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On fuzzy inverse systems of fuzzy topological spaces

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Abstract

In this paper, we have initiated and studied the concepts of fuzzy inverse systems of fuzzy topological spaces and fuzzy continuous functions along with their fuzzy inverse limits. We have shown that every fuzzy inverse system possesses fuzzy inverse limit and fuzzy inverse limit is unique in some sense. The fuzzy inverse limit $(X, \psi_i)_I$ of a fuzzy inverse system $(X_i, \psi_{ij})_I$ enjoys many satisfactory desired properties. Further, we have introduced a covariant functor $F\lim_{\leftarrow}$ from the category $\mathbf{FIS}(I)$ of fuzzy inverse systems of fuzzy topological spaces and fuzzy continuous functions to the category $\mathbf{FTS}$ of fuzzy topological spaces and fuzzy continuous functions.

Keywords

Fuzzy inverse system, fuzzy inverse limit, category, covariant functor.

AMS Subject Classification

26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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Contents

1. Introduction

In 1965, Zadeh [10] generalized the notion of "sets" to "fuzzy sets", which was a great achievement not only in pure mathematics, but also in the whole world of mathematical sciences because it has several direct applications in different branches of science. "Fuzzy topology", initiated by Chang [3], becomes a mature field in fuzzy mathematics. To build the foundation of fuzzy topology, the works of Lowen [6, 7], Pao-Ming and Ying-Ming [8], Guojun [5] are worths to be mentioned.

The notions of inverse systems and their inverse limits has several application in different branches of mathematics, specially in category theory [4] and finite group theory [9]. The main purpose of this article is to introduce the concepts of fuzzy inverse systems of fuzzy topological spaces and fuzzy continuous functions and their fuzzy inverse limits. In section 3, We have achieved a bridge result between the fuzzy inverse system of the topological spaces and the inverse system of the fuzzy topological spaces. We have also established that every fuzzy inverse system of fuzzy topological spaces and fuzzy continuous functions has fuzzy inverse limit. Further we have shown that every fuzzy inverse system $(X_i, \psi_{ij})_I$ possesses the unique fuzzy inverse limit in the sense that if $(X, \psi_i)_I$ and $(X', \psi'_i)_I$ are two fuzzy inverse limits of $(X_i, \psi_{ij})_I$, then there exists a fuzzy homeomorphism $\mathscr{F} : X \to X'$ such that $\psi'_i \mathscr{F} = \psi_i$ for each $i \in I$. We have studied several other basic properties of the notion of fuzzy inverse limit $(X, \psi_i)_I$ of a fuzzy inverse system $(X_i, \psi_{ij})_I$. Besides these, in section 4, we have initiated a new covariant functor $F\lim_{\leftarrow}$ from the category $\mathbf{FIS}(I)$ to the category $\mathbf{FTS}$ of fuzzy topological spaces and fuzzy continuous functions.

2. Preliminaries

Zadeh [10] initiated the concept of "fuzzy set".

Definition 2.1. *Suppose X be a non-empty set. Then any function with domain X and codomain $J = [0, 1]$ is said to be a* fuzzy subset *of the set X and the collection of all fuzzy subsets of the X is denoted by J^X.*

Let A be a fuzzy set of X. Then the set $supp(A) = \{y \in X : A(y) > 0\}$ is called the support of A.

Let t be a real number with $0 < t \leq 1$ and $y \in X$. Then the fuzzy set y_t of X is defined by:

$$y_t(z) = \begin{cases} 1 & z = y \\ 0 & z \neq y \end{cases}$$

is said to be a *fuzzy point*. We use the symbol $Pt(X)$ to denote the collection of all fuzzy points of a set X.

Let A and B be two fuzzy sets of X. Then we define the relation $\leq$ on the set of all fuzzy sets of X as follows:

$$A \leq B \Longleftrightarrow A(y) \leq B(y) \ \forall y \in X.$$

Let A be a fuzzy set of X and $y_t \in Pt(X)$ satisfying $y_t \leq A$. Then we say that "A contains the fuzzy point y_t " and in symbol, we write it as $y_t \in A$.

Let $\alpha \in [0,1]$. Then $\underline{\alpha}$ is the fuzzy set (called a constant fuzzy) of X defined by:

$$\underline{\alpha}(y) = \alpha \ \forall y \in X.$$

Let A and B be two fuzzy sets of X. If there exists a $y \in X$ satisfying $A(y) + B(y) > 1$, then A and B are called *quasi-coincident* [8] fuzzy sets. In this case, in symbol, we write $A\hat{q}B$. Otherwise, A and B are called *not quasi-coincident* [8] fuzzy sets and then we use the symbol $A\bar{q}B$.

Let μ and ν be two fuzzy sets of X. Then the *join* of μ and ν (denoted by $\mu \vee \nu$) is a fuzzy set of X such that $(\mu \vee \nu)(y) = \max\{\mu(y), \nu(y)\} \ \forall y \in X$. And the *meet* of μ and ν is defined by $(\mu \vee \nu)(y) = \min\{\mu(y), \nu(y)\} \ \forall y \in X$. It is well-known that the fuzzy set μ' of X with the property $\mu'(y) = 1 - \mu(y)$ for all $y \in X$ is called the fuzzy complement of μ.

Chang [3] initiated first the concept of fuzzy topology.

Definition 2.2. *Let $\delta \subset J^X$ satisfies:*
(i) $\underline{0}, \underline{1} \in \delta$,
(ii) $\mu \wedge \nu \in \delta$ for all $\mu, \nu \in \delta$,
(iii) $\vee\{\mu \in \delta_0\} \in \delta$ for any subfamily δ_0 of δ.
Then X is called a fuzzy topological space *with* fuzzy topology δ. *A fuzzy topological space X with a fuzzy topology δ is denoted by (X, δ). A $\mu \in J^X$ is called a fuzzy open set of the fuzzy topological space X if $\mu \in \delta$. And if $\nu' \in \delta$, then ν is called a* fuzzy closed *set of the fuzzy topological space X.*

In the entire paper, (X, δ) and (Y, σ) (or simply X and Y) are non-empty fuzzy topological spaces.

Let $y_t \in Pt(X)$. If a fuzzy open set μ of the fuzzy topological space X has the property $y_t\hat{q}\mu$, then μ is said to be *fuzzy quasi-neighborhood* of y_t. We denotes the set of all fuzzy quasi-neighborhoods of y_t in the fuzzy topological space X by $\mathscr{Q}(X, y_t)$.

Let $\eta \in J^X$ and $\Omega \subset \delta$. Then Ω is said to be a *fuzzy open cover* of η if $\vee\{\mu : \mu \in \Omega\} \geq \eta$.

Lowen [6] introduced the notion of *fuzzy compact space*.

Definition 2.3. *Let (X, δ) be a fuzzy topological space. Then An $\eta \in J^X$ is said to be* fuzzy compact *[6] if for every $\varepsilon > 0$ and for every fuzzy open cover Ω of η, there exists a finite subfamily $\Omega_0 \subset \Omega$ such that $\bigvee_{\mu \in \Omega_0} \mu \geq \eta - \underline{\varepsilon}$. If all the constant fuzzy subsets of X are fuzzy compact, then X is said to be a* fuzzy compact space.

A mapping $\psi : (X, \delta) \to (Y, \sigma)$ is said to be *fuzzy continuous* if $\psi^{-1}(\mu) \in \delta$ for all $\mu \in \sigma$, or equivalently, for each $x_t \in Pt(X)$ and for each $\zeta \in \mathscr{Q}(Y, \psi(x_t))$, there is a $\nu \in \mathscr{Q}(X, x_t)$ such that $\psi(\nu) \leq \zeta$. A bijective fuzzy continuous mapping $\psi : X \to Y$ is called *fuzzy homeomorphism* if ψ^{-1} is fuzzy continuous.

3. Fuzzy inverse systems of fuzzy topological spaces

In this section, we have introduced the notions of fuzzy inverse systems of fuzzy topological spaces and their fuzzy inverse limits. We have studied several basic properties of fuzzy inverse limits of fuzzy inverse systems of fuzzy topological spaces.

Throughout this section, symbol I stands for a directed index set.

Definition 3.1. *A fuzzy inverse system consists of*
(i) a family $\{X_i : i \in I\}$ of fuzzy topological spaces and
(ii) a family $\{\psi_{ij} : (i,j) \in I \times I\}$ of fuzzy continuous mappings $\psi_{ij} : X_j \to X_i$ such that
(a) ψ_{ii} is the identity mapping on X_i for each $i \in I$,
(b) $\psi_{ij}\psi_{jk} = \psi_{ik}$ for all $i,j,k \in I$ with $i \leq j \leq k$.
This fuzzy inverse system is denoted by $(X_i, \psi_{ij})_I$ or simply, by (X_i, ψ_{ij}).

Let (X, τ) be a given topological space and $\tau_{\mathbf{R}} = \{]r, \infty[: r \in \mathbf{R}\} \cup \{\emptyset\}$. Consider the space $J = [0,1]$ with subtopology $\tau_{\mathbf{R}}|_J$ and $\omega(\tau) = \{\mu \in X^J : \mu \text{ is continuous}\}$. Lowen [7] showed that $\omega(\tau)$ is a fuzzy topology on X.

Lemma 3.2. *The mapping $\psi : (X, \tau) \to (Y, \tau')$ is continuous if and only if $\psi : (X, \omega(\tau)) \to (Y, \omega(\tau'))$ is fuzzy continuous.*

Proof. In [1], Afsan has showed that the continuity of $\psi : (X, \tau) \to (Y, \tau')$ ensures the fuzzy continuity of $\psi : (X, \omega(\tau)) \to (Y, \omega(\tau'))$.

Conversely, suppose $\psi : (X, \omega(\tau)) \to (Y, \omega(\tau'))$ is fuzzy continuous. Let V be an open set containing $\psi(x) \in Y$. Then for each $\varepsilon > 0$, $\chi_V \in \mathscr{Q}(Y, \psi(x_\varepsilon))$. Then there exists $\mu \in \mathscr{Q}(X, x_\varepsilon)$ such that $\psi(\mu) \leq \chi_V$. Now the continuity of $\mu : (X, \tau) \to J$ ensures that $U = \{p \in X : \mu(p) > 1 - \varepsilon\}$ is open in (X, τ) and containing x. Also $\psi(U) \subset V$. So, $\psi : (X, \tau) \to (Y, \tau')$ is continuous. $\square$

Following theorem is the direct consequence of the result of above lemma.

Theorem 3.3. *Let $((X_i,\tau_i),\psi_{ij})_I$ be an inverse system of topological spaces and continuous functions. Then $((X_i,\omega(\tau_i)),\psi_{ij})_I$ is a fuzzy inverse system of fuzzy topological spaces and fuzzy continuous functions.*

Definition 3.4. *Let $(X_i,\psi_{ij})_I$ be a fuzzy inverse system and Y be a any fuzzy topological space. Then a family $\{\psi_i : i\in I\}$ of fuzzy continuous mappings $\psi_i : Y\to X_i$ is called fuzzy compatible with this fuzzy inverse system if*

$$\psi_{ij}\psi_j = \psi_i$$

for all $i,j\in I$ with $i\leq j$.

Definition 3.5. *Let $(X_i,\psi_{ij})_I$ be a fuzzy inverse system. A fuzzy inverse limit of this fuzzy inverse system consists of*
(i) A fuzzy topological space X and
(ii) a family $\{\psi_i : i\in I\}$ of fuzzy continuous mappings $\psi_i : X\to X_i$ fuzzy compatible with the fuzzy inverse system $(X_i,\psi_{ij})_I$ such that

($\star$) for each fuzzy compatible family $\{\theta_i : i\in I\}$ of fuzzy continuous mappings $\theta_i : Y\to X_i$, there exists a unique fuzzy continuous mapping $\psi : Y\to X$ such that $\psi_i\psi=\theta_i$ for all $i\in I$.
This fuzzy inverse limit of the fuzzy inverse system $(X_i,\psi_{ij})_I$ is denoted by $(X,\psi_i)_I$ or simply by

$$F-\lim_{\leftarrow} X_i.$$

Remark 3.6. *Let $((X,\tau),\psi_i)_I$ be an inverse limit of an inverse system $((X_i,\tau_i),\psi_{ij})_I$ of topological spaces and continuous functions. Then $((X,\omega(\tau)),\psi_i)_I$ is a fuzzy inverse limit of the fuzzy inverse system $((X_i,\omega(\tau_i)),\psi_{ij})_I$ of fuzzy topological spaces and fuzzy continuous functions.*

Lemma 3.7. *Let $\psi : (X,\delta)\to(Y,\sigma)$ be a fuzzy continuous function and $A\subset X$. Then $\psi\mid_A : (A,\delta_A)\to(Y,\sigma)$ is fuzzy continuous.*

Proof. We note that $\delta_A=\{\mu\mid_A : \mu\in\delta\}$. Suppose $\nu\in\sigma$. Then $\psi^{-1}(\nu)\in\delta$. Now we shall show that $\psi^{-1}(\nu)\mid_A=(\psi\mid_A)^{-1}(\nu)$. Let $a\in A$. Then $(\psi\mid_A)^{-1}(\nu)(a)=\nu(\psi\mid_A(a))=\nu(\psi(a))=\psi^{-1}(\nu)(a)$. Thus $(\psi\mid_A)^{-1}(\nu)=\psi^{-1}(\nu)\mid_A\in\delta_A$ and so $\psi\mid_A$ is fuzzy continuous. □

Following theorem shows that every fuzzy inverse system has a fuzzy inverse limit.

Theorem 3.8. *Let $(X_i,\psi_{ij})_I$ be a fuzzy inverse system. Then $(X_i,\psi_{ij})_I$ has a fuzzy inverse limit.*

Proof. (i) Let $X=\prod\{X_i : i\in I\}$. Let $Y=\{x\in X : \psi_{ij}\pi_j(x)=\pi_i(x) : \text{for all } i,j\in I \text{ with } j\geq i\}$ and $\psi_i=\pi_i\mid_X$ for each $i\in I$. We claim that $(Y,\psi_i)_I$ is a fuzzy inverse limit of $(X_i,\psi_{ij})_I$. Lemma 3.7 ensures that ψ_i is fuzzy continuous for each $i\in I$. It is also obvious that $\psi_{ij}\psi_j=\psi_i$ for all $i,j\in I$ with $j\geq i$. Now suppose $\{\theta_i : i\in I\}$ be a fuzzy compatible family of fuzzy continuous mappings $\theta_i : Z\to X_i$. We define the mapping $\omega : Y\to X$ by $\omega(y)=(\psi_i(y))$. Then $\pi_i\omega=\psi_i$ for all $i\in I$. Then ω is fuzzy continuous. We note that for $j\geq i$, $\pi_i(\omega(z))=\psi_i(z)=(\psi_{ij}\psi_j)(z)=(\psi_{ij}\pi_j\omega)(z)=\phi_{ij}\pi_j(\omega(z))$ for all $z\in Z$. So $\omega(z)\in Y$ for all $z\in Z$. Now we define the mapping $\varpi : Z\to Y$ by $\varpi(z)=\omega(z)$ for all $z\in Z$. So ϖ is fuzzy continuous and $\psi_i\varpi=\theta_i$ for all $i\in I$. Let $\varpi_1 : Z\to Y$ be another fuzzy continuous mapping satisfying $\psi_i\varpi_1=\theta_i$ for all $i\in I$. Then for all $i\in I$ and for all $z\in Z$, $[\varpi_1(z)]_i=i$-th component of $\varpi_1(z)=\psi_i(z)$. So $\varpi_1(z)=\varpi(z)$ for all $z\in Z$. Hence $\varpi_1=\varpi$ and consequently, $(Y,\psi_i)_I$ is a fuzzy inverse limit of $(X_i,\psi_{ij})_I$. □

Remark 3.9. *The fuzzy inverse limit $(Y,\psi_i)_I$ of the fuzzy inverse system $(X_i,\psi_{ij})_I$ is denoted by*

$$F-s\lim_{\leftarrow} X_i.$$

Theorem 3.10. *Let $(X,\psi_i)_I$ and $(X',\psi_i')_I$ be two fuzzy inverse limits of a fuzzy inverse system $(X_i,\psi_{ij})_I$. Then there exists a fuzzy homeomorphism $\mathscr{F} : X\to X'$ such that $\psi_i'\mathscr{F}=\psi_i$ for each $i\in I$.*

Proof. Since $(X,\psi_i)_I$ is a fuzzy inverse limit of $(X_i,\psi_{ij})_I$ and $\{\psi_i' : i\in I\}$ is compatible family of continuous mappings with the fuzzy inverse system $(X_i,\psi_{ij})_I$, there exists a unique fuzzy continuous mapping $\varpi : X'\to X$ such that $\psi_i\varpi=\psi_i'$ for each $i\in I$. Again since $(X',\psi_i')_I$ ia a fuzzy inverse limit of $(X_i,\psi_{ij})_I$ and $\{\psi_i : i\in I\}$ is compatible family of continuous mappings with the fuzzy inverse system $(X_i,\psi_{ij})_I$, there exists a unique fuzzy continuous mapping $\varpi' : X\to X'$ such that $\psi_i'\varpi'=\psi_i$ for each $i\in I$. So, $\psi_i(\varpi\varpi')=\psi_i$ for all $i\in I$. Since $\{\psi_i' : i\in I\}$ is compatible family of continuous mappings with the fuzzy inverse system $(X_i,\psi_{ij})_I$, there exists a unique fuzzy continuous mapping $id_X : X\to X$ such that $\psi_i id_X=\psi_i$ for each $i\in I$. Thus we must have $\varpi\varpi'=id_X$. Similarly, we can show that $\varpi'\varpi=id_X$. Then $\mathscr{F}=\varpi'$ is a fuzzy homeomorphism satisfying $\psi_i'\mathscr{F}=\psi_i$ for each $i\in I$. □

Remark 3.11. *(i) Every fuzzy inverse system $(X_i,\psi_{ij})_I$ possesses the unique fuzzy inverse limit in the sense that if $(X,\psi_i)_I$ and $(X',\psi_i')_I$ are two fuzzy inverse limits of $(X_i,\psi_{ij})_I$, then there exists a fuzzy homeomorphism $\mathscr{F} : X\to X'$ such that $\psi_i'\mathscr{F}=\psi_i$ for each $i\in I$.*

Theorem 3.12. *Let $(X_i,\psi_{ij})_I$ be a fuzzy inverse system of fuzzy Hausdorff topological spaces and*

$$Y=F-s\lim_{\leftarrow} X_i.$$

Then χ_Y is fuzzy closed subset of $X=\prod\{X_i : i\in I\}$.

Proof. Let

$$x_\lambda=(x_i)_\lambda\in 1_X-\chi_Y.$$

Then $(x_i) \notin Y$. Then there exist $i, j \in I$ with $j \geq i$ such that $\psi_{ij}(x_j) \neq x_i$. Consider the fuzzy points $(\psi_{ij}(x_j))_\lambda \in FP(X_i)$ and $(x_i)_\lambda \in FP(X_i)$. Since X_i is fuzzy Housdorff, we can find $U \in \mathscr{Q}(X_i, (\psi_{ij}(x_j))_\lambda)$ and $V \in \mathscr{Q}(X_i, (x_i)_\lambda)$ such that $U \bigwedge V = \underset{\sim}{0}$. Since ψ_{ij} is fuzzy continuous, there exists $U^\star \in \mathscr{Q}(X_j, (x_j)_\lambda)$ such that $\psi_{ij}(U^\star) \leq U$. Consider the fuzzy open set $W = \prod\{V_k : k \in I, V_i = V, V_j = U^\star, V_k = 1_{X_k} \text{ for } k \neq i, j\}$. Then W is a fuzzy quasi-neighbourhood of $x = (x_i)$ in X such that $W \leq 1_X - \chi_Y$. So $1_X - \chi_Y$ is fuzzy open subset of X and so χ_Y fuzzy closed subset of X. □

Theorem 3.13. *Let $(X_i, \psi_{ij})_I$ be a fuzzy inverse system and*

$$Y = F \lim_{\leftarrow} X_i.$$

(a) If each X_i is fuzzy Hausdorff, so is Y.
(b) If each X_i is fuzzy compact topologically generated and fuzzy Hausdorff, then Y is fuzzy compact.
(c) If each X_i is fuzzy compact topologically generated, fuzzy totally disconnected fuzzy Hausdorff space, then Y is so.
(d) If for each $i \in I$, $X_i \neq \underset{\sim}{0}$, fuzzy compact topologically generated and Hausdorff, then $Y \neq \underset{\sim}{0}$.

Proof. It is sufficient to prove the results for

$$Y = Fs \lim_{\leftarrow} X_i.$$

(a) Since each X_i is Hausdorff, $X = \prod\{X_i : i \in I\}$ is Hausdorff. Since Y is a subspace of X, Y is also Hausdorff.
(b) Since each X_i is fuzzy compact, $X = \prod\{X_i : i \in I\}$ is fuzzy compact. We claim that X is topologically generated. Since each (X_i, δ_i) is topologically generated, there exists a topology τ_i on X_i such that $\delta_i = \omega(\tau_i)$. Let δ be the fuzzy product topology on X and τ be the product topology on X. We know that $\pi_{k_i}^{-1}(\omega(\tau_{k_i})) = \omega(\pi_{k_i}^{-1}(\tau_{k_i}))$ and so $\bigwedge_{i=1}^{n} \pi_{k_i}^{-1}(\omega(\tau_{k_i})) = \bigwedge_{i=1}^{n} \omega(\pi_{k_i}^{-1}(\tau_{k_i}))$ for each finite $n \in \mathbb{N}$. So, $\delta = \omega(\tau)$. Therefore Y is a closed subspace of the fuzzy compact topologically generated Hausdorff space X. Thus Y is fuzzy compact.
(c) In [2], Balasubharamanian shaw that the fuzzy product of fuzzy totally disconnected spaces is fuzzy totally disconnected and subspace of fuzzy totally disconnected spaces is fuzzy totally disconnected. Thus the result follows.
(d) For each $i \in I$, we define the set $Y_i = \{x = (x_j) \in X : \psi_{ij}(x_i) = x_j \text{ whenever } j \leq i\}$. Then it can be shown that χ_{Y_i} is closed subset of X with $\chi_{Y_i} \neq \underset{\sim}{0}$. □

Theorem 3.14. *Let $(X_i, \psi_{ij})_I$ be a fuzzy inverse system and $(Y, \psi_i)_I$ be fuzzy inverse limit of the fuzzy inverse system $(X_i, \psi_{ij})_I$. Then $\{\psi_i^{-1}(U) : U$ is open in $X_i\}$ is a base for Y.*

Proof. Consider a fuzzy open set $P = [\psi_{i_1}^{-1}(U_1) \wedge \psi_{i_2}^{-1}(U_2) \wedge \ldots \wedge \psi_{i_n}^{-1}(U_n)] \mid Y$, where U_j is a fuzzy open set of X_{i_j}, $j = 1, 2, \ldots, n$. Let $x_\lambda = (x_\lambda^i) \in Pt(Y)$ with $P \hat{q} x_\lambda$. Choose a $k \in I$ such that $k \geq i_1, i_2, \ldots, i_t$. Then thee set $\psi_{i_r k}^{-1}(U_r)$ $(r = 1, 2, \ldots, n)$ in open in X_k since $\psi_{i_r j} : X_k \to X_{i_r}$ is a continuous mapping. Since $\psi_{ik}(x^k) = x^i$ for all $i \leq k$, $x_\lambda^k \hat{q} \psi_{i_r k}^{-1}(U_r)$ $(r = 1, 2, \ldots, n)$. Consider $U = \bigwedge\{\psi_{i_r k}^{-1}(U_r) : r = 1, 2, \ldots, n\}$. Then U is a fuzzy open set of X_k and $x_\lambda^k \hat{q} U$. Thus $\psi_k^{-1}(U)$ is a fuzzy open set of Y and $x_\lambda \hat{q} \psi_k^{-1}(U)$. Clearly $\psi_k^{-1}(U) \leq P$. So $\{\psi_i^{-1}(U) : U$ is open in $X_i\}$ is a base for Y. □

Theorem 3.15. *If X is a subset of Y satisfying $\psi_i(X) = X_i$ for each $i \in I$. Then χ_X is dense in Y.*

Proof. Let U be any fuzzy open set of X_i. Then $\psi_i^{-1}(U)$ is a basic open set in Y. Clearly $\psi_i(\chi_X) \hat{q} U$, i.e. $\chi_X \hat{q} \psi_i^{-1}(U)$. So every basic open set of Y is quasi-coincident with χ_X and hence χ_X is dense in Y. □

4. Category of fuzzy inverse systems of fuzzy topological spaces and fuzzy continuous functions

In this section, we have introduced a category $\mathscr{C} = \mathscr{F}\mathscr{I}\mathscr{S}(I)$ of fuzzy inverse systems of fuzzy topological spaces and fuzzy continuous functions and a covariant functor $F \lim_{\leftarrow}$ from the category **FIS**(I) to the category **FTS** of fuzzy topological spaces and fuzzy continuous functions.

Lemma 4.1. *A mapping $\theta : X \to Y$ is a fuzzy continuous if and only if $\psi_i \theta$ is fuzzy continuous.*

Proof. If the mapping θ is a fuzzy continuous, then clearly $\psi_i \theta$ is fuzzy continuous for each $i \in I$. Now let $\psi_i \theta$ is fuzzy continuous for each $i \in I$. Let U be any fuzzy open set of X_i. Then $\psi_i^{-1}(U)$ is a basic open set in Y. Now $\theta^{-1}(\psi_i^{-1}(U)) = (\psi_i \theta)^{-1}(U)$ is open in X. So θ is fuzzy continuous. □

Definition 4.2. *Let $(X_i, \psi_{ij})_I$ and $(X'_i, \psi'_{ij})_I$ be two fuzzy inverse systems. A fuzzy morphism $\Theta : (X_i, \psi_{ij})_I \to (X'_i, \psi'_{ij})_I$ of the fuzzy inverse systems consists of a collection of fuzzy continuous functions $\{\vartheta_i : X_i \to X'_i : i \in I\}$ such that $\vartheta_i \psi_{ij} = \psi'_{ij} \vartheta_j$, i.e. the diagram*

$$\begin{array}{ccc} X_j & \xrightarrow{\psi_{ij}} & X_i \\ \vartheta_j \downarrow & & \vartheta_i \downarrow \\ X'_j & \xrightarrow{\psi'_{ij}} & X'_i \end{array}$$

is commutative whenever $j \leq i$.

Here the functions of the set $\{\vartheta_i : X_i \to X'_i : i \in I\}$ are called the components of the morphism Θ. A homomorphism $\Theta : (X_i, \psi_{ij})_I \to (X_i, \psi_{ij})_I$ is called the identity homomorphism of the system $(X_i, \psi_{ij})_I$ if the component $\vartheta_i = id_{X_i}$ for all $i \in I$.
Let $(X_i, \psi_{ij})_I$, $(X'_i, \psi'_{ij})_I$ and $(X''_i, \psi''_{ij})_I$ be three fuzzy inverse systems. Let $\Theta : (X_i, \psi_{ij})_I \to (X'_i, \psi'_{ij})_I$ and $\Psi : (X'_i, \psi'_{ij})_I \to (X''_i, \psi''_{ij})_I$ be homomorphism with respective sets $\{\vartheta_i : X_i \to$

$X'_i : i \in I\}$ and $\{\varphi_i : X'_i \to X''_i : i \in I\}$ of the components. Then the composition of morphism is defined as the morphism $\Psi\Theta : (X_i, \psi_{ij})_I \to (X''_i, \psi''_{ij})_I$ between the fuzzy inverse systems $(X_i, \psi_{ij})_I$ and (X''_i, ψ''_{ij}) whose set of components is $\{\varphi_i \vartheta_i : X_i \to X''_i : i \in I\}$.

Definition 4.3. *Let the category $\mathscr{C} = \mathscr{F}\mathscr{I}\mathscr{S}(I)$*
(a) whose objects are the set of all fuzzy inverse systems indexed by I and
(b) morphisms are the morphisms of fuzzy inverse systems.
This category is called the standard category of fuzzy inverse systems.

Let $(X_i, \psi_{ij})_I$, $(X'_i, \psi'_{ij})_I$ be two members of the category $\mathscr{F}\mathscr{I}\mathscr{S}(I)$ such that

$$X = F \lim_{\leftarrow} X_i = (X, \psi_i)_I \text{ and } X' = F \lim_{\leftarrow} X'_i = (X', \psi'_i)_I.$$

Assume that $\Theta : (X_i, \psi_{ij})_I \to (X'_i, \psi'_{ij})_I$ be a morphism with the set of components $\{\vartheta_i : X_i \to X'_i : i \in I\}$. Then the collection of compilable mappings $\vartheta_i \psi_i : X \to X'_i$ yields the unique continuous mapping

$$F \lim_{\leftarrow} \Theta = \psi'_{\Theta} : X \to X'$$

such that

$$\psi'_i \psi'_{\Theta} = \vartheta_i \psi_i$$

for all $i \in I$. Moreover, we see that if $\Theta : (X_i, \psi_{ij})_I \to (X'_i, \psi'_{ij})_I$ and $\Psi : (X'_i, \psi'_{ij})_I \to (X''_i, \psi''_{ij})_I$ be homomorphism with respective sets $\{\vartheta_i : X_i \to X'_i : i \in I\}$ and $\{\varphi_i : X'_i \to X''_i : i \in I\}$ of the components, then

$$F \lim_{\leftarrow}(\Psi\Theta) = F \lim_{\leftarrow}(\Psi) F \lim_{\leftarrow}(\Theta).$$

Thus

$$F \lim_{\leftarrow}$$

is a covariant functor from the category **FIS**(I) to the category **FTS** of fuzzy topological spaces and fuzzy continuous functions.

Above results can be summarized in the following theorem:

Theorem 4.4. *Let* **FTS** *be the category of all fuzzy topological spaces and fuzzy continuous functions. Then*

$$F \lim_{\leftarrow} : \mathbf{FTS} \to \mathbf{FTS}$$

is a covariant functor from the category **FIS**(I) *to the category* **FTS**.

5. Conclusion

In this paper, we have fuzzified the classical notions of inverse systems of topological spaces and its inverse limit in fuzzy topology. We have investigated and archived several basic properties of these notions. There are several applications of inverse systems and their inverse limits in different branches of mathematics, specially in category theory [4] and finite group theory [9]. Therefore, as the case of non-fuzzy mathematics, several research in fuzzy mathematics can be done in future on the basis of the notions introduced in this paper.

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