma},\; \tau\in\mathscr{J}. \quad (3.7)$$

Let $\mathscr{Q}_{\nu} = \{y \in \mathscr{H} : \|y\| < \nu\}$. From the assumption of $(\mathscr{S}_5)$, $\exists$ a number $\nu_0 > 0 \ni$

$$\frac{\nu_0}{\Theta_1\|\vartheta\|\xi(\nu_0)} > \frac{1}{1-\Theta_2 k}. \quad (3.8)$$

We will continue to prove that operators Ψ_1,Ψ_2 meet all Theorem 2.14 requirements.

Step 1. The set $\Psi(\overline{\mathscr{Q}}_{\nu_0})$ is bounded. For any $y \in \overline{\mathscr{Q}}_{\nu_0}$, we procure

$$\begin{aligned}\|\Psi_1 y\| &\leq \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho-\upsilon^\rho)^{1-\rho}}|h(\upsilon,y(\upsilon))|d\upsilon \\ &\quad + \frac{T^\rho}{\rho|\sigma|}\left[|\varpi|\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi \frac{\upsilon^{\rho-1}}{(\varphi^\rho-\upsilon^\rho)^{1-(\rho+\varsigma)}}\right. \\ &\quad \left.\times|h(\upsilon,y(\upsilon))|d\upsilon\right] \\ &\leq \|\vartheta\|\xi(\nu_0)\left\{\frac{T^{\rho\rho}}{\rho^\rho\Gamma(\rho+1)}\right. \\ &\quad \left.+\frac{T^\rho}{\rho|\sigma|}\left[\frac{|\varpi|\ \varphi^{\rho(\rho+\varsigma)}}{\rho^{\rho+\varsigma}\Gamma(\rho+\varsigma+1)}\right]\right\} \\ &\leq \|\vartheta\|\xi(\nu_0)\Theta_1.\end{aligned}$$

It demonstrate that $\Psi_1(\overline{\mathscr{Q}}_{\nu_0})$ is uniformly bounded. The assumptions of $(\mathscr{S}_2)$
and $(\mathscr{S}_4)$ imply that

$$\|\Psi_2 y\| \leq \frac{T^\rho}{\rho|\sigma|}k\nu_0,$$

for any $y \in \overline{\mathscr{Q}}_{\nu_0}$. Thus, the set $\Psi(\overline{\mathscr{Q}}_{\nu_0})$ is bounded.

Step 2. Continuous and completely continuous operator P. Step 1 means that $\Psi_1(\overline{\mathscr{Q}}_{\nu_0})$ is uniformly bounded. Furthermore, for any $\tau_1,\tau_2 \in \mathscr{J}$, we have

$$\begin{aligned}&\|(\Psi_1 y)(\tau_2)-(\Psi_1 y)(\tau_1)\| \\ &\leq \frac{\rho^{1-\rho}}{\Gamma(\rho)}\left|\int_0^{\tau_2}\frac{\upsilon^{\rho-1}}{(\tau_2^\rho-\upsilon^\rho)^{1-\rho}}|h(\upsilon,y(\upsilon))|d\upsilon\right. \\ &\quad \left.-\int_0^{\tau_1}\frac{\upsilon^{\rho-1}}{(\tau_1^\rho-\upsilon^\rho)^{1-\rho}}|h(\upsilon,y(\upsilon))|d\upsilon\right| \\ &\quad +\frac{|\tau_2^\rho-\tau_1^\rho|}{\rho|\sigma|}\left[|\varpi|\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}}{(\varphi^\rho-\upsilon^\rho)^{1-(\rho+\varsigma)}}\right. \\ &\quad \left.\times|h(\upsilon,y(\upsilon))|d\upsilon\right] \\ &\leq \frac{\|\vartheta\|\xi(\nu_0)}{\rho^\rho\Gamma(\rho+1)}\left[2|\tau_2^\rho-\tau_1^\rho|^\rho+|\tau_2^{\rho\rho}-\tau_1^{\rho\rho}|\right] \\ &\quad +\frac{|\tau_2^\rho-\tau_1^\rho|\|\vartheta\|\xi(\nu_0)}{\rho|\sigma|}\left[\frac{|\varpi|\ \varphi^{\rho(\rho+\varsigma)}}{\rho^{\rho+\varsigma}\Gamma(\rho+\varsigma+1)}\right],\end{aligned}$$

which is independent of y and tends to zero as $\tau_2-\tau_1 \to 0$. Thus, Ψ_1 is equicontinuous. Hence, by the Arzela-Ascoli Theorem, $\Psi_1(\overline{\mathscr{Q}}_{\nu_0})$ is relatively compact. Now let $y_m, y \in \overline{\mathscr{Q}}_{\nu_0}$ with $\|y_m-y\| \to 0$. Then the limit $|y_m(\tau)-y(\tau)| \to 0$ is uniformly valid on $\mathscr{J}$. It follows that $\|h(\tau,y_m(\tau))-h(\tau,y(\tau))\| \to 0$ is uniformly valid on $\mathscr{J}$ from the uniform continuity of $h(\tau,y)$ on the compact set $\mathscr{J}\times[-\nu_0,\nu_0]$. Then $\|\Psi_1 y_m-\Psi_1 y\| \to 0$ as $m \to \infty$ that proves Ψ_1 is continuity. The operator Ψ_1 is continuous and completely continuous.

Step 3. The operator $\Psi_2 : \overline{\mathscr{Q}}_{\nu_0} \to \mathscr{H}$ is contractive. Observe that

$$\begin{aligned}|(\Psi_2 y)(\tau)-(\Psi_2 z)(\tau)| &= \frac{\tau^\rho}{\rho|\sigma|}|g(y)-g(z)| \\ &\leq \frac{T^\rho}{\rho|\sigma|}k\|y-z\| = \lambda\|y-z\|,\end{aligned}$$

with $\lambda = \Theta_2 k < 1$ by the assumptions of $(\mathscr{S}_2)$. Hence Ψ_2 is contractive.

Step 4. Furthermore, it will be shown that the case (ii) in Theorem 2.14 does not occur. For this, we presume that case (ii) is true. Then, we have that $\exists\ \zeta \in (0,1)$ and $y \in \partial\mathscr{Q}_{\nu_0} \ni y = \zeta\Psi y$. So, we have $\|y\| = \nu_0$ and for $\tau \in \mathscr{J}$,

$$\begin{aligned}y(\tau) &= \zeta\left\{\frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau\frac{\upsilon^{\rho-1}}{(\tau^\rho-\upsilon^\rho)^{1-\rho}}h(\upsilon,y(\upsilon))d\upsilon+\frac{\tau^\rho}{\rho\sigma}g(y)\right. \\ &\quad -\frac{\tau^\rho}{\rho\sigma}\left[\varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}}{(\varphi^\rho-\upsilon^\rho)^{1-(\rho+\varsigma)}}\right. \\ &\quad \left.\left.\times h(\upsilon,y(\upsilon))d\upsilon\right]\right\}.\end{aligned}$$

Next, we get

$$\begin{aligned}|y(\tau)| &\leq \xi(\|y\|)\left\{\frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^T\frac{\upsilon^{\rho-1}}{(T^\rho-\upsilon^\rho)^{1-\rho}}\vartheta(\upsilon)d\upsilon\right. \\ &\quad \left.+\frac{T^\rho}{\rho|\sigma|}\left[|\varpi|\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}\vartheta(\upsilon)d\upsilon}{(\varphi^\rho-\upsilon^\rho)^{1-(\rho+\varsigma)}}\right]\right\} \\ &\quad +\frac{T^\rho}{\rho|\sigma|}k\|y\|,\end{aligned}$$

with the hypothesis $(\mathscr{S}_3)-(\mathscr{S}_5)$. Taking the maximum over $\tau \in \mathscr{J}$, we get

$$\begin{aligned}\|y\| &\leq \xi(\|y\|)\left\{\frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^T\frac{\upsilon^{\rho-1}}{(T^\rho-\upsilon^\rho)^{1-\rho}}\vartheta(\upsilon)d\upsilon\right. \\ &\quad \left.+\frac{T^\rho}{\rho|\sigma|}\left[|\varpi|\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}\vartheta(\upsilon)d\upsilon}{(\varphi^\rho-\upsilon^\rho)^{1-(\rho+\varsigma)}}\right]\right\} \\ &\quad +\frac{T^\rho}{\rho|\sigma|}k\|y\|.\end{aligned}$$

Which implies that

$$\nu_0 \leq \Theta_1\|\vartheta\|\xi(\nu_0)+k\Theta_2\nu_0.$$

Thus,

$$\frac{\nu_0}{\Theta_1\|\vartheta\|\xi(\nu_0)} \leq \frac{1}{1-k\Theta_2},$$

which contradicts (3.8). Therefore, we have demonstrated that operators Ψ_1 and Ψ_2 have fulfilled all the Theorem 2.14 assumptions. Thus, the Theorem 3.2 conclusion implies that operator Ψ has at least one fixed point $y \in \overline{\mathscr{Q}}_{\nu_0}$, which is the solution of the BVP (1.1). □

Remark 3.3. *Setting $\rho \to 1$, in the problem (1.1), then the problem reduces to*

$$\begin{array}{l} {}_{\mathscr{C}}\mathscr{D}^{\rho} y(\tau) = h(\tau, y(\tau)),\ \tau \in \mathscr{J}, \\ y(0) = 0,\ \varpi \mathscr{I}^{\varsigma} y(\varphi) = g(y), \\ 1 < \rho \leq 2, 0 < \varsigma < 1,\ \varphi \in (0,T). \end{array} \tag{3.9}$$

In this case the values of $\widehat{\Theta}_1$ and $\widehat{\Theta}_2$ are found to be

$$\widehat{\Theta}_1 = \frac{T^{\varsigma}}{\Gamma(\varsigma+1)} + \frac{T}{|\widehat{\sigma}|}\left(\frac{|\varpi|\ \varphi^{\rho+\varsigma}}{\Gamma(\rho+\varsigma+1)}\right), \tag{3.10}$$

$$\widehat{\Theta}_2 = \frac{T}{|\widehat{\sigma}|}, \tag{3.11}$$

and The operator form (2.5) changes

$$\widehat{\Psi}(y)(\tau) = \int_0^{\tau} \frac{(\tau-\upsilon)^{\rho-1}}{\Gamma(\rho)} h(\upsilon, y(\upsilon)) d\upsilon + \frac{\tau}{\widehat{\sigma}}\left[g(y) - \varpi \int_0^{\varphi} \frac{(\varphi-\upsilon)^{\rho-1}}{\Gamma(\rho)} h(\upsilon, y(\upsilon)) d\upsilon\right], \tau \in \mathscr{J},$$

$$\text{where } \widehat{\sigma} = \frac{\varpi \varphi^{\varsigma+1}}{\Gamma(2+\varsigma)} \neq 0.$$

Corollary 3.4. *Suppose that $(\mathscr{S}_1)$ and $(\mathscr{S}_2)$ holds. Then the BVP (3.9), provided that*

$$\eta_1 := K\widehat{\Theta}_1 + k\widehat{\Theta}_2 < 1,$$

where $\widehat{\Theta}_1, \widehat{\Theta}_2$ is described by (3.10) and (3.11) respectively, has a unique solution on J.

Corollary 3.5. *Suppose that $(\mathscr{S}_2)$, $(\mathscr{S}_3)$,$(\mathscr{S}_4)$ and $(\mathscr{S}_5)$ holds. Then the BVP (3.9), provided that*

$$\frac{\nu_0}{\widehat{\Theta}_1 \|\vartheta\| \xi(\nu_0)} > \frac{1}{1 - k\widehat{\Theta}_2},$$

has at least one solution on $\mathscr{J}$.

Examples are given in this section to illustrate the feasibility of the results obtained.

4. Examples

Example 4.1. *Consider the following BVP*

$$\begin{cases} {}_{\frac{1}{2}}\mathscr{D}^{\frac{7}{4}} y(\tau) = 1 + \left(\frac{|y|}{|y|+1}\right) \cdot \frac{e^{-\tau^2}}{4(\tau+\sqrt{4})^2}, \\ y(0) = 0,\ \frac{1}{10} y(\frac{3}{2}) = {}_{\frac{1}{2}}\mathscr{I}^{\frac{3}{4}} y(\frac{8}{5}),\ \tau \in [0,2]. \end{cases} \tag{4.1}$$

Here $\rho = \frac{7}{4}$, $\varsigma = \frac{3}{4}$, $\varpi = 1$, $T = 2$, $\rho = \frac{1}{2}$, $\varphi = \frac{8}{5}$, $k = \frac{1}{10}$, $K = \frac{1}{16}$, $g(y) = \frac{1}{10} y\left(\frac{3}{2}\right)$ and $h(\tau, y) = 1 + \left(\frac{|y|}{|y|+1}\right) \cdot \frac{e^{-\tau^2}}{4(\tau+\sqrt{4})^2}$. We can acquire values using specified information: $\sigma \cong 3.991024198349486$, $\Theta_1 \cong 6.006235934230121$, $\Theta_2 \cong 0.7086970622518162$, $\eta \cong 0.4462594521145642$, and

$$\begin{aligned} |h(\tau,y) - h(\tau,z)| &\leq \frac{1}{16}\left|\frac{|y|}{|y|+1} - \frac{|z|}{|z|+1}\right| \\ &\leq \frac{1}{16}|y - z|. \end{aligned}$$

With $K\Theta_1 + k\Theta_2 < 1 \cong 0.4462594521145642$, The Theorem 3.1 presumptions are fulfilled. The BVP (4.1) has a unique solution for $[0,2]$.

Example 4.2. *Consider the following BVP*

$$\left\{ {}_{\frac{1}{2}}\mathscr{D}^{\frac{9}{5}} y(\tau) = \left(|y| + \frac{1+|y|}{2+|y|}\right)\frac{\tau}{200},\ \tau \in [0,2], \right. \tag{4.2}$$

Example 4.1 boundary conditions augmented.
Here $\rho = \frac{9}{5}$, $\varsigma = \frac{3}{4}$, $\varpi = 1$, $T = 2$, $\rho = \frac{1}{2}$, $\varphi = \frac{8}{5}$, $k = \frac{1}{10}$, $g(y) = \frac{1}{10} y\left(\frac{3}{2}\right)$ and $h(\tau, y) = \left(|y| + \frac{1+|y|}{2+|y|}\right)\frac{\tau}{200}$. Using the specified information, we can acquire values: $\sigma \cong 4.038195686699365$, $\Theta_1 \cong 6.001791053992137$, $\Theta_2 \cong 0.700418539414$, and $|h(\tau,y)| = |\left(|y| + \frac{1+|y|}{2+|y|}\right)\frac{\tau}{200}| \leq (1 + 3|y| + |y^2|)\frac{\tau}{200}$, we choose $\vartheta(\tau) = \frac{\tau}{200}$ and $\xi(|y|) = 1 + 3|y| + |y^2|$, and we find that $\sup_{\nu \in (0,\infty)} \frac{\nu}{\Theta_1 \|\vartheta\| \xi(\nu)} \cong 3.33233860 0274471 > 1.0753172110360703 \cong \frac{1}{1 - \Theta_2 k}$. The Theorem 3.2 presumptions are fulfilled. The BVP (4.2) has at least one solution for $[0,2]$.

In this section, we obtain existence results for the BVP (1.2) by using the nonlinear alternative for contractive maps.

5. Main Results : Multi-valued case

Theorem 5.1. *Suppose that $(\mathscr{S}_2)$, $(\mathscr{S}_6)$, $(\mathscr{S}_7)$ and $(\mathscr{S}_8)$ holds. Then the BVP (1.2) has at least one solution on $\mathscr{J}$.*

Proof. In order to convert the problem (1.2) into a fixed point question, the operator $\Omega : \mathscr{H} \to \mathscr{E}(\mathscr{H})$ specified by

$$\Omega(y) = \left\{ \begin{array}{l} q \in \mathscr{H}: \\ q(\tau) = \begin{cases} \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_0^{\tau} \frac{\upsilon^{\rho-1}}{(\tau^{\rho} - \upsilon^{\rho})^{1-\rho}} h(\upsilon) d\upsilon \\ + \frac{\tau^{\rho}}{\rho\sigma} g(y) \\ - \frac{\tau^{\rho}}{\rho\sigma}\left[\varpi \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \int_0^{\varphi} \frac{\upsilon^{\rho-1} h(\upsilon) d\upsilon}{(\varphi^{\rho} - \upsilon^{\rho})^{1-(\rho+\varsigma)}}\right] \end{cases} \end{array} \right\}$$

is considered. For $h \in \Psi_{H,y}$. Next, we intimate two operators: $\Omega_1 : \mathscr{H} \to \mathscr{H}$ by

$$\Omega_1 y(\tau) = \frac{\tau^{\rho}}{\rho\sigma} g(y), \tag{5.1}$$

and the multivalued operator $\Omega_2 : \mathscr{H} \to \mathscr{E}(\mathscr{H})$ by

$$\Omega_2(y) = \left\{ \begin{array}{l} q \in \mathscr{H} : \\ q(\tau) = \left\{ \dfrac{\rho^{1-\rho}}{\Gamma(\rho)} \displaystyle\int_0^\tau \dfrac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}} h(\upsilon) d\upsilon \right. \\ -\dfrac{\tau^\rho}{\rho\sigma} \left[\varpi \dfrac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \displaystyle\int_0^\varphi \dfrac{\upsilon^{\rho-1} h(\upsilon) d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}} \right] \end{array} \right\}.$$

Observe that $\Omega = \Omega_1 + \Omega_2$. We define the operators Ω_1 and Ω_2 that follow all of Theorem 2.15 assumptions on $\mathscr{J}$. For that, we consider the operators $\Omega_1, \Omega_2 : \mathscr{B}_\nu \to \mathscr{E}_{cp,c}(\mathscr{H})$, where $\mathscr{B}_\nu = \{y \in \mathscr{H} : \|y\| \le \nu\}$ is a bounded set in $\mathscr{H}$. Next, we prove that Ω_2 is compact valued on $\mathscr{B}_\nu$. Considering that operator Ω_2 is the $\mathscr{G} \circ \Psi_H$ composition where $\mathscr{G}$ is the linear continuous operator $\mathscr{L}^1(\mathscr{J}, \mathbb{R})$ into $\mathscr{H}$, as described by

$$\begin{aligned} \mathscr{G}(\theta)(\tau) &= \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}} \theta(\upsilon) d\upsilon \\ &\quad - \frac{\tau^\rho}{\rho\sigma} \left[\varpi \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \int_0^\varphi \frac{\upsilon^{\rho-1}\theta(\upsilon) d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}} \right]. \end{aligned}$$

Suppose $y \in \mathscr{B}_\nu$ is arbitrary and allow $\{\theta_m\}$ to be a sequence in $\Psi_{H,y}$. In this case, we have $\theta_m(\tau) \in H(\tau, y(\tau))$ for almost all $\tau \in \mathscr{J}$, by definition of $\Psi_{H,y}$. Because $H(\tau, y(\tau))$ is compact for all $\forall\, \tau \in \mathscr{J}$, it has a convergent subsequence of $\{\theta_m(\tau)\}$, which converges for almost all $\tau \in \mathscr{J}$ to some $\theta(\tau) \in \Psi_{H,y}$. First, $\mathscr{G}$ is continuous, so $\mathscr{G}(\theta_m)(\tau) \to \mathscr{G}(\theta)(\tau)$ point-wise on $\mathscr{J}$. To prove that the convergence is uniform, we retain to demonstrate that the equicontinuous sequence is $\{\mathscr{G}(\theta_m)\}$. Let $\tau_1, \tau_2 \in \mathscr{J}$ with $\tau_1 < \tau_2$. Then, we find that $\|\mathscr{G}(\theta_m)(\tau_2) - \mathscr{G}(\theta_m)(\tau_1)\|$

$$\begin{aligned} &\le \frac{\rho^{1-\rho}}{\Gamma(\rho)} \left| \int_0^{\tau_2} \frac{\upsilon^{\rho-1}}{(\tau_2^\rho - \upsilon^\rho)^{1-\rho}} \theta_m(\upsilon) d\upsilon \right. \\ &\quad \left. - \int_0^{\tau_1} \frac{\upsilon^{\rho-1}}{(\tau_1^\rho - \upsilon^\rho)^{1-\rho}} \theta_m(\upsilon) d\upsilon \right| \\ &\quad + \frac{|\tau_2^\rho - \tau_1^\rho|}{\rho|\sigma|} \left[\varpi \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \int_0^\varphi \frac{\upsilon^{\rho-1}\theta_m(\upsilon) d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}} \right] \\ &\le \frac{\|\vartheta\|\xi(\nu)}{\rho^\rho \Gamma(\rho+1)} \left[2|\tau_2^\rho - \tau_1^\rho|^\rho + |\tau_2^{\rho\rho} - \tau_1^{\rho\rho}| \right] \\ &\quad + \frac{|\tau_2^\rho - \tau_1^\rho| \|\vartheta\| \xi(\nu)}{\rho|\sigma|} \left[\frac{|\varpi|\ \varphi^{\rho(\rho+\varsigma)}}{\rho^{\rho+\varsigma}\Gamma(\rho+\varsigma+1)} \right]. \end{aligned}$$

The inequality above tends to be zero like $\tau_2 \to \tau_1$. Therefore, the sequence $\{\mathscr{G}(\theta_m)\}$ is equicontinuous and we obtain a uniform convergent subsequence using the Arzela-Ascoli theorem. So, there is a subsequence of $\{\theta_m\}$, $\ni \mathscr{G}(\theta_m) \to \mathscr{G}(\theta)$. Considering $\mathscr{G}(\theta) \in \mathscr{G}(\Psi_{H,y})$. Consequently, $\Omega_2(y) = \mathscr{G}(\Psi_{H,y})$ is compact $\forall\, y \in \mathscr{B}_\nu$. So $\Omega_2(y)$ is compact. Now, we depict that $\Omega_2(y)$ is convex $\forall\, y \in \mathscr{H}$. Let $v_1, v_2 \in \Omega_2(y)$. We nominate $h_1, h_2 \in \Psi_{H,y} \ni$

$$\begin{aligned} v_j(\tau) &= \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}} h_j(\upsilon) d\upsilon \\ &\quad - \frac{\tau^\rho}{\rho\sigma} \left[\varpi \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \int_0^\varphi \frac{\upsilon^{\rho-1} h_j(\upsilon) d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}} \right], \end{aligned}$$

$j = 1, 2$ for almost all $\tau \in \mathscr{J}$. Let $0 \le \gamma \le 1$. Then, we have

$$\begin{aligned} &[\gamma v_1 + (1-\gamma) v_2](\tau) \\ &= \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}} [\gamma h_1(\upsilon) + (1-\gamma) h_2(\upsilon)] d\upsilon \\ &\quad - \frac{\tau^\rho}{\rho\sigma} \left[\varpi \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \int_0^\varphi \frac{\upsilon^{\rho-1}}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}} \right. \\ &\quad \left. \times [\gamma h_1(\upsilon) + (1-\gamma) h_2(\upsilon)] d\upsilon \right]. \end{aligned}$$

Since H has convex values, so $\mathscr{G}_{H,y}$ is convex and $\gamma h_1(\upsilon) + (1-\gamma) h_2(\upsilon) \in \mathscr{G}_{H,y}$. Thus, $\gamma v_1 + (1-\gamma) v_2 \in \Omega_2(y)$. Hence, Ω_2 is convex-valued. Undoubtedly, Ω_1 is compact and convex-valued. The remaining facts contains multiple phases and statements.

Step 1. We demonstrate that Ω_1 is a contraction on $\mathscr{H}$. This is a result of $(\mathscr{S}_2)$, and the evidence in Step 2 of Theorem 3.2 is similar to that of Operator Ψ_2.

Step 2. We will proceed to demonstrate that the operator Ω_2 is compact and upper semicontinuous. This is shown in number of claims.

Claim I: Ω_2 maps bounded sets into bounded sets in $\mathscr{H}$. To see this, let $\mathscr{B}_\nu = \{y \in \mathscr{H} : \|y\| \le \nu\}$ be a bounded set in $\mathscr{H}$. Then, for each $q \in \Omega_2(y)$, $y \in \mathscr{B}_\nu$, $\exists\, h \in \Psi_{H,y} \ni$

$$\begin{aligned} q(\tau) &= \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}} h(\upsilon) d\upsilon \\ &\quad - \frac{\tau^\rho}{\rho\sigma} \left[\varpi \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \int_0^\varphi \frac{\upsilon^{\rho-1} h(\upsilon) d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}} \right]. \end{aligned}$$

Then, for $\tau \in \mathscr{J}$, we have

$$\begin{aligned} &|q(\tau)| \\ &\le \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}} |h(\upsilon)| d\upsilon \\ &\quad + \frac{\tau^\rho}{\rho|\sigma|} \left[|\varpi| \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \int_0^\varphi \frac{\upsilon^{\rho-1}|h(\upsilon)| d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}} \right] \\ &\le \xi(\|y\|) \left\{ \frac{\rho^{1-\rho}}{\Gamma(\rho)} \int_0^\tau \frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}} \vartheta(\upsilon) d\upsilon \right. \\ &\quad \left. + \frac{\tau^\rho}{\rho|\sigma|} \left[|\varpi| \frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)} \int_0^\varphi \frac{\upsilon^{\rho-1}\vartheta(\upsilon) d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}} \right] \right\} \\ &\le \xi(\nu) \left\{ \frac{T^{\rho\rho}}{\rho^\rho \Gamma(\rho+1)} + \frac{T^\rho}{\rho|\sigma|} \left[\frac{|\varpi|\ \varphi^{\rho(\rho+\varsigma)}}{\rho^{\rho+\varsigma}\Gamma(\rho+\varsigma+1)} \right] \right\} \|\vartheta\|. \end{aligned}$$

Thus, $\|q\| \leq \xi(\nu)\Theta_1\|\vartheta\|$.

Claim II: Next, we show that Ω_2 maps bounded sets into equicontinuous sets. Let $\tau_1, \tau_2 \in \mathscr{J}$, and $y \in \mathscr{B}_\nu$. For each $q \in \Omega_2(y)$, we get

$$\begin{aligned}
&|q(\tau_2) - q(\tau_1)| \\
&\leq \frac{\rho^{1-\rho}}{\Gamma(\rho)}\left|\int_0^{\tau_2}\frac{\upsilon^{\rho-1}}{(\tau_2^\rho - \upsilon^\rho)^{1-\rho}}h(\upsilon)d\upsilon - \int_0^{\tau_1}\frac{\upsilon^{\rho-1}}{(\tau_1^\rho - \upsilon^\rho)^{1-\rho}}h(\upsilon)d\upsilon\right| \\
&\quad + \frac{|\tau_2^\rho - \tau_1^\rho|}{\rho|\sigma|}\left[\frac{\varpi\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}}h(\upsilon)d\upsilon\right] \\
&\leq \frac{\|\vartheta\|\xi(\nu)}{\rho^\rho\Gamma(\rho+1)}\left[2|\tau_2^\rho - \tau_1^\rho|^\rho + |\tau_2^{\rho\rho} - \tau_1^{\rho\rho}|\right] \\
&\quad + \frac{|\tau_2^\rho - \tau_1^\rho|\|\vartheta\|\xi(\nu)}{\rho|\sigma|}\left[\frac{|\varpi|\ \varphi^{\rho(\rho+\varsigma)}}{\rho^{\rho+\varsigma}\Gamma(\rho+\varsigma+1)}\right].
\end{aligned}$$

The right side of the above inequality obviously tends to be zero, independent of $y \in \mathscr{B}_\nu$ as $\tau_2 - \tau_1 \to 0$. Therefore, the Arzela-Ascoli theorem that $\Omega_2 : \mathscr{H} \to \mathscr{E}_{cp,c}(\mathscr{H})$, is completely continuous. By Lemma 2.12, Ω_2 is semi-continuous if we prove it has a closed graph since Ω_2 is already shown to be absolutely continuous. We set it out in the next paragraph.

Claim III: Ω_2 has a closed graph. Let $y_m \to y_*$ and $q_m \in \Omega_2(y_m)$ and $q_m \to q_*$. Then, we need to show that $q_* \in \Omega_2(y_*)$ associated with $q_m \in \Omega_2(y_m)$, $\exists\ h_m \in \Psi_{H,y_m} \ni$ for each $\tau \in \mathscr{J}$,

$$\begin{aligned}
q_m(\tau) &= \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau\frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}}h_m(\upsilon)d\upsilon \\
&\quad - \frac{\tau^\rho}{\rho\sigma}\left[\varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}h_m(\upsilon)d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}}\right].
\end{aligned}$$

And it is enough to show $\exists\ h_* \in \Psi_{H,y_*} \ni$ for each $\tau \in \mathscr{J}$,

$$\begin{aligned}
q_*(\tau) &= \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau\frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}}h_*(\upsilon)d\upsilon \\
&\quad - \frac{\tau^\rho}{\rho\sigma}\left[\varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}h_*(\upsilon)d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}}\right].
\end{aligned}$$

Let us consider the linear operator $\Omega : \mathscr{L}^1(\mathscr{J}, \mathbb{R}) \to \mathscr{H}$ defined by

$$\begin{aligned}
h \mapsto \Omega(h)(\tau) &= \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau\frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}}h(\upsilon)d\upsilon \\
&\quad - \frac{\tau^\rho}{\rho\sigma}\left[\varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}h(\upsilon)d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}}\right].
\end{aligned}$$

Observe that

$$\begin{aligned}
&\|q_m(\tau) - q_*(\tau)\| \\
&= \left\|\frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau\frac{\upsilon^{\rho-1}(h_m(\upsilon) - h_*(\upsilon))d\upsilon}{(\tau^\rho - \upsilon^\rho)^{1-\rho}} - \frac{\tau^\rho}{\rho\sigma}\right. \\
&\quad \left.\times\left[\frac{\varpi\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}(h_m(\upsilon) - h_*(\upsilon))d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}}\right]\right\| \to 0,
\end{aligned}$$

as $m \to \infty$. Thus, using by Lemma 2.13 that $\Omega \circ \Psi_{H,y}$ is a closed graph operator. Furthermore, we have $q_m(\tau) \in \Omega(\Psi_{H,y_m})$. Since $y_m \to y_*$, therefore, we have

$$\begin{aligned}
q_*(\tau) &= \frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau\frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}}h_*(\upsilon)d\upsilon \\
&\quad - \frac{\tau^\rho}{\rho\sigma}\left[\varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}h_*(\upsilon)d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}}\right],
\end{aligned}$$

for some $h_* \in \Psi_{H,y_*}$. Consequently Ω_2 has a closed graph. Ω_2 is upper semi-continuous. Operators Ω_1 and Ω_2 satisfy all the criteria of the theorem and its implementation, thus results, in either case, (i) or case (ii). We define the case (ii), as not probable. If $y \in \mu\Omega_1(y) + \mu\Omega_2(y)$ for $\mu \in (0,1)$, then $\exists\ h \in \Psi_{H,y} \ni$

$$\begin{aligned}
y(\tau) &= \mu\left\{\frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^\tau\frac{\upsilon^{\rho-1}}{(\tau^\rho - \upsilon^\rho)^{1-\rho}}h(\upsilon)d\upsilon + \frac{\tau^\rho}{\rho\sigma}g(y)\right. \\
&\quad \left. - \frac{\tau^\rho}{\rho\sigma}\left[\varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}h(\upsilon)d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}}\right]\right\},
\end{aligned}$$

for $\tau \in \mathscr{J}$. Consequently, we have

$$\begin{aligned}
|y(\tau)| &\leq \xi(\|y\|)\|\vartheta\|\left[\frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^T\frac{\upsilon^{\rho-1}}{(T^\rho - \upsilon^\rho)^{1-\rho}}d\upsilon\right. \\
&\quad \left. + \frac{T^\rho}{\rho|\sigma|}\left(\varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}}\right)\right] \\
&\quad + \frac{T^\rho}{\rho|\sigma|}k\|y\|, \tau \in \mathscr{J},
\end{aligned}$$

which, on taking supremum over $\tau \in \mathscr{J}$, yields

$$\begin{aligned}
\|y\| &\leq \xi(\|y\|)\left[\frac{\rho^{1-\rho}}{\Gamma(\rho)}\int_0^T\frac{\upsilon^{\rho-1}}{(T^\rho - \upsilon^\rho)^{1-\rho}}\vartheta(\upsilon)d\upsilon\right. \\
&\quad \left. + \frac{T^\rho}{\rho|\sigma|}\left(\varpi\frac{\rho^{1-(\rho+\varsigma)}}{\Gamma(\rho+\varsigma)}\int_0^\varphi\frac{\upsilon^{\rho-1}\vartheta(\upsilon)d\upsilon}{(\varphi^\rho - \upsilon^\rho)^{1-(\rho+\varsigma)}}\right)\right] \\
&\quad + \frac{T^\rho}{\rho|\sigma|}k\|y\| \\
&\leq \xi(\|y\|)\|\vartheta\|\Theta_1 + \Theta_2 k\|y\|.
\end{aligned}$$

If case (ii) of Theorem 2.15 holds, then $\exists\ \mu \in (0,1)$ and $y \in \partial\mathscr{B}_\alpha$ with $y = \mu\Omega(y)$. Then, y is a solution of (3.5) with $\|y\| = \alpha$. Now, the last inequality implies that

$$\frac{\alpha(1 - k\Theta_2)}{\xi(\alpha)\|\vartheta\|\Theta_1} \leq 1,$$

which contradicts (3.3). Hence Ω has a fixed point in $\mathscr{J}$ by Theorem 2.15, and hence the problem (1.2) has a solution. □

The problem comparison results (4.1) and (3.9) are presented in this section.

6. Discussion

Here, two cases, where been discussed, and also $\rho = 0.5$ happens to be case 1, case 2 is for $\rho = 1$. Rest of the values are kept in common for problems (4.1) and (3.9). Problem (4.1) signifies the generalized Caputo case and problem (3.9) delineates the Caputo case. The assumption value of the uniqueness of solutions for the problem (4.1) is η and is illustrated in Figure. 1. Likewise, η_1 depicts the assumption value for the problem (3.9) and is represented in Figure.2. Here we demonstrate the comparison results of assumption values of the problem (4.1), and (3.9) is represented in Figure.3. From the above-said figure, we justify the values of η, η_1 by showing the influence of ρ for its differing values on the characteristics of a fractional derivative. It is evident from the figure that when $\rho > 0$, we can get positive solutions under the assumptions of Theorem 3.1. According to Figure.1 and Figure.2, the behavior of the fractional derivative concerning ρ leads to a new path regarding control applications. We came to realize that in favor of the results stated at the top, the problem of GCFDEs with non-local GRLFI boundary condition holds good for existing conditions. Therefore, the reader can build the problem with ample ideas with certain consistent estimates of the problem parameters. We enrolled below a few special cases.

- If $\rho \to 1$, after that we acquire the solution for the problem of Caputo type FDEs with non-local Riemann-Liouville fractional integral boundary conditions.

- If $\rho \to 0$, in that case, we come to have the results (by noting that $\lim_{\rho\to 0}\left(\frac{\tau^\rho - b^\rho}{\rho}\right) = \ln\left(\frac{\tau}{b}\right)$ and $\lim_{\rho\to 0}\left(\frac{c^\rho - \tau^\rho}{\rho}\right) = \ln\left(\frac{c}{\tau}\right)$) for the problem of Caputo-Hadamard type FDEs with non-local Hadamard fractional integral boundary conditions. Make note that the generalized fractional derivative and integral in the problem reduces to the Caputo-Hadamard fractional derivative and Hadamard integral in the limit $\rho \to 0$ with the help of L'Hospital's rule.

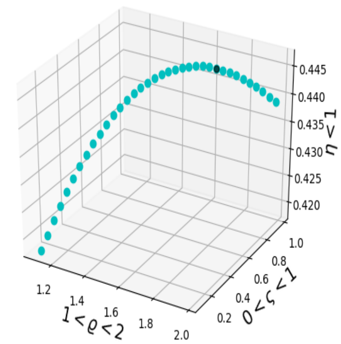

Figure 1. case 1

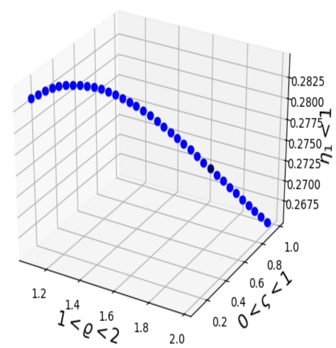

Figure 2. case 2

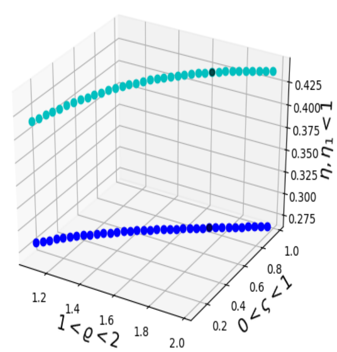

Figure 3. case 1 & case 2

Acknowledgment

The third author was supported by the minor research project funded by University Grants Commissions (F.No.4-4/2015-16 (MRP/UGC-SERO)).

References

[1] B. Ahmad, M. Alghanmi, SK. Ntouyas and A. Alsaedi, Fractional differential equations involving generalized derivative with Stieltjes and fractional integral boundary conditions, *Applied Mathematics Letters,* 84(2018), 111–117.

[2] B. Ahmad and SK. Ntouyas, Existence results for fractional differential inclusions with Erdelyi-Kober fractional integral conditions, *Analele Stiintifice ale Universitatii Ovidius Constanta Seria Matematica,* 25 (2017), 5–24.

[3] B. Ahmad, SK. Ntouyas, J. Tariboon and A. Alsaedi, Caputo type fractional differential equations with non-local Riemann-Liouville and Erdelyi-Kober type integral boundary conditions, *Filomat,* 31(14)(2017), 4515–4529.

[4] B. Ahmad, SK. Ntouyas, Y. Zhou and A. Alsaedi, A study of fractional differential equations and inclusions with non-local Erdelyi-Kober type integral boundary conditions, *Bulletin of the Iranian Mathematical Society,* 44(5)(2018), 1315–1328.

[5] A. Almeida, A.B. Malinowska and T. Odzijewicz, Fractional Differential Equations with Dependence on the Caputo-Katugampola derivative, *Journal of Computational Dynamics* **11** (2016), DOI: 10.1115/1.4034432.

[6] P.R.S. Antunes and R.A.C. Ferreira, Analysis of a class of boundary value problems depending on left and right Caputo fractional derivatives, *Communications in Nonlinear Science and Numerical Simulation,* 48(2017), 398–413.

[7] A. Atangana and D. Baleanu, Application of fixed point theorem for stability analysis of a nonlinear Schrodinger with Caputo-Liouville derivative, *Filomat* **31**(8)(2017), 2243–2248.

[8] K. Deimling, *Multivalued Differential Equations*, Walter DeGruyter, Berlin-New York, 1992.

[9] X. Ding and J.J. Nieto, Analytical solutions for multiterm time-space fractional partial differential equations with non-local damping terms, *Fractional Calculus and Applied Analysis,* 21(2018), 312–335.

[10] H. Dong, B. Guo and B. Yin, Generalized fractional supertrace identity for Hamiltonian structure of NLS-MKdV hierarchy with self-consistent sources, *Analysis and Mathematical Physics,* 6(2)(2016), 199–209.

[11] P. Duraisamy, A.R. Vidhyakumar, T. Nandha Gopal and M. Subramanian, Influence of nonlocal discrete and integral boundary conditions involving Caputo derivative in boundary value problem, *Journal of Physics : Conference Series,* 1139(1)(2018), 012014.

[12] P. Duraisamy and T. Nandha Gopal, Existence results for fractional delay integro-differential equations with multipoint boundary conditions, *Malaya Journal of Matematik*, 7(2019), 96–103.

[13] Y.Y. Gambo, F. Jarad, D. Baleanu and T. Abdeljawad, On Caputo modification of the Hadamard fractional derivatives, *Advances in Difference Equations,* 2014 (10)(2014), 1–10.

[14] F. Jarad, T. Abdeljawad and D. Baleanu, On the generalized fractional derivatives and their Caputo modification, *Journal of Nonlinear Sciences and Applications,* 10(2017), 2607–2619.

[15] U.N. Katugampola, New approach to a generalized fractional integral , *Applied Mathematics and Computation,* 218(3)(2011), 860–865.

[16] U.N. Katugampola, A new approach to a generalized fractional derivatives , *Bulletin of Mathematical Analysis and Applications,* 6(4)(2014), 1–15.

[17] U.N. Katugampola, Existence and Uniqueness Results for a Class of Generalized Fractional Differential Equations, *arXiv*: 1411.5229 (2016).

[18] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Amsterdam, Boston, Elsevier, 2006.

[19] A. Lasota and Z. Opial, An application of the Kakutani-Ky Fan theorem in the theory of ordinary differential equations, *Bulletin L'Academie Polonaise des Science, Serie des Sciences Mathematiques, Astronomiques et Physiques,* 13(1965), 781–786.

[20] B. Lupinska and T. Odzijewicz A Lyapunov-type inequality with the Katugampola fractional derivative, *Mathematical Methods in the Applied Sciences,* (2018), 1–12,
DOI: 10.1002/mma.4782.

[21] S. Muthaiah, D. Baleanu, Existence of Solutions for Nonlinear Fractional Differential Equations and Inclusions Depending on Lower-Order Fractional Derivatives, *Axioms*, 9(2020), 44.

[22] S. Muthaiah, M. Murugesan and N. Thangaraj, Existence of solutions for nonlocal boundary value problem of Hadamard fractional differential equations, *Advances in the Theory of Nonlinear Analysis and its Applications*, 3(3)(2019), 162–173.

[23] SK. Ntouyas and Sina. Etemad, On the existence of solutions for fractional differential inclusions with sum and integral boundary conditions, *Applied Mathematics and Computation,* 266 (2015), 235–246.

[24] D. O'Regan, Fixed-point theory for the sum of two operators, *Applied Mathematics Letters,* 9(1996), 1–8.

[25] W.V. Petryshyn and P. M. Fitzpatric, A degree theory, fixed point theorems, and mapping theorems for multivalued noncompact maps, *Transactions of the American Mathematical Society,* 194 (1974), 1–25.

[26] I. Podlubny, *Fractional Differential Equations Mathematics in Science and Engineering*, Vol. 198, Academic Press, San Diego, CA, 1999.

[27] M. Subramanian, and D. Baleanu, Stability and Existence Analysis to a Coupled System of Caputo Type Fractional Differential Equations with Erdelyi-Kober Integral Boundary Conditions, *Applied Mathematics & Information Sciences*, 14(3), 2020, 415–424.

[28] M. Subramanian, A.R. Vidhyakumar and T. Nandhagopal, Analysis of fractional boundary value problem with non local flux multi-point conditions on a Caputo fractional differential equation, *Studia Universitatis Babes-Bolyai Mathematica*, 64(4)(2019), 511-527.

[29] M. Subramanian, A.R. Vidhyakumar and T. Nandhagopal, Analysis of fractional boundary value problem with non-local integral strip boundary conditions, *Nonlinear Studies*, 26(2)(2019), 1-10.

[30] M. Subramanian, A.R. Vidhyakumar and T. Nandhagopal, A Fundamental Approach on Non-integer Order Differential Equation Using Nonlocal Fractional Sub-Strips Boundary Conditions, *Discontinuity, Nonlinearity, and Complexity*, 8(2)(2019), 189-199.

[31] A.R. Vidhyakumar, P. Duraisamy, T. Nandha Gopal and M. Subramanian, Analysis of fractional differential equation involving Caputo derivative with non-local discrete and multi-strip type boundary conditions, *Journal of Physics : Conference Series* 1139(1) (2018), 012020.

[32] S. Zeng, D. Baleanu, Y. Bai and G. Wu Fractional differential equations of Caputo-Katugampola type and numerical solutions, *Applied Mathematics and Computation,* 315 (2017), 549–554.

[33] Y. Zhou, J. Wang and L. Zhang *Basic Theory of Fractional Differential Equations*, World Scientific, 2016.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆