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Common fixed point theorems in rs_b-distance with property C and weakly commuting maps in probabilistic s_b-metric space

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Abstract
We establish the conception of rs_b-distance with property C on a Menger Probabilistic s_b-metric space. Moreover, we have proved a few fixed point theorems in a Complete Menger Probabilistic s_b-metric space. Also we display the Weakly Commuting maps in same space.

Keywords
Menger Probabilistic s_b-metric space, rs_b-distance, rs_b-distance with property C, Weakly Commuting maps.

AMS Subject Classification
47H10, 54E35, 54E40, 54H25.

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Contents

1. Introduction

In this work, there are more conjunctions about the approach of metric spaces (MS). Fixed point (fd-pt.) concept in S-metric spaces $(S-MS)$ and b-metric spaces $(b-MS)$ has been published in more papers like [4],[5],[8], etc. In our work, we scrutinize a new approach of $S-MS$ called probabilistic s_b-MS, which is an expansion of the $S-MS$ using the concept of self to be different from zero. Rouse by crafted by Bakhtin in [4], we initially present the Ps_b-MS as a generalization of the $b-MS$. Recently, R.Saadati,[9] introduced the idea of r-distance on a Menger Ps_b-MS. Through an idea of r-distance, we have defined rs_b-distance and have proved a few fixed pt. theorems in the same space.

2. Preliminaries

Definition 2.1. *A probabilistic metric space [9] (PMS) be a triple $(M,\mathscr{F},\tau)$, here M is a nonempty set, $\mathscr{F}$ is a function from $M^2 \to \Delta^+$, τ is a triple function and the coming properties were convinced $\forall$ s,u,w in M;*

(a) $\mathscr{F}_{ss} = \varepsilon_0$

(b) $\mathscr{F}_{su} \neq \varepsilon_0$ *if* $p \neq q$

(c) $\mathscr{F}_{su} = F_{us}$

(d) $\mathscr{F}_{sw} \geq \tau(F_{su}, F_{uw})$

If $\tau = \tau_T$ any t-norm $T \Rightarrow (M,\mathscr{F},\tau_T)$ termed as Menger space(MS).

Definition 2.2. *A probabilistic b-metric space [1] (briefly PbMS) be a quadruple (M,F,τ,s), here M is a non empty set, $\mathscr{F}$ is a function from $M^2 \to \Delta^+$, τ is a triangle function $s \geq 1$ is a real number and the following conditions are fulfilled; $\forall$ $s,u,w \in M$ and $r > 0$,*

(a) $\mathscr{F}_{ss} = \mathscr{H}$

(b) $\mathscr{F}_{su} = \mathscr{H} \Rightarrow s = u$

(c) $\mathscr{F}_{su} = \mathscr{F}_{us}$

(d) $\mathscr{F}_{su}(dr) \geq \tau(F_{sw},)F_{wu})(r)$.

If $\tau = \tau_T$ any t-norm $T \Rightarrow (M, F, \tau_T, s)$ be termed as b-MS.

Definition 2.3. *Take $\mathscr{X}$ is a non-empty set and $b \geq 1$ be a given number Suppose that a mapping $b \geq 1$ be a given number. Let us take a mapping $s_b : X^3 \to R^+$ be a function fulfilled the Coming properties:*

(i) $s_b(m,o,p) = 0 \iff m = o = p$ *and*

(ii) $s_b(m,o,p) \leq b[s_b(m,m,a) + s_b(o,o,a) + s_b(p,p,a)]$ $\forall$ $m,o,p,a \in \mathscr{X}$.

$\therefore$ the function s_b be termed as s_b-metric on X [8] and the pair $(\mathscr{X}, s_b)$ is a s_bMS.

Definition 2.4. *Take $(\mathscr{X}, \mathscr{F}, \mathscr{T})$ is a Menger PMS. Then the function $h : \mathscr{X}^2 \times [0,\infty] \to [0,1]$ be termed as r-distance [8] on $\mathscr{X}$ if the comings are fulfilled:*

(R1) $h_{m,p}(e+h) \geq \mathscr{T}(h_{m,o}(e), h_{o,p}(h))$, $\forall\, m,o,p \in \mathscr{X}$ $e,h \geq 0$;

(R2) Any $m \in \mathscr{X}$ *and* $e \geq 0$, $h_{m,.} : \mathscr{X} \times [0,\infty] \to [0,1]$ *be continuous;*

(R3) Any $\varepsilon > 0$, $\exists\, \rho > 0 \ni h_{p,m}(t) \geq 1-\rho$ *and* $h_{p,o}(f) \geq 1-\rho$ *imply* $\mathscr{F}_{m,o}(e+f) \geq 1-\varepsilon$.

Example 2.5. *Take $(\mathscr{X}, F, \mathscr{T})$ is a Menger PMS. Then $h = \mathscr{F}$ is an r-distance on X.*

Proof. Properties $(R1)$ and $(R2)$ are accessible. Take $\varepsilon > 0$ and elect $\rho > 0 \ni \mathscr{T}(1-\rho, 1-\rho) \geq 1-\varepsilon$. Then $h_{p,m}(e) \geq 1-\rho$ and $h_{p,o}(f)\rho 1-\rho$, we've,

$$\begin{aligned} \mathscr{F}_{m,o}(e+f) &\geq \mathscr{T}(\mathscr{F}_{p,m}(e), \mathscr{F}_{p,o}(f)) \\ &\geq \mathscr{T}(1-\delta, 1-\delta) \geq 1-\varepsilon. \end{aligned}$$

□

Definition 2.6. *Take $\mathscr{X}$ as a MS and $\mathscr{T}$ is a mapped, a point $u \in \mathscr{X}$ is termed as*

(i) Fd-pt[6] of $\mathscr{T}$ if it is arrangement of the functional Equation $\mathscr{T}(q) = q$.

(ii) ε-Fd-pt[6] of $\mathscr{T}$ if $d(u, \mathscr{T}(u)) < \varepsilon$ $\forall$ $\varepsilon > 0$.

3. Common Fixed Point Theorems with rs_b- distance

Definition 3.1. *A mapping $s : [0,1]^2 \to [0,1]$ is continuous s-norm if s fulfills the coming properties:*

(i) s is associative and commutative.

(ii) s is continuous.

(iii) $s(g,0) = a$ $\forall$ $g \in [0,1]$.

(iv) $s(g,i) \leq s(k,l)$ *whenever* $g \leq k$ *and* $i \leq l$ $\forall$ $g,i,k,l \in [0,1]$

the classical ex: of continuous t-norms were

$$s(g,i) = \min(g+i-1) \text{ and } s(g,i) = max(g,i)$$

Definition 3.2. *A Menger probabilistic s_b normed space (briefly Menger $Ps_b - NS$) is a triple $(\mathscr{X}, \eta, \mathscr{T})$ here X is a vector space, T is a continuous t-norm and η is a mapping from $\mathscr{X}$ into $D^+ \ni$ the coming properties hold, $\forall$ m,o,p in $\mathscr{X}$:*

(i) $\eta_m(e) = \varepsilon_0(e)$ $\forall$ $e > 0$ *iff* $m = 0$.

(ii) $\mu_{\alpha x}(t) = \eta_x(\frac{t}{|\alpha|}))$ *for* $\alpha \neq 0$.

(iii) $\eta_{m+o+p}(e_1+e_2+e_3) \geq \mathscr{T}(\eta_m(e_1), \eta_o(e_2), \eta_p(e_3))$ $\forall$ $m,n,o \in \mathscr{X}$ *and* $e_1,e_2,e_3 \geq 0$.

Remark 3.3. *Assume for all $\eta \in [0,1]$ $\exists$ a $\sigma \in]0,1[$ which doesn't rely upon n, with $\mathscr{T}^{n-1}(1-\sigma, ..., 1-\sigma) > 1-\eta$ for each $n \in \{1,2,3,...\}$.*

Definition 3.4. *Take (X,F,T) is a Menger Ps_bMS. Then the function $h : \mathscr{X}^3 \times [0,\infty] \to [0,1]$ is termed as rs_b - distance on $\mathscr{X}$ if the coming were fulfilled.*

(i) $h_{m,o,p}(e_1+e_2+e_3) \geq \mathscr{T}(h_{mov}(e_1), h_{mvp}(e_2), h_{vop}(e_3))$ $\forall$ $m,o,p \in \mathscr{X}$ *and* $e_1,e_2,e_3 \geq 0$;

(ii) any $m \in \mathscr{X}$ *and* $e \geq 0$, $h_m : \mathscr{X} \times [0,\infty) \to [0,1]$ *is continuous;*

(iii) any $\varepsilon > 0\, \exists\, \rho > 0 \ni h_{vop}(e_1) \geq 1-\rho, h_{mvp}(e_2) \geq 1-\rho$ *and* $h_{mov}(e_3) \geq 1-\rho$ *imply* $\mathscr{F}_{mop}(e_1+e_2+e_3) \geq 1-\varepsilon$.

Example 3.5. *Take $(\mathscr{X}, \mathscr{F}, \mathscr{T})$ is a Menger $Ps_b - MS$. Then $h = \mathscr{F}$ is an rs_b - distance on $\mathscr{X}$.*

Proof. By definition (3.1), properties (i) and (ii) are obvious. For property (iii), Give $\varepsilon > 0$ and elect $\rho > 0 \ni \mathscr{T}(1-\rho, 1-\rho, 1-\rho) \geq 1-\varepsilon$.

$\Rightarrow \mathscr{F}_{m,o,v}(e_1) \geq 1-\rho, \mathscr{F}_{m,v,p}(e_2) \geq 1-\rho$ and $\mathscr{F}_{v,o,p}(e_3) \geq 1-\rho$, we've

$\mathscr{F}_{m,o,p}(e) \geq \mathscr{T}(\mathscr{F}_{m,o,v}(e_1), \mathscr{F}_{m,v,p}(e_2), \mathscr{F}_{v,o,p}(e_3)) \geq \mathscr{T}(1-\rho, 1-\rho, 1-\rho) \geq 1-\varepsilon$

$\Rightarrow h = \mathscr{F}$ is an rs_b- distance on X. □

Example 3.6. *Take $(\mathscr{X}, \mathscr{F}, \mathscr{T})$ is a Menger $Ps_b - S$ and let $\mathscr{A}$ is a continuous mapping from $\mathscr{X}$ into $\mathscr{X}$. Then the function $h : \mathscr{X}^3 \to [0,\infty) \to [0,1]$ characterized by $h_{m,o,p}(e) = \min(\mathscr{F}_{\mathscr{A}m,o,v}(e_1), \mathscr{F}_{\mathscr{A}m,o,\mathscr{A}o}(e_2), \mathscr{F}_{\mathscr{A}v,\mathscr{F}o,\mathscr{F}p}(e_3))$, $\forall\, m,o,p \in \mathscr{X}$ and $e_1,e_2,e_3 > 0$ is an rs_b - distance on $\mathscr{X}$.*

Proof. Take $m,o,p,v \in \mathscr{X}$ and $e_1,e_2,e_3 > 0$ is an rs_b- distance on $\mathscr{X}$. If $\mathscr{F}_{\mathscr{A}m,o,p}(e) \leq \mathscr{F}_{\mathscr{A}m,\mathscr{A}o,\mathscr{A}p}(e)$ then we've

$h_{m,o,p}(e_1+e_2+e_3) = \mathscr{F}_{\mathscr{A}m,o,p}(e_1+e_2+e_3)$

$\geq \mathscr{T}(\mathscr{F}_{\mathscr{A}m,o,v}(e_1), \mathscr{F}_{\mathscr{A}m,v,\mathscr{A}p}(e_2), \mathscr{F}_{v,\mathscr{A}o,\mathscr{F}p}(e_3))$

$\geq \mathscr{T}(min(\mathscr{F}_{\mathscr{A}m,o,p}(e_1), \mathscr{F}_{\mathscr{A}m,\mathscr{A}o,v}(e_1))min(\mathscr{F}_{\mathscr{A}m,o,p}(e_2),$
$\mathscr{F}_{\mathscr{A}m,o,\mathscr{A}p}(e_2))$
$min(\mathscr{F}_{v,\mathscr{A}o,\mathscr{A}p}(e_3), \mathscr{F}_{v,\mathscr{A}o,\mathscr{A}p}(e_3)))$

$= \mathscr{T}(h_{m,o,p}(e_1), h_{m,o,p}(e_2), h_{v,o,p}(e_3))$

with this inequality, we've

$h_{m,o,p}(e_1+e_2+e_3) = \mathscr{F}_{\mathscr{A}m,\mathscr{A}o,\mathscr{F}p}(e_1+e_2+e_3)$
$\geq \mathscr{T}(\mathscr{F}_{\mathscr{A}m,\mathscr{A}o,\mathscr{A}p}(e_1), \mathscr{F}_{\mathscr{A}m,\mathscr{A}o,\mathscr{A}p}(e_2), \mathscr{A}_{\mathscr{A}v,\mathscr{A}o,\mathscr{A}p}(e_3))$
$h_{m,o,p}(e_1+e_2+e_3) = \mathscr{F}_{\mathscr{A}m,\mathscr{A}o,\mathscr{A}p}(e_1+e_2+e_3)$
$\geq \mathscr{T}(\mathscr{F}_{\mathscr{A}m,\mathscr{A}o,v}(e_1), \mathscr{F}_{\mathscr{A}m,v,\mathscr{A}p}(e_2), \mathscr{F}_{v,\mathscr{A}o,\mathscr{A}p}(e_3))$
$\geq \mathscr{T}(\min \mathscr{F}_{\mathscr{A}m,o,p}(e_1), \mathscr{F}_{\mathscr{A}m,\mathscr{A}o,v}(e_1)), min(\mathscr{F}_{\mathscr{A}m,o,p}$
$(e_2), \mathscr{F}_{\mathscr{A}m,v,\mathscr{A}p}(e_2)),$
$min(\mathscr{F}_(v,\mathscr{A}y,Az)(t_3), F_(u,Ay,Az)(t_3)))$
$= T(f_{x,y,u}(t_1), f_{x,u,z}(t_2), f_(u,y,z)(t_3))$

Hence (i) holds. As A is continuous then (ii) is clear. To prove (ii)

take $\varepsilon > 0$ be given and elect $\rho > 0 \ni \mathscr{T}(1-\rho, 1-\rho, 1-\rho) \geq 1-\varepsilon$.

Then from $h_{m,o,v}(e_1) \geq 1-\rho, h_{m,v,p}(e_2) \geq 1-\rho$ and $h_{v,o,p}$ $(e_3) \geq 1-\rho$ we've $\mathscr{F}_{\mathscr{A}m,o,v}(e_1) \geq 1-\rho, \mathscr{F}_{\mathscr{A}m,v,\mathscr{A}p}(e_2) \geq 1-\rho$ and $\mathscr{F}_{\mathscr{A}m,\mathscr{A}o,v}(e_3) \geq 1-\rho$.

Therefore $\mathscr{F}_{m,o,p}(e_1+e_2+e_3) \geq \mathscr{T}(\mathscr{F}_{\mathscr{A}m,o,v}(e_1),$
$\mathscr{F}_{\mathscr{A}m,v,\mathscr{A}o}(e_2), \mathscr{F}_{\mathscr{A}m,\mathscr{A}o,v}(e_3))$
$\geq \mathscr{T}(1-\rho, 1-\rho, 1-\rho)$
$\geq 1-\varepsilon$.

Thence h is an rs_b- distance on $\mathscr{X}$. □

Example 3.7. *Take $(\mathscr{X}, \beta, \mathscr{T})$ is a Menger $Ps_b - S$. Then the function $h : \mathscr{X}^3 \times [0,\infty] \to [0,1]$ characterized by $h_{m,o,p}(e) = \beta_m(e)$ $\forall$ $m,o,p \in \mathscr{X}$ and $t > 0$ rs_b- distance on $\mathscr{X}$, here $\beta = \eta$.*

Proof. Take $m,o,p \in \mathscr{X}$ and $e_1, e_2, e_3 > 0$. Then we've
$h_{m,o,p}(e_1+e_2+e_3) = \beta_v(e_1+e_2+e_3)$
$\geq \mathscr{T}(\beta_{m,o,v}(e_1), \beta_{m,v,p}(e_2), \beta_{v,o,p}(e_3))$
$= \mathscr{T}(e_{m,o,v}(e_1), h_{m,v,p}(e_2), h_{v,o,p}(e_3))$

Hence (i) holds. Also (ii) is clear. To prove (iii), give $\varepsilon > 0$ and elect $\rho > 0 \ni \mathscr{T}(1-\rho, 1-\rho, 1-\rho) \geq 1-\varepsilon$

$\Rightarrow h_{m,o,v}(e_1) \geq 1-\rho, h_{m,v,p}(e_2) \geq 1-\rho$ and $h_{v,o,p}(e_3) \geq 1-\rho$ we've

$\mathscr{F}_{m,o,p}(e_1+e_2+e_3) = \beta_{m-o-p}(e_1+e_2+e_3)$
$\geq \mathscr{T}(\beta_m(e_1), \beta_o(e_2), \beta_p(e_3))$
$= \mathscr{T}(h_{m,o,v}(e_1), h_{m,v,p}(e_2), h_{v,o,p}(e_3))$
$\geq \mathscr{T}(1-\rho, 1-\rho, 1-\rho)$
$\geq 1-\varepsilon$

Hence h is an rs_b- distance on $\mathscr{X}$. □

Lemma 3.8. *Take $(\mathscr{X}, \mathscr{F}, \mathscr{T})$ is a Menger Ps_bMS and h is a rs_b– distance. Take $\{m_n\}$, $\{o_n\}$ and $\{p_n\}$ be sequence in $\mathscr{X}$. And take $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be a sequences in $[0,\infty)$ converging to zero and $m,o,p \in \mathscr{X}$ and $e_1, e_2, e_3 > 0$. Then the coming hold:*

(i) if $h_{m_n,o_n,v}(e_1) \geq 1-\alpha_n, h_{m_n,v,o_n}(e_2) \geq 1-\beta_n, h_{v,o_n,p_n}(e_3)$ $\geq 1-\gamma_n$ for any $n \in N$ then $\mathscr{F}_{m_n,o_n,p_n}(e_1+e_2+e_3) \to 1$

(ii) if $h_{m_n,o_n,v)(e_1} \geq 1-\alpha_n, h_{m_n,v,o_n}(e_2) \geq 1-\beta_n$ and h_{v,o_n,p_n} $(e_3) \geq 1-\gamma_n$ some $n \in N \Rightarrow m = o = p$.

(iii) if $h_{x_n,x_m,x_k}(e) \geq 1-\alpha_n$ any $n,m,k \in N$ with $k > m > n$ formerly $\{x_n\}$ be a CS.

Proof. (i) Take $\varepsilon > 0$. From the definition of rs_b– distance, $\exists$ $\rho > 0 \ni h_{m,o,v}(e_1) \geq 1-\rho, h_{m,v,p}(e_2) \geq 1-\rho$ and $e_{v,o,p}(e_3) \geq 1-\rho$ implies $e_{m,o,p}(e_1+e_2+e_3) \geq 1-\varepsilon$.

Elect $n_0 \in N \ni \{\alpha_n\} \leq \rho$, $\{\beta_n\} \leq \rho$ and $\{\gamma_n\} \leq \rho$ $\forall$ $n \geq n_0$. Then we've, any $n \geq n_0, h_{m,o,v}(e_1) \geq 1-\alpha_n \geq 1-\rho, h_{v,o,p}(e_2) \geq 1-\beta_n \geq 1-\rho$ and $h_{v,m,o}(e_3) \geq 1-\gamma_n \geq 1-\rho$ and hence $h_{m,o,p}(e_1+e_2+e_3) \geq 1-\varepsilon$. This implies that $\mathscr{F}_{m_n,o_n,p_n}(e_1+e_2+e_3) \to 1$. Thence we've that $\{m_n\}$ converges to x. It follows from (i) that (ii) hold.

(iii) Take $\varepsilon > 0$. By (i), Elect $\rho > 0$ and $n_0 \in N$. For any $n,m,k \geq n_0+1$, $h_{m_n,o_n,p_{n_0}}(e_1) \geq 1-\alpha_{n_0} \geq 1-\rho, h_{m_n,m_{n_0},o_n}(e_2) \geq 1-\alpha_{n_0} \geq 1-\rho$ and $h_{m_n,o_n,m_{n_0}}(e_3) \geq 1-\alpha_{n_0} \geq 1-\rho$ and thence $\mathscr{F}_{x_n,x_m,x_k}(e_1+e_2+e_3) \geq 1-\varepsilon$. $\Rightarrow \{m_n\}$ is a CS.

□

Theorem 3.9. *Take $(\mathscr{X}, \mathscr{F}, \mathscr{T})$ is a complete Menger Ps_bMS, h is a rs_b– distance and a mapping from $\mathscr{X}$ into itself. Assume that $\exists$ $k \in]0,1[\ni h_{\mathscr{A}m,\mathscr{A}^2m,\mathscr{A}^3m}(e) \geq h_{m,\mathscr{A}m,\mathscr{A}^2m}(\frac{e}{k})$, $\forall$ $m \in \mathscr{X}, e > 0 \ni sup\{\mathscr{T}(h_{m,o,p}(e), h_{m,\mathscr{A}m}(e)) : m \in \mathscr{X}\} < 1$ $\forall$ $p,o \in \mathscr{X}$ with $o \neq \mathscr{A}o, p \neq \mathscr{A}p$. Then we've*

(i) If t-norm holds and $\exists$ a $v \in \mathscr{X}$ with $\mathscr{E}_h(v, \mathscr{A}v, \mathscr{B}v) = \sup\{(\mathscr{E}_\gamma, h)(v, \mathscr{A}v, \mathscr{B}v) : \gamma \in]0,1[< \infty\}$ then $\exists$ $p \in \mathscr{X}$ $\ni p = \mathscr{A}p$.

(ii) If s-norm holds then $\exists$ $p \in \mathscr{X} \ni p = \mathscr{A}p$. Furthermore if $r = \mathscr{A}r, q = \mathscr{B}q$ and $h \in D^+$ then $h_{r,r,r} =_0$

Proof. (i) Take $v \in \mathscr{X}$ is $\ni \mathscr{E}_h(v, \mathscr{A}v) < \infty$. Characterize $v_n = \mathscr{A}^n$ u any $n \in N$. Then we've, for some $n \in N$,
$h_{v_n,v_{n+1},v_{n+2}}(e) \geq h_{v_{n-1},v_n,v_{n+1}}(\frac{e}{k^n})$
...
$\geq h_{v,v_1,v_2}(\frac{e}{k^n})$
$\therefore \mathscr{E}_{\sigma,h}(v_n, v_{n+1}, v_{n+2}) = inf\{t > 0 : h_{v_n,v_{n+1},v_{n+2}}(e) > 1-\sigma\}$
$\leq inf\{e > 0 : h_{v,v_1,v_2}(\frac{e}{k^n}) > 1-\sigma\}$
$= k^n \mathscr{E}_{\sigma,h}(v, v_1, v_2)$
hence $m > n$ and $\sigma \in [0,1]$ $\exists$ $\gamma \in [0,1] \ni$
$\mathscr{E}_{\sigma,h}(v_n, v_m, v_k) \leq \mathscr{E}_{\gamma,h}(v_n, v_{n+1}) + ... + \mathscr{E}_{h(v_{m-1},v_m)} + ... + \mathscr{E}_{h(v_k,v_{k+1})}$
$\leq \frac{k^n}{1-k} \mathscr{E}_{\gamma,h}(v, v_1, v_2)$

Then $\exists$ $n_0 \in N \ni \forall$ $n > n_0$ we've $\mathscr{E}_{\sigma,h)(v_n,v_m,v_k} \to 0$ and hence $\{v_n\}$ is a CS.

Also for any sequence $\{m_n\}$ is a CS w.r.to h iff it is a CS with $\mathscr{E}_{\sigma,h}$.

$\therefore$ $\{\rho_n\} \to 0 \ni$ for $n \geq max\{n_0, n_1\}, n_1 \in N$, we've $h_{v_n,v_m,v_k}(e) \geq 1-\rho_n$.

By the reason of $\mathscr{X}$ is complete, then $\{v_n\} \to$ some point $p \in \mathscr{X}$.

Thence by definition of Menger probabilistic s_b-normed space, we've $h_{v_n,v_{m,p}} = lim_{\mathscr{T}}(k \to \infty) h_{v_n,v_m,v_k} \geq 1-\rho_n$

and $h_{v_n,v_{n+1},v_{n+2}} \geq 1-\rho_n$. Assume that $p \neq \mathscr{A}_p$.
By statement, we've

$1 \geq sup\{\mathscr{T}(h_{m,o,v}(e), h_{m,v,p}(e), h_{v,o,p}(e)) : m \in \mathscr{X}\}$
$\geq sup\{\mathscr{T}(h_{m,o,v}(e), h_{\mathscr{A}m,v,\mathscr{A}p}(e), h_{v,\mathscr{A}o,\mathscr{A}p}(e)) : m \in \mathscr{M}\}$
$\geq sup\{\mathscr{T}(h_{\mathscr{A}m,\mathscr{A}o,v}(e), h_{\mathscr{A}m,v,\mathscr{A}p}(e), h_{v,\mathscr{A}o,\mathscr{A}p}(e)) : m \in \mathscr{X}\}$
$\geq sup\{\mathscr{T}(h_{v_n,\mathscr{A}o,v}(e), h_{v_n,v,v_{n+2}}(e), h_{v,v_{n+1},v_{n+2}}(e)) : n \in N\}$
$\geq sup\{\mathscr{T}(1-\rho_n, 1-\rho_n, 1-\rho_n : n \in N)\} = 1$,
Which is inconsistency. $\therefore$ we've $p = \mathscr{A}p$.

(ii) The proof is same as (i) but σ does not depend on k.

Presently if $v = \mathscr{A}v$ and $h \in D^+$ then we've
$h_{r,r,r}(e) = h_{\mathscr{A}r,\mathscr{A}^2r,\mathscr{A}^3r}(e)$
$\geq h_{r,\mathscr{A}r,\mathscr{A}^2r}(\frac{e}{k})$
$= h_{r,r,r}(\frac{e}{k})$
Enduring this process, we've
$h_{r,r,r}(e) = h_{r,r,r}(\frac{e}{k^n})$
. Also we've $e_{r,r,r} = \varepsilon_0$.

□

Theorem 3.10. *Take $(\mathscr{X}, \mathscr{F}, \mathscr{T})$ is a complete Menger Ps_b MS and $\mathscr{A}$ is a mapping from $\mathscr{X}$ into itself. Assume $\exists\, \beta \in]0,1[\, \ni$*
$\mathscr{F}_{\mathscr{A}m,\mathscr{M}o,\mathscr{A}p}(e) \geq \mathscr{S}(\mathscr{F}_{m,\mathscr{A}m,\mathscr{A}^2m}(\frac{e}{\beta}), \mathscr{F}_{o,\mathscr{A}o,\mathscr{A}^2o}(\frac{e}{\beta}), \mathscr{F}_{p,\mathscr{A}p,\mathscr{A}^2p}(\frac{e}{\beta}))\ \forall\, m,o,p \in \mathscr{X}$ *and* $e > 0$.

(i) If t-norm holds and $\exists$ a $u \in X$ with $\mathscr{E}(v, \mathscr{A}v, \mathscr{A}^2v) < \infty$ then $\mathscr{A}$ has a unique fd-pt..

(ii) If s-norm holds then $\mathscr{A}$ has a unique fd-pt.

Proof. (i) Take $m \in \mathscr{X}$. From the difference (I), we've $\mathscr{F}_{\mathscr{A}m,\mathscr{A}^2m,\mathscr{A}^3x}(e) \geq \mathscr{S}(\mathscr{F}_{m,\mathscr{A}m,\mathscr{A}^2m}(\frac{e}{\beta})), \mathscr{F}_{\mathscr{A}m,\mathscr{A}^2m,\mathscr{A}^3m}(\frac{e}{\beta}), \mathscr{F}_{\mathscr{A}^2m,\mathscr{A}^3m,\mathscr{A}^4m}(\frac{e}{\beta})$ and hence $\mathscr{F}_{\mathscr{A}m,\mathscr{A}^2m,\mathscr{A}^3m}(e) \geq \mathscr{F}_{m,\mathscr{A}m,\mathscr{A}^2m}(\frac{t}{\beta})$ By the reason of the probabilistic metric $\mathscr{F}$ is an r_{s_b}-distance, assume that $\exists\, o \in \mathscr{X}$ with $o \neq \mathscr{A}o$ and $sup\{\mathscr{F}_{m,o,p}(e) : \mathscr{F}_{m,\mathscr{A}m,\mathscr{A}^2m}(e) : m \in \mathscr{X}\} = 1$.

By the reason $\mathscr{F}_{m_n,o,\mathscr{A}o}(e) \to 1$ and $\mathscr{F}_{m_n,\mathscr{A}m_n,\mathscr{A}^2_{m_n}}(e) \to 1$, then by lemma (3.9), we've $\{\mathscr{A}m_n\} \to o$.

another way, by the reason of $\mathscr{A}$ fulfills the condition (I) then we've, $\mathscr{F}_{\mathscr{A}m_n,\mathscr{A}^2m_n,\mathscr{A}o}(e) \geq \mathscr{S}(\mathscr{F}_{m_n,\mathscr{A}m_n,\mathscr{A}^2m_n}(\frac{e}{\beta}), \mathscr{F}_{o,\mathscr{A}o,\mathscr{A}^2o}(\frac{e}{\beta})) \to 1$ as $n \to \infty$ i.e $o = \mathscr{A}o$. inconsistency. Thence if $o \neq \mathscr{A}o$ then $sup\{\mathscr{A}_{m,o,p}(e) : \mathscr{A}_{m,\mathscr{A}m,\mathscr{A}^2m}(e) : m \in \mathscr{X}\} < 1$.

Then by theorem (3.10), $\exists\, p \in \mathscr{X} \ni p = \mathscr{A}p$. By the reason of $\mathscr{F} \in D^+$ then the uniqueness is trivial.

(ii) The proof is same as (i).

□

4. rs_b-distance with Property $\mathscr{C}$ and Weakly Commuting maps in Ps_bM-Space

Definition 4.1. *State rs_b- distance h has property $(\mathscr{C}')$ if it fulfills the coming condition: $h_{m,o,p}(e) = \mathscr{C}'\ \forall\, t > 0 \Rightarrow \mathscr{C}' = 1$.*

Theorem 4.2. *Take $(\mathscr{X}, \mathscr{F}, \mathscr{T})$ is a complete Menger Ps_bM-space, h is r-distance on it and $\mathscr{A}, \mathscr{B}, \mathscr{C} : \mathscr{X} \to X$ be maps that fulfill the coming properties:*

(i) $\mathscr{C}(\mathscr{X}) \subseteq \mathscr{B}(\mathscr{X}) \subseteq \mathscr{A}(\mathscr{X})$.

(ii) $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ are continuous

(iii) $h_{\mathscr{C}(m),\mathscr{C}(o)}(e) \geq h_{\mathscr{F}(m),\mathscr{B}(o)}(e) \geq h_{\mathscr{A}(m),\mathscr{A}(o)}(\frac{e}{k})\ \forall\, m,o \in \mathscr{X}$, $t > 0$, $0 < k < 1$.

Suppose $m \in \mathscr{X}$
$\mathscr{E}_h(\mathscr{A}(m), \mathscr{B}(m), \mathscr{C}(m)) + \mathscr{E}_h(\mathscr{A}(m), \mathscr{B}(m), p) + \mathscr{E}_h(\mathscr{B}(m), \mathscr{C}(m), p) + \mathscr{E}_h(\mathscr{B}(m), \mathscr{B}(\mathscr{B}(m)), p) + \mathscr{E}_h(\mathscr{C}(m), \mathscr{C}(\mathscr{C}(m)), p) + \mathscr{E}_h(\mathscr{C}(m), \mathscr{B}(m), \mathscr{A}(m)) < \infty$, $\forall\, p \in \mathscr{X}$ with $\mathscr{B}(\mathscr{Z}) \neq \mathscr{B}(\mathscr{B}(p)))$ and $\mathscr{C}(p) \neq \mathscr{C}(\mathscr{C}(p))$ here $\mathscr{E}_h(z,y,x) = sup\{\mathscr{E}_{\gamma,h}(z,y,x) : \gamma \in (0,1)\}$.
Also suppose if $\{m_n\}$ is a sequence in $\mathscr{X}$ with $\lim_{\mathscr{T}}(n \to \infty)\mathscr{A}(m_n) = o \in \mathscr{X}$, then $\forall \eta \in (0,1)$, we have $\mathscr{E}_{\eta,h}(\mathscr{A}(m_n), \mathscr{A}(m_s), o) \leq \lim_{s,u\to\infty} \mathscr{E}\eta, h(\mathscr{A}(m_n), \mathscr{A}(m_p), \mathscr{A}(m_u))$.
In addition,

(i) If t-norm holds and $\exists$ a m_0 with $\mathscr{E}_h(\mathscr{A}(m_0), \mathscr{B}(m_0), \mathscr{C}(m_0)) = \sup\{\mathscr{E}_{\gamma,h}(\mathscr{A}(m_0), \mathscr{B}(m_0), \mathscr{C}(m_0)) : \gamma \in (0,1)\} < \infty$ and $\mathscr{E}_h(\mathscr{B}(m_0), \mathscr{B}(\mathscr{B}(m_0)), \mathscr{C}(\mathscr{C}(m_0))) = sup\{\mathscr{E}_{\gamma,h}(\mathscr{B}(m_0), \mathscr{B}(\mathscr{B}(m_0)), \mathscr{C}(\mathscr{C}(m_0))) : \gamma \in (0,1)\} < \infty$ then $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ have a Common fixed point given that $\mathscr{A}, \mathscr{B}$ and C commute each other.

(ii) If s-norm holds then A,B,C have a common fixed point given that $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ commute one another. In addition if h has the property $\mathscr{C}'$, $h(.)$ is non decreasing and $\mathscr{B}(r) = \mathscr{B}(\mathscr{B}(r))$, $\forall\, r \in \mathscr{X}$ then $h_{\mathscr{B}(r),\mathscr{B}(r)}(r) = 1$ and $\mathscr{C}(w) = \mathscr{C}(\mathscr{C}(w))\ \forall\, w \in X$ and $h_{\mathscr{C}(w),\mathscr{C}(w)}(e) = 1$.

Proof. (i) First $\forall\, m \in \mathscr{X}$, $\inf\{\mathscr{E}_h(\mathscr{A}(m), \mathscr{B}(m), \mathscr{C}(m)) + \mathscr{E}_h(\mathscr{A}(m), \mathscr{B}(m), p) + \mathscr{E}_h(\mathscr{B}(m), \mathscr{C}(m), p) + \mathscr{E}_h(\mathscr{B}(m), \mathscr{B}(\mathscr{B}(m)), p) + \mathscr{E}_h(\mathscr{C}(m), \mathscr{C}(\mathscr{C}(m)), p) + \mathscr{E}_h(\mathscr{C}(m), \mathscr{B}(m), \mathscr{A}(m))\} > 0$, $\forall\, p \in \mathscr{X}$ with $\mathscr{B}(p) \neq \mathscr{B}(\mathscr{B}(p))$ and $\mathscr{C}(p) \neq \mathscr{C}(\mathscr{C}(p))$.

Assume this is true. For that, let $m_0 \in \mathscr{X}$ with $\mathscr{E}_h(\mathscr{A}(m_0), \mathscr{B}(m_0), \mathscr{C}(m_0))$
$< \infty$, $\mathscr{E}_h(\mathscr{B}(m_0), \mathscr{B}(\mathscr{B}(m_0)), p) < \infty$ and $\mathscr{E}_h(\mathscr{C}(m_0), \mathscr{C}(\mathscr{C}(m_0)), p) < \infty$.

But (i), we find $m_1, m_2 \ni \mathscr{A}(m_1) = \mathscr{B}(m_0) = \mathscr{C}(m_2)$.
By acceptance we can characterize a sequence $\{m_n\}_n \ni \mathscr{A}(m_n) = \mathscr{B}(m_{n-1}) = \mathscr{C}(m_{n+1})$.
By acceptance again,
$h_{\mathscr{A}(m_n),\mathscr{B}(m_{n+1}),\mathscr{C}(m_{n+2})}(e) = h_{\mathscr{B}(m_{n-1}),\mathscr{B}(m_n),\mathscr{B}(m_{n+1})}(e) \geq h_{\mathscr{A}(m_{n-1}),\mathscr{A}(m_n),\mathscr{A}(m_{n+1})}(\frac{e}{k})) \geq \ldots \geq h_{\mathscr{A}(m_0),\mathscr{A}(m_1),\mathscr{A}(m_2)}(\frac{e}{k^n})$ and therefore,

$\mathscr{E}_{\gamma,h}(\mathscr{A}(m_n),\mathscr{A}(m_{n+1}),\mathscr{A}(m_{n+2}))\leq k^n\mathscr{E}_{\gamma,h}(\mathscr{A}(m_0),\mathscr{A}(m_1),\mathscr{A}(m_2))$, for $n=1,2,...\Rightarrow s>n$ and for $\eta\in]0,1[$ $\exists\,\sigma\in]0,1[\ni\mathscr{E}_{\eta,h}(\mathscr{A}(m_n),\mathscr{A}(m_s),\mathscr{A}(m_u))\leq\mathscr{E}_{\sigma,h}(\mathscr{A}(m_{s-1}),\mathscr{A}(m_s),\mathscr{A}(m_{s+1}))+\mathscr{E}_{\gamma,h}(\mathscr{A}(m_{s-2}),\mathscr{A}(m_{s-1}),\mathscr{A}(m_s))+...+\mathscr{E}_{\sigma,h}(\mathscr{A}(m_n),\mathscr{A}(m_{n+1}),\mathscr{A}(m_{n+2}))\leq\mathscr{E}_h(\mathscr{A}(m_0),\mathscr{A}(m_1),\mathscr{A}(m_2))\sum_{j=n}^{s-1}k^j\leq(\frac{k^n}{1-k})\mathscr{E}_h(\mathscr{A}(m_0),\mathscr{A}(m_1),\mathscr{A}(m_2))$.

Thence $\{\mathscr{A}(m_n)\}$ is a CS. By the reason of $\mathscr{X}$ is complete then $\exists\ o\in\mathscr{X}\ni lim_{\mathscr{F}}(n\to\infty)\mathscr{A}(m_n)=o$ If $\mathscr{B}(m_{n-1})=\mathscr{A}(m_n)\to o$ then $\{\mathscr{B}(\mathscr{A}(m_n))\}_n\to\mathscr{B}(o)$.

Be that as it may $\mathscr{B}(\mathscr{A}(m_n))=\mathscr{A}(\mathscr{B}(m_n)),\mathscr{C}(\mathscr{B}(m_n))=\mathscr{B}(\mathscr{C}(m_n))$ and $\mathscr{A}(\mathscr{C}(m_n))=\mathscr{C}(\mathscr{A}(m_n))$, by the commutative condition and so $\mathscr{A}(\lfloor(m_n)),\mathscr{B}(\mathscr{C}(m_n))$ and $\mathscr{C}(\mathscr{A}(m_n))\to\mathscr{A}(o)$. By the reason the limits are unique, $\mathscr{A}(o)=\mathscr{B}(o)=\mathscr{C}(o)$ and so $\mathscr{A}(\mathscr{A}(o))=\mathscr{A}(\mathscr{B}(o))=\mathscr{A}(\mathscr{C}(o))$.

Differently, we've
$\mathscr{E}_{\eta,h}(\mathscr{A}(m_n)),\mathscr{A}(m_s,o)\leq\lim_{s,u\to\infty}\mathscr{E}\eta,h(\mathscr{A}(m-n),\mathscr{A}(m_s,\mathscr{A}(m_u))\leq\mathscr{E}_{\eta,h}(\mathscr{A}(m_n),\mathscr{A}(m_s),o)\leq$
Since $\forall\ \eta\in]0,1[$ then we've $\mathscr{E}_h(\mathscr{A}(m_n),\mathscr{A}(m_s),o)\leq\frac{k^n}{1-k}\mathscr{E}_h(\mathscr{A}(m_0),\mathscr{A}(m_1),\mathscr{A}(m_2))$. Also, by the reason $\mathscr{B}(m_n)=\mathscr{A}(m_{n+1})=\mathscr{C}(m_{n+2})$ then we've $\mathscr{E}_h(\mathscr{B}(m_n),\mathscr{B}(m_s),o)\leq\frac{k^{n+1}}{1-k}\mathscr{E}_h(\mathscr{A}(m_0),\mathscr{A}(m_1),\mathscr{A}(m_2))$ and $\mathscr{E}_h(\mathscr{C}(m_n),\mathscr{C}(m_s),o)\leq\frac{k^{n+2}}{1-k}\mathscr{E}_h(\mathscr{A}(m_0),\mathscr{A}(m_1),\mathscr{A}(m_2))$ and
$h_{\mathscr{B}(m_n),\mathscr{B}(\mathscr{B}(m_n)),\mathscr{C}(m_n)}(e)\geq h_{\mathscr{A}(m_n),\mathscr{A}(\mathscr{E}B(m_n)),\mathscr{C}(m_n)}(\frac{e}{k})$
$=h_{\mathscr{B}(m_{n-1}),\mathscr{B}(\mathscr{B}(m_{n-1})),\mathscr{C}(m_{n-1})}(\frac{e}{k})$
$\geq h_{\mathscr{A}(m_{n-1}),\mathscr{A}(\mathscr{B}(m_{n-1})),\mathscr{C}(m_{n-1})}(\frac{e}{k^2})$
$=h_{\mathscr{B}(m_{n-2}),\mathscr{B}(\mathscr{B}(m_{n-2})),\mathscr{E}(m_{n-1})}(\frac{e}{k^2})$
$=h_{\mathscr{B}(m_{n-2}),\mathscr{B}(\mathscr{B}(m_{n-2})),\mathscr{C}(m_{n-1})}(\frac{e}{k^2})$
$\geq...\geq h_{\mathscr{A}(m_1),\mathscr{B}(\mathscr{A}(m_1)),\mathscr{C}(m_1)}((\frac{e}{k^n})$
$\Rightarrow\mathscr{E}_{\eta,h}(\mathscr{B}(m_n),\mathscr{B}(\mathscr{B}(m_n)),\mathscr{C}(m_n))\geq k^n\mathscr{E}_{\eta,h}(\mathscr{A}(m_1),\mathscr{B}(\mathscr{A}(m_1)),\mathscr{C}(m_1))\leq k^n\mathscr{E}_h(\mathscr{A}(m_1),\mathscr{B}(\mathscr{A}(m_1)),\mathscr{C}(m_1))$ and so, $\mathscr{E}_h(\mathscr{B}(m_n),\mathscr{B}(\mathscr{B}(\mathscr{A}(m_n)),\mathscr{C}(m_n))\leq k^n\mathscr{E}_h(\mathscr{A}(m_1),\mathscr{B}(\mathscr{A}(m_1)),\mathscr{C}(m_1))$.

Presently $\mathscr{B}(o)=\mathscr{B}(\mathscr{B}(o))$ and $\mathscr{C}(o)=\mathscr{C}(\mathscr{C}(o))$. Suppose $\mathscr{B}(o)\neq\mathscr{B}(\mathscr{B}(o))$. By above, we've $0<\inf\{\mathscr{E}_h(\mathscr{A}(m),\mathscr{B}(m),\mathscr{C}(m))+\mathscr{E}_h(\mathscr{A}(m),\mathscr{B}(m),o)+\mathscr{E}_h(\mathscr{B}(m),\mathscr{C}(m),o)+\mathscr{E}_h(\mathscr{B}(m),\mathscr{B}(\mathscr{B}(m)),o)+\mathscr{E}_h(\mathscr{C}(m),\mathscr{C}(\mathscr{C}(m)),o)+\mathscr{E}_h(\mathscr{C}(m),\mathscr{B}(m),\mathscr{A}(m)):m\in\mathscr{X}\}\leq\inf\{\mathscr{E}_h(\mathscr{A}(m_n),\mathscr{B}(m_n),\mathscr{C}(m_n))+\mathscr{E}_h(\mathscr{A}(m_n)\mathscr{B}(m_n),o)+\mathscr{E}_h(\mathscr{B}(m_n),\mathscr{C}(m_n),o)+\mathscr{E}_h(\mathscr{B}(m_n),\mathscr{B}(\mathscr{B}(m_n)),o)+\mathscr{E}_h(\mathscr{C}(m_n),\mathscr{C}(\mathscr{C}(m_n)),o)+\mathscr{E}_h(\mathscr{C}(m_n),\mathscr{B}(m_n),\mathscr{A}(m_n)):n\in N\}=\inf\{\mathscr{E}_h(\mathscr{A}(m_n),\mathscr{A}(m_{n+1}),o)+\mathscr{E}_h(\mathscr{A}(m_n),\mathscr{A}(m_s),\mathscr{A}(m_u))+\mathscr{E}_h(\mathscr{B}(m_n),\mathscr{B}(m_s),o)+\mathscr{E}_h(\mathscr{B}(m_1),\mathscr{B}(\mathscr{B}(m_1)),o)+\mathscr{E}_h(\mathscr{C}(m_n),\mathscr{C}(m_s),\mathscr{C}(m_s))+\mathscr{E}_h(\mathscr{C}(m_s),\mathscr{B}(m_u),\mathscr{A}(m_n))\}=\inf\{\mathscr{E}_h(\mathscr{A}(m_n),\mathscr{A}(m_{n+1}),o)+\mathscr{E}_h(\mathscr{A}(m_n),\mathscr{A}(m_s),\mathscr{A}(m_u))+\mathscr{E}_h(\mathscr{B}(m_n),\mathscr{B}(m_s),o)+\mathscr{E}_h(\mathscr{B}(m_1),\mathscr{B}(\mathscr{B}(m_1)),o)+\mathscr{E}_h(\mathscr{C}(m_n),\mathscr{C}(m_s),\mathscr{C}(m_u))+\mathscr{E}_h(\mathscr{C}(m_u),\mathscr{B}(m_u),\mathscr{A}(m_n))\}$
$\leq\inf\{k^n\mathscr{E}_h(\mathscr{A}(m_0),\mathscr{A}(m_1),o)+(\frac{k^n}{1-k})\mathscr{E}_h(\mathscr{A}(m_0),\mathscr{A}(m_1),\mathscr{A}(m_2))+(\frac{k^{n+1}}{1-k})\mathscr{E}_h(\mathscr{A}(m_0),\mathscr{A}(m_1),\mathscr{A}(m_2))+k^n\mathscr{E}_h(\mathscr{B}(m_1),\mathscr{B}(\mathscr{B}(m_1)),o)+(\frac{k^n}{1-k})\mathscr{E}_h(\mathscr{C}(m_0),\mathscr{C}(m_1),\mathscr{C}(m_2))+(\frac{k^{n+1}}{1-k})\mathscr{E}_f(\mathscr{C}(m_0),\mathscr{C}(m_1),\mathscr{C}(m_2)):n\in N\}=0$

This is a conflict. Thus $\mathscr{B}(o)=\mathscr{B}(\mathscr{B}(o))$ and $\mathscr{C}(o)=\mathscr{C}(\mathscr{C}(o))$. Thus $\mathscr{B}(o)=\mathscr{B}(\mathscr{B}(o)=\mathscr{A})(\mathscr{B}(o))$ and accordingly $\mathscr{B}(o)$ is a common fixed point of $\mathscr{A},\mathscr{B}$ and $\mathscr{C}$. Moreover if $\mathscr{B}(o)$ is a common fixed point of $\mathscr{A},\mathscr{B}$ and $\mathscr{C}$ $\mathscr{B}(v)=\mathscr{B}(\mathscr{B}(v))\ \forall\ v\in X$, then we have $h_{(\mathscr{B}(o),\mathscr{B}(o),\mathscr{B}(o)}(e)=h_{\mathscr{B}(\mathscr{B}(o)),\mathscr{B}(\mathscr{B}(o)),\mathscr{B}(\mathscr{B}(o))}(e)\geq h_{\mathscr{A}(\mathscr{B}(\mathscr{C}(o))),\mathscr{A}(\mathscr{B}(\mathscr{C}(o))),\mathscr{A}(\mathscr{B}(\mathscr{C}(o)))}(\frac{e}{k})\geq h_{\mathscr{A}(\mathscr{B}(o)),\mathscr{A}(\mathscr{B}(o)),\mathscr{A}(\mathscr{B}(o))}(\frac{e}{k})=h_{\mathscr{B}(o),\mathscr{B}(o),\mathscr{B}(o)}(\frac{e}{k})$,
Since $\mathscr{A}(\mathscr{B}(\mathscr{C}(o)))=\mathscr{C}(\mathscr{A}(\mathscr{B}(o)))\Rightarrow(\mathscr{B}(o))=\mathscr{C}(\mathscr{B}(o))\Rightarrow\mathscr{B}(o)=\mathscr{B}(o)$ and $\mathscr{B}(\mathscr{C}(o))=\mathscr{B}(o)$ and $\mathscr{B}(\mathscr{C}(o))=\mathscr{B}(o)$. Differently, known h is decreasing, then we've $h_{\mathscr{B}(o),\mathscr{B}(o),\mathscr{B}(o)}(e)\leq h_{\mathscr{B}(o),\mathscr{B}(o),\mathscr{B}(o)}(\frac{e}{k})$
Thus we've $h_{\mathscr{B}(o),\mathscr{B}(o),\mathscr{B}(o)}(e)=h_{\mathscr{B}(o),\mathscr{B}(o),\mathscr{B}(o)}(\frac{e}{k})\Rightarrow h_{\mathscr{B}(o),\mathscr{B}(o),\mathscr{B}(o)}(e)=\mathscr{C}'\ \forall\ e>0$. Hence by property $(\mathscr{C}')$ we've $h_{\mathscr{B}(o),\mathscr{B}(o),\mathscr{B}(o)}(e)=1$.
To prove the assert, consider that $\exists\,o\in\mathscr{X}$ with $\mathscr{B}(o)\neq\mathscr{B}(\mathscr{B}(o))$ and $\mathscr{C}(o)\neq\mathscr{C}(\mathscr{C}(o))$ and $\inf\{\mathscr{E}_h(\mathscr{A}(m),\mathscr{B}(m),\mathscr{C}(m))+\mathscr{E}_h(\mathscr{A}(m),\mathscr{B}(m),o)+\mathscr{E}_h(\mathscr{B}(m),\mathscr{C}(m),o)+\mathscr{E}_h(\mathscr{B}(m),\mathscr{B}(\mathscr{B}(m)),o)+\mathscr{E}_h(\mathscr{C}(m),\mathscr{C}(\mathscr{C}(m)),o)+\mathscr{E}_h(\mathscr{C}(m),\mathscr{B}(m),\mathscr{A}(m)):m\in\mathscr{X}\}=0$.
Then $\exists\,\{m_n\}\ni lim_{n\to\infty}\{\mathscr{E}_h(\mathscr{A}(m_n),\mathscr{B}(m_n),\mathscr{C}(m_n))+\mathscr{E}_h(\mathscr{A}(m_n),\mathscr{B}(m_n),o)+\mathscr{E}_h(\mathscr{B}(m_n),\mathscr{C}(m_n),o)+\mathscr{E}_h(\mathscr{B}(m_n),\mathscr{B}(\mathscr{B}(m_n)),o)+\mathscr{E}_h(\mathscr{C}(m_n),\mathscr{C}(\mathscr{C}(m_n)),o)+\mathscr{E}_h(\mathscr{C}(m_n),\mathscr{B}(m_n),\mathscr{A}(m_n))\}=0$. We know that,
$h_{\mathscr{A}(m_n),\mathscr{B}(m_n),\mathscr{C}(m_n)}(e)\to 1$ and $h_{\mathscr{A}(m_n),\mathscr{B}(m_n),o}(e)\to 1$ and by lemma (3.9) we've $\lim_{\mathscr{B}\to\infty}\mathscr{B}(m_n)=o$ and $\lim_{n\to\infty}\mathscr{C}(m_n)=o$.
Also $h_{\mathscr{B}(m_n),\mathscr{C}(m_n),o}(e)\to 1$, $h_{\mathscr{A}(m_n),\mathscr{B}(m_n),\mathscr{B}(\mathscr{B}(m_n))}(e)\to 1$ and $h_{\mathscr{A}(m_n),\mathscr{B}(m_n),\mathscr{C}(\mathscr{C}(m_n))}(e)\to 1$
$\therefore$ by lemma (3.9), we've
$\lim_{n\to\infty}\mathscr{B}(\mathscr{B}(m_n))=y$ and $\lim_{n\to\infty}\mathscr{C}(\mathscr{C}(m_n))=0$.
Therefore $\mathscr{B}(o)=\mathscr{B}(\mathscr{B}(o))$ and $\mathscr{C}(o)=\mathscr{C}(\mathscr{C}(o))$, which is a contradiction. Hence if $\mathscr{B}(o)\neq\mathscr{B}(\mathscr{B}(o))$ and $\mathscr{C}(o)\neq\mathscr{C}(\mathscr{C}(o))$ then $\inf\{\mathscr{E}_h(\mathscr{A}(m),\mathscr{B}(m),\mathscr{C}(m))+\mathscr{E}_h(\mathscr{A}(m),\mathscr{B}(m),o)+\mathscr{E}_h(\mathscr{B}(m),\mathscr{C}(m),o)+\mathscr{E}_h(\mathscr{B}(m),\mathscr{B}(\mathscr{B}(m)),o)+\mathscr{E}_h(\mathscr{C}(m)\mathscr{C}(\mathscr{C}(m)),o)+\mathscr{E}_h(\mathscr{C}(m),\mathscr{B}(m),\mathscr{A}(m)):m\in\mathscr{X}\}>0$.

□

Definition 4.3. *Take h and k be maps from a Menger Ps_bM-space $(\mathscr{X},\mathscr{F},\mathscr{T})$. The maps h and k are termed as be Weakly Commuting if $\mathscr{F}_{hkm,khm}(e)\geq\mathscr{F}_{hm,km}(e)$ for each m in $\mathscr{X}$ and $e>0$.*

Remark 4.4. *Consider φ denote the set of all onto and strictly non-decreasing function φ from $[0,\infty)\to[0,\infty)$ which gratify $lim_{n\to\infty}\varphi^n(e)=0$ for $e>0$. Here $\varphi^n(e)$ stands for nth iterative function of $\varphi(e)$.*

If $\varphi\in\phi$ then $\varphi(e)<t$ for $e>0$. suppose that $\exists\ e_o>0$ with $e_0\leq\varphi(e_0)$. Then since φ is non decreasing we have $e_0\leq\varphi^n(e_0)$ φ $\forall\ n\in\{1,2,...\}$ which is inconsistency. Also $\varphi(0)=0$.

Lemma 4.5. *Assume a Menger Ps_bM-space $(\mathscr{X},\mathscr{F},\mathscr{T})$ fulfills the coming condition: $\mathscr{F}_{m,o,p}(e)=\mathscr{C}\ \forall\ e>0$ then we've*

$\mathscr{C} = \varepsilon_0(e)$ *and* $m = o$.

Theorem 4.6. *Take* $(\mathscr{X}, \mathscr{F}, \mathscr{T})$ *is a complete Menger* Ps_b*-space and* h, k *and* l *be weakly commuting self mappings of* X *fulfilling the coming properties:*

(i) $h(\mathscr{X}) \subseteq k(\mathscr{X}) \subseteq l(\mathscr{X})$

(ii) h and k or l is continuous.

(iii) $\mathscr{F}_{hm,ho,hp}(\varphi(e)) \geq \mathscr{F}_{km,ko,kp}(e) \geq \mathscr{F}_{lm,lo,lp}(e)$*, here* $\varphi \in \phi$.

(a) If t-norm holds and $\exists\, m_0 \in X$ *with* $\mathscr{E}_{\mathscr{F}}(lm_0, km_0, hm_0) = sup\{\mathscr{E}_{\gamma,\mathscr{F}(lm_0,km_0,hm_0)} : \gamma \in (0,1)\} < \infty$*, thus h and k have exclusive common fixed point.*

(b) If s-norm holds then h and k have a unique common fixed point.

Proof. Elect $m_0 \in \mathscr{X}$ with $\mathscr{E}_{\mathscr{F}}(lm_0, km_0, hm_0) < \infty$. Take $m_1 \in X$ with $hm_0 = km_1 = lm_2$. In general, pick $m_{n+1}, m_{n+2} \ni hm_n = km_{n+1} = lm_{n+2}$. Presently $\mathscr{F}_{hm_n,hm_{n+1},hm_{n+2}}(\varphi^{n+1}(e)) \geq \mathscr{F}_{hm_{n-1},hm_n,hm_{n+1}}(\varphi^n(e)) \geq \mathscr{F}_{km_0,km_1,km_2}(e) \geq \mathscr{E}_{lm_0,lm_1,lm_2}(e)$. every $\sigma \in (0,1)$, $\mathscr{F}_{\sigma,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+2}) = \inf\{\varphi^{n+1}(e) > 0 : \mathscr{F}_{hm_n,hm_{n+1},hm_{n+2}}(\varphi^{n+1}(e)) > 1-\sigma\}$
$\leq inf\{\varphi^{n+1}(e) > 0 : \mathscr{F}_{lm_0,km_0,hm_0}(e) > 1-\sigma\}$
$\leq \varphi^{n+1}(inf\{e > 0 : \mathscr{F}_{lm_0,km_0,hm_0}(e) > 1-\sigma\})$
$= \varphi^{n+1}(\mathscr{E}_{\sigma,\mathscr{F}}(lm_0, km_0, hm_0))$
$\leq \varphi^{n+1}(\mathscr{E}_{\mathscr{F}}(lm_0, km_0, hm_0))$
Hence $\mathscr{F}_{\sigma,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+2}) \leq \varphi^{n+1}(\mathscr{E}_{\mathscr{F}}(lm_0, km_0, hm_0))$ Take $\varepsilon > 0$ and $n \in \{1,2,3,\ldots\}$ so $\mathscr{F}_{\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+2}) < \varepsilon - 2\varphi(\varepsilon)$. For $\sigma \in (0,1)$, $\exists\, \eta \in (0,1)$ with $\mathscr{E}_{\sigma,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+3}) \leq \mathscr{E}_{\eta,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+1}) + \mathscr{E}_{\eta,\mathscr{F}}(hm_{n+1}, hm_{n+2}, hm_{n+2}) + \mathscr{E}_{\eta,\mathscr{F}}(hm_{n+2}, hm_{n+3}, hm_{n+3})$
$\leq \mathscr{E}_{\eta,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+1})$
$+ \varphi(\mathscr{E}_{\eta,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+1})) + \varphi(\mathscr{E}_{\eta,\mathscr{F}}(hm_{n+1}, hm_{n+1}, hm_{n+2})) \leq \mathscr{E}_{\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+1}) + \varphi(\mathscr{E}_{\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+1})) + \varphi(\mathscr{E}_{\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+2}))$
$\leq [\varepsilon - 2\varphi(\varepsilon)] + \varphi(\varepsilon) + \varphi(\varepsilon)$
$\leq \varepsilon$
Then $\mathscr{E}_{\sigma,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+3}) \leq \varepsilon$. For $\sigma \in (0,1)$, $\exists\, \eta \in (0,1)$ with $\mathscr{E}_{\sigma,\mathscr{F}}(hm_n, hm_{n+2}, hm_{n+4}) \leq \mathscr{E}_{\eta,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+2}) + \mathscr{E}_{\eta,\mathscr{F}}(hm_{n+1}, hm_{n+2}, hm_{n+3}) + \mathscr{E}_{\eta,\mathscr{F}}(hm_{n+2}, hm_{n+3}, hm_{n+4}) \leq \mathscr{E}_{\eta,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+2}) + \varphi(\mathscr{E}_{\eta,\mathscr{F}}(hm_n, hm_{n+1}, hm_{n+2})) + \mathscr{E}_{\eta,\mathscr{F}}(hm_{n+2}, hm_{n+3}, hm_{n+4}) \leq (\varepsilon - 2\varphi(\varepsilon)) + \varphi(\varepsilon - \varphi(\varepsilon)) + \varphi(\mathscr{E}_{\eta,\mathscr{F}}(hm_{n+1}, hm_{n+2}, hm_{n+3})$
$\leq (\varepsilon - 2\varphi(\varepsilon)) + \varphi(\varepsilon - \varphi(\varepsilon)) + \varphi(\varepsilon - \varphi(\varepsilon))$
$\leq \varepsilon$.
Similarly for each $\sigma \in (0,1)$, we've $\mathscr{E}_{\sigma,\mathscr{F}}(hm_n, hm_{n+2}, hm_{n+4}) \leq \varepsilon$.
Note that $\mathscr{F}_(hm_{n+1}, hm_{n+2}, hm_{n+3})\varepsilon(e) \leq \mathscr{F}_{km_{n+1},km_{n+2},km_{n+3}}(e) = \mathscr{F}_{hm_n,hm_{n+2},hm_{n+4}}$
$\Rightarrow \mathscr{E}_{\sigma,\mathscr{E}}(hm_{n+1}, hm_{n+2}, hm_{n+3}) \leq \varphi(\mathscr{E}_{\eta,\mathscr{F}}(hm_n, hm_{n+2}, hm_{n+4}))$. Therefore, $\mathscr{E}_{\mathscr{F}}(hm_n, hm_{n+2}, hm_{n+4}) \leq \varepsilon$.
By induction, $\mathscr{E}_{\mathscr{F}}(hm_n, hm_{n+k}, hm_{n+k+2}) \leq \varepsilon$ for $k \in \{1,2,3,.\}$ Therefore $\{hm_n\}$ is a CS and by the reason $\mathscr{X}$ is complete, $\{hm_n\}$ converges to r in $\mathscr{X}$. Also $\{km_n\}$ and $\{lm_n\}$ converges to p. let us now presume that the mapping h is continuous. Then $\lim_n hm_n = hp$, $\lim_n km_n = fz$ and $\lim_n hlm_n = hp$. Since h, k and l are weakly commuting each other, we've,
$\mathscr{F}_{hkm_n,khm_n}(e) \geq \mathscr{F}_{hm_n,km_n}(e), \mathscr{F}_{hlm_n,lhm_n}(e) \geq \mathscr{F}_{hm_n,lm_n}(e)$ and $\mathscr{F}_{klm_n,lkm_n}(e) \geq \mathscr{F}_{km_n,lm_n}(e)$.
Take $n \to \infty$ in the above disparity and $\lim_{n\to\infty} khm_n = hp$ and $\lim_{n\to\infty} klm_n = hp$ continuity of h.
Presently prove $p = hp$. Consider $p \neq hp$. By (iii) some $e > 0$, we've $\mathscr{F}_{hm_n,hhm_n,hp_n}(\varphi^{k+1}(e)) \geq \mathscr{F}_{km_n,khm_n,kp_n}(\varphi^k(e)) \geq \mathscr{F}_{lm_n,lhm_n,lp_n}(e)$
$\Rightarrow \mathscr{F}_{p,hp,p}(\varphi^{k+1}(e)) \geq \mathscr{F}_{p,kp,p}(\varphi^k(e)) \geq \mathscr{F}_{p,hp,p}(e)$
Also we've $\mathscr{F}_{p,hp,p}(\varphi^k(e)) \geq \mathscr{F}_{p,hp,p}(\varphi^{k-1}(e))$ and $\mathscr{F}_{p,hp,p}(\varphi(e)) \geq \mathscr{F}_{p,hp,p}(e)$
Thus we've $\mathscr{F}_{p,hp,p}(\varphi^{k+1}(e)) \geq \mathscr{F}_{p,hp,p}(e)$
Differently, $\mathscr{F}_{p,hp,p}(\varphi^{k+1}(e)) \leq \mathscr{F}_{p,hp,p}(t)$.
Then $\mathscr{F}_{p,hp,p}(e) = \mathscr{C}$ and by lemma (2.3.5) $p = hp$. Since $h(\mathscr{X}) \subseteq k(\mathscr{X}) \subseteq l(\mathscr{X})$ Thus the locate $p_1, p_2 \in \mathscr{X} \ni p = hp = hp_1 = hp_2 = kp_1 = lp_2$.

Presently $\mathscr{F}_{hhm_n,hp_1,hp_2}(e) \geq \mathscr{F}_{khm_n,kp_1,lp_2}(varphi^{-1}(e))$.
Taking limit as $n \to \infty$, we've,
$\mathscr{F}_{hp,hp_1,hp_2}(e) \geq \mathscr{F}_{p,kp_1,lp_2}(\varphi^{-1}(e)) \Rightarrow \mathscr{F}_{hp,p,p}(e) \geq \mathscr{F}_{p,hp,p}(\varphi^{-1}(e)) = \varepsilon_0(e), \Rightarrow hp = hp_1$.
That is $p = hp = hp_1 = kp_1 = lp_2 = lp_1$.
Also $e > 0$, known h, k and l are weakly commuting each other, we've $\mathscr{F}_{hp,kp,lp}(e) = \mathscr{F}_{hkp_1,klp_1,hlp_1}(e) \geq \mathscr{F}_{hp_1,kp_1,lp_1}(e) = \varepsilon_0(e)$

Thence $hp = kp = lp$. Thus p is a common fixed point of h, k and l. To prove the uniqueness, suppose $p_1 \neq p_2 \neq p$ is another common fixed point of h, k and l. Then some $e > 0$ and $n \in N$ we've
$\mathscr{F}_{z,z_1,z_2}(\varphi^{n+1}(t)) = F_{fz,fz_1,fz_2}(\varphi^{n+1}(t))$
$\geq F_{gz,gz_1,gz_2}(\varphi^n(t)) = F_{hz,hz_1,hz_2}(\varphi^n(t))$
$= F_{z,z_1,z_2}(\varphi^n(t))$ Also we have $\mathscr{F}_{p,p_1,p_2}(\varphi^n(e)) \geq \mathscr{F}_{p,p_1,p_2}(\varphi^{n-1}(e))$ and $\mathscr{F}_{p,p_1,p_2}(\varphi^n(e)) \geq \mathscr{F}_{p,p_1,p_2}(e)$.

Thence we've $\mathscr{F}_{p,p_1,p_2}(\varphi^{n+1}(e)) \geq \mathscr{F}_{p,p_1,p_2}(e)$.

Differently, we've $\mathscr{F}_{p,p_1,p_2}(e) \leq \mathscr{F}_{p,p_1,p_2}(\varphi^{n+1}(e))$

Then $\mathscr{F}_{p,p_1,p_2}(e) = \mathscr{C}$ and by lemma (4.5) $p = p_1 = p_2$ which is a conflict.

Thus p is the unique common fixed point of h, k and l. □

5. Conclusion

Main consequence of this work is,

(i) r-distance in Menger PMS can be extended to rs_b- distance in Menger probabilistic s_b-metric spaces.

(ii) A few fixed point theorems were proved in complete Menger Ps_bMS.

(iii) Also some statements were proved in both rs_b-distance with property $\mathscr{C}$ and weakly commuting maps.

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