Malaya Journal of Matematik, Vol. 8, No. 3, 1132-1137, 2020
https://doi.org/10.26637/MJM0803/0068

Some results on weaker class of compatible mappings in S-metric space

V. Srinivas[1]* and K. Mallaiah[2]

Abstract
The theme of this paper is to establish two common fixed point theorems in S- metric space using the weaker class of compatible mappings such as semi compatible, weakly reciprocally continuous (WRC), weakly compatible mappings and occasionally weakly compatible (OWC) mappings. Further some examples are discussed on these concepts to strengthen our results.

Keywords
Common Fixed point, S-metric space, semi compatible, WRC, OWC and weakly compatible mappings.

AMS Subject Classification
54H25.

[1] *Department of Mathematics, University College of Science, Saifabad, OU, Hyderabad. T.S, India.*
[2] *Department of Mathematics, JN Government polytechnic, Ramanthapur, Hyderabad. T.S, India.*
***Corresponding author**: [1] srinivasmaths4141@gmail.com.; [2] kamanilayam95@gmail.com
Article History: Received **14** March **2020**; Accepted **28** June **2020**
©2020 MJM.

Contents

1. Introduction

The theory of metric space has been developed in many fields of Mathematics. One of the generalizations of metric space is S-metric space. In 2012,Sedhi.S, Shobe. N, and Aliouche.A[1] developed the notion of S-metric space and proved some theorems. Sharma and Sahu[4] introduced the notion of semi compatible mappings and established various fixed point theorems in metric space .On the other hand Pant used the concept of reciprocal continuity which is weaker than continuity. Junck[7] gave a generalized concept of compatibility with the introduction of weakly compatible mappings .The concept of OWC maps in metric space has been developed by Al-Thagafi and Shahzed[8] which is the most general among all the commutativity concepts. Now in this paper we present two common fixed point theorems in S-metric space using the new contraction condition along with the weaker form of compatible mappings such as semi compatible, WRC, weakly compatible mappings and OWC mappings. These results generalize and extend some existing theorems in S-metric space [2] ,[3] ,[5] and [6]. Further some examples are also discussed to support our theorems.

2. Preliminaries

Before establishing our theorems we present some definitions and examples.

Definition 2.1. *[1] A non empty set X defined on a function $S : X^3 \to [0,\infty)$ holding the following conditions:*
(i) $S(\alpha,\beta,\gamma) \geq 0$,
(ii) $S(\alpha,\beta,\gamma) = 0$; *if and only if* $\alpha = \beta = \gamma$,
(iii) $S(\alpha,\beta,\gamma) \leq S(\alpha,\alpha,a)+S(\beta,\beta,a)+S(\gamma,\gamma,a)$
*for all $\alpha,\beta,\gamma,a \in X$. Then the pair (X,S) is called an **S-metric space**.*

Remark 2.2. ***(i)*** *In an S-metric space, we observe that*
$S(\alpha,\alpha,\beta) = S(\beta,\beta,\alpha)$
(ii)*In an S-metric space, by triangle inequality*
$S(\alpha,\alpha,\beta) = 2S(\alpha,\alpha,\gamma)+S(\beta,\beta,\gamma)$
(iii) *In an S-metric space, if there exist sequences $\{\alpha_n\}$ and $\{\beta_n\}$ such that $lim_{k\to\infty}\alpha_k = \alpha$ and $lim_{k\to\infty}\beta_k = \beta$ then $S(\alpha_k,\alpha_k,\beta_k) = S(\alpha,\alpha,\beta)$.*

Definition 2.3. *Let(X,S) be an S-metric space and $A \subset X$*
(i)*the set A is said to be **S-bounded** if there exists $r > 0$ such that $S(\alpha,\alpha,\beta) < r, \forall \alpha,\beta \in X$.*

(ii) *A sequence $\{\alpha_k\}$ in X **converges to x** if $S(\alpha_k,\alpha_k,\alpha)\to 0$ as $k\to\infty$ that is for every $\varepsilon>0$ there exists $k_0\in$ **N** such that $S(\alpha_k,\alpha_k,\alpha)<\varepsilon$, for $k\geq k_0$.*
(iii) *A sequence $\{\alpha_k\}$ in X is said to be a **cauchy sequence** if for each $\varepsilon>0$ there exists $k_0\in$ **N** such that $S(\alpha_k,\alpha_k,\alpha_l)<\varepsilon$ for all $k,l\geq k_0$.*
(iv)*A **complete** S-metric space is one in which every cauchy sequence is convergent .*

Definition 2.4. *Define G and I are two self maps of an S-metric space ,then G and I are said to be **commuting** if and only if $GI\alpha=IG\alpha$ for all $\alpha\in X$.*

Definition 2.5. *We define mappings G and I of an S-metric space as **weakly commuting** on α if $S(GI\alpha,GI\alpha,IG\alpha)\leq S(G\alpha,G\alpha,I\alpha)$ for all $\alpha\in X$.*

Definition 2.6. *We define mappings G and I of an S-metric space as **compatible** if $S(GI\alpha_k,GI\alpha_k,IG\alpha_k)=0$ as $k\to\infty$ when ever there is a sequence $\{\alpha_k\}$ in X such that $G\alpha_k=I\alpha_k=\mu$ as $k\to\infty$ for all $\mu\in X$.*

Definition 2.7. *[7] Suppose G and I are mappings of S-metric space in which $G\mu=I\mu$ for some $\mu\in X$ such that $GI\mu=IG\mu$ holds.Then G and I are known as **Weakly Compatible mappings**.*

Now we give an example in which the mappings are weakly compatible but not compatible.

Example 2.8. *Let X=$[0,\infty)$ be an S-metric space with δ_1 and δ_2 are two metrics on X and $S(\alpha,\beta,\gamma)$=$\delta_1(\alpha,\gamma)+\delta_2(\beta,\gamma)$.*
Define G and I as

$$G(\alpha)=\begin{cases}1-\alpha & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ \frac{3\alpha-1}{2} & \text{if } \frac{1}{2}<\alpha\leq 1\end{cases}\quad \text{and}$$

$$I(\alpha)=\begin{cases}3\alpha-1 & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ \frac{3\alpha+1}{4} & \text{if } \frac{1}{2}<\alpha\leq 1.\end{cases}$$

Take a sequence $\{\alpha_k\}$ as $\alpha_k=\frac{1}{2}-\frac{1}{k}$ for $k\geq 0$.
Now $G(\alpha_k)$=$G(\frac{1}{2}-\frac{1}{k})$=$1-(\frac{1}{2}+\frac{1}{k})$=$\frac{1}{2}$ and
$I(\alpha_k)=I(\frac{1}{2}-\frac{1}{k})$=$3(\frac{1}{2}-\frac{1}{k})-1$=$\frac{1}{2}$ as $k\to\infty$.
Therefore $G(\alpha_k)=I(\alpha_k)=\frac{1}{2}$ as $k\to\infty$.
Further $GI(\alpha_k)=GI(\frac{1}{2}-\frac{1}{k})$=$G(\frac{1}{2}-\frac{3}{k})=\frac{1}{2}$ and
$IG(\alpha_k)=IG(\frac{1}{2}-\frac{1}{k})$=$I(\frac{1}{2}+\frac{1}{k})=(\frac{5}{8}+\frac{3}{4k})=\frac{5}{8}$ as $k\to\infty$.
Therefore $S(GI\alpha_k,GI\alpha_k,IG\alpha_k)=S(\frac{1}{2},\frac{1}{2},\frac{5}{8})\neq 0$.
showing that the mappings G and I are not compatible.
Also $G(\frac{1}{2})=I(\frac{1}{2})=\frac{1}{2}$ and $GI(\frac{1}{2})=G(\frac{1}{2})=\frac{1}{2}$
and $IG(\frac{1}{2})=I(\frac{1}{2})=\frac{1}{2}$ implies $GI(\frac{1}{2})=IG(\frac{1}{2})$.
This gives the pair (G,I) is weakly compatible.

Definition 2.9. *[8] We define mappings G and I of an S-metric space as **OWC** if there exists a point $\mu\in X$ which is a coincidence point of G and I at which they commute.*

Now we give an example in which the mappings are OWC but not weakly compatible.

Example 2.10. *Let X=$[0,1]$ be an S-metric ,δ_1 and δ_2 are two metrics on X and $S(\alpha,\beta,\gamma)$=$\delta_1(\alpha,\gamma)+\delta_2(\beta,\gamma)$.*
Define G and I as

$$G(\alpha)=\begin{cases}\frac{1+\alpha}{5} & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ 1-\alpha & \text{if } \frac{1}{2}<\alpha\leq 1\end{cases}\quad \text{and}$$

$$I(\alpha)=\begin{cases}\frac{3\alpha+1}{5} & \text{if } 0\leq\alpha\leq\frac{1}{2}\\ \frac{2\alpha+1}{4} & \text{if } \frac{1}{2}<\alpha\leq 1.\end{cases}$$

Now $G(0)=I(0)$=$\frac{1}{5}$ and $G(\frac{1}{2})$=$I(\frac{1}{2})$=$\frac{1}{2}$.
Thus 0 and $\frac{1}{2}$ are coincidence points.
Further GI(0)=$G(\frac{1}{5})$=$\frac{6}{25}$ and IG(0)=$I(\frac{1}{5})$=$\frac{8}{25}$, which implies $GI(0)\neq IG(0)$.
But $GI(\frac{1}{2})$=$IG(\frac{1}{2})$=$(\frac{1}{2})$ which gives $GI(\frac{1}{2})$=$IG(\frac{1}{2})$.
Therefore G and I are only OWC but not weakly compatible mappings.

Definition 2.11. *[4] We define mappings G and I of an S-metric space as **semi-compatible** if $S(GI\alpha_k,GI\alpha_k,I\mu)=0$ as $k\to\infty$ when ever there is a sequence $\{\alpha_k\}$ in X such that $G\alpha_k=I\alpha_k=\mu$ as $k\to\infty$ for all $\mu\in X$.*

Definition 2.12. *[**?**] We define mappings G and I of an S-metric space as **G-WRC** if $S(GI\alpha_k,GI\alpha_k,G\mu)=0$ as $k\to\infty$ when ever there is a sequence $\{\alpha_k\}$ in X such that $G\alpha_k=I\alpha_k=\mu$ as $k\to\infty$ for all $\mu\in X$.*

Definition 2.13. *[**?**]We define mappings G and I of an S-metric space as **I-WRC** if $S(IG\alpha_k,IG\alpha_k,I\mu)=0$ as $k\to\infty$ when ever there is a sequence $\{\alpha_k\}$ in X such that $G\alpha_k=I\alpha_k=\mu$ as $k\to\infty$ for all $\mu\in X$.*

Now we give an example in which the mappings are only WRC but not compatible.

Example 2.14. *Let X=$[0,\infty)$ be an S-metric space ,δ_1 and δ_2 are two metrics on X ,$S(\alpha,\beta,\gamma)$=$\delta_1(\alpha,\gamma)+\delta_2(\beta,\gamma)$.*
Define G and I such that $G\{\alpha\}$ =$3\alpha+1$ and $I\{\alpha\}$ =$2\alpha+2$.
Take a sequence $\{\alpha_k\}$ as $\alpha_k=1+\frac{1}{k}$ for $k\geq 0$.
Now $G(\alpha_k)$=$G(1+\frac{1}{k})$=$3(1+\frac{1}{k})+1$=4 and
$I(\alpha_k)$=$I(1+\frac{1}{k})$=$2(1+\frac{1}{k})+2$=4 as $k\to\infty$.
Therefore $G(\alpha_k)=I(\alpha_k)=4=\mu$ (say) as $k\to\infty$.
Further $GI(\alpha_k)$=$GI(1+\frac{1}{k})$=$G2(1+\frac{1}{k})+2$=$G(4+\frac{2}{k})=3(4+\frac{2}{k})+1=13$ as $k\to\infty$.
Also $IG(\alpha_k)$=$IG(1+\frac{1}{k})$=$I3(1+\frac{1}{k})+1$=$I(4+\frac{3}{k})=2(4+\frac{2}{k})+2=10$ as $k\to\infty$.
This gives $S(GI\alpha_k,GI\alpha_k,IG\alpha_k)$=$S(13,13,10)\neq 0$ and this gives the pair (G,I) is not compatible. Further $G\mu$=G(4)=13 and $I\mu$=I(4)=10.
Therefore $S(GI\alpha_k,GI\alpha_k,G\mu)$=$S(13,13,13)=0$
as $k\to\infty$, this gives the pair (G,I) is G-WRC.
Also $S(IG\alpha_k,IG\alpha_k,I\mu)$=S(10,10,10)=0
as $k\to\infty$, this gives the pair (G,I) is I- WRC.

Now we proceed for our main theorems.

3. Main Results

Theorem 3.1. *Let (X,S) be a complete S-metric space and there are four mappings G, H, I and J holding the conditions*

(C-1) *$G(X) \subseteq J(X)$ and $H(X) \subseteq I(X)$*
(C-2) *$S(G\alpha, G\alpha, H\beta) \leq \lambda \ max\ \{S(I\alpha, I\alpha, J\beta), \frac{S(G\alpha,G\alpha,I\alpha)S(H\beta,H\beta,J\beta)}{S(I\alpha,I\alpha,J\beta)}, S(H\beta, H\beta, G\alpha)\}$*
for all $\alpha, \beta \in X$, where $\lambda \in (0,1)$
(C-3) *the pair(G,I) is G-WRC and semi compatible*
(C-4) *the pair (H,J) is weakly compatible on X.*
Then the above mappings will be having a unique common fixed point.

Proof. Begin with using the condition (C-1), there is a point $\alpha_0 \in X$ such that $G\alpha_0 = J\alpha_1 = \beta_0$ (say) for some $\alpha_1 \in X$.For this point α_1 then $\exists$ a point $\alpha_2 \in X$ such that $H\alpha_1 = I\alpha_2 = \beta_1$(say). Continuing this process, it is possible to construct a sequence $\{\beta_k\} \in X$ such that
β_{2k}=$G\alpha_{2k} = J\alpha_{2k+1}$ and.β_{2k+1}=$H\alpha_{2k+1} = I\alpha_{2k+2}$ for $k \geq 0$.
We now prove $\{\beta_k\}$ is a Cauchy sequence in S-metric space.
Consider $S(\beta_{2k}, \beta_{2k}, \beta_{2k+1})$=

$$S(G\alpha_{2k}, G\alpha_{2k}, H\alpha_{2k+1}) \leq \lambda max\{S(I\alpha_{2k}, I\alpha_{2k}, J\alpha_{2k+1}), \frac{S(G\alpha_{2k}, G\alpha_{2k}, I\alpha_{2k})S(H\alpha_{2k+1}, H\alpha_{2k+1}, J\alpha_{2k+1})}{S(I\alpha_{2k}, I\alpha_{2k}, J\alpha_{2k+1})}, S(H\alpha_{2k+1}, H\alpha_{2k+1}, G\alpha_{2k})\}$$

this gives

$$S(\beta_{2k}, \beta_{2k}, \beta_{2k+1}) \leq \lambda max\{S(\beta_{2k-1}, \beta_{2k-1}, \beta_{2k}), \frac{S(\beta_{2k}, \beta_{2k}, \beta_{2k-1})S(\beta_{2k+1}, \beta_{2k+1}, \beta_{2k})}{S(\beta_{2k-1}, \beta_{2k-1}, \beta_{2k})}, S(\beta_{2k}, \beta_{2k}, \beta_{2k+1})\}$$

on simplification

$$S(\beta_{2k}, \beta_{2k}, \beta_{2k+1}) \leq \lambda S(\beta_{2k-1}, \beta_{2k-1}, \beta_{2k}). \quad (3.1)$$

By Similar arguments we have

$$S(\beta_{2k-1}, \beta_{2k-1}, \beta_{2k}) \leq \lambda(S(\beta_{2k-2}, \beta_{2k-2}, \beta_{2k-1}). \quad (3.2)$$

Now from (3.1) and (3.2) we have
$S(\beta_k, \beta_k, \beta_{k-1}) \leq \lambda S(\beta_{k-1}, \beta_{k-1} for \beta_{k-2})$ for $k \geq 2$
where $0 < \lambda < 1$.
Therefore in general

$$S(\beta_k, \beta_k, \beta_{k-1}) \leq \lambda^{k-1} S(\beta_1, \beta_1, \beta_0). \quad (3.3)$$

Hence for $k > l$ on using the multiplicative triangle inequality we get

$$S(\beta_k, \beta_k, \beta_l) \leq 2S(\beta_l, \beta_l, \beta_{l+1}) + 2S(\beta_{l+1}, \beta_{l+1}, \beta_{l+2}) + 2S(\beta_{l+2}, \beta_{l+2}, \beta_{l+3}) + \cdots + 2S(\beta_{k-1}, \beta_{k-1}, \beta_k).$$

Hence from (3.3) and $0 < \lambda < 1$
$S(\beta_k, \beta_k, \beta_l) \leq 2(\lambda^l + \lambda^{l+1} + \lambda^{l+2} + \cdots + \lambda^{k-1}) S(\beta_1, \beta_1, \beta_0)$
this implies
$S(\beta_k, \beta_k, \beta_l) \leq 2\lambda^l (1 + \lambda + \lambda^2 + \cdots) S(\beta_1, \beta_1, \beta_0)$
also gives
$S(\beta_k, \beta_k, \beta_l) \leq 2\frac{\lambda^l}{1-\lambda} S(\beta_1, \beta_1, \beta_0) \to 0$ as $l \to \infty$.
This results $\{\beta_k\}$ as a cauchy sequence in S-metric space.
By the completeness of X, $\{\beta_k\}$ converges to some point in X as $k \to \infty$.
Consequently the sub sequences $\{G\alpha_{2k}\}$,$\{I\alpha_{2k}\}$,$\{J\alpha_{2k+1}\}$ and $\{H\alpha_{2k+1}\}$ of $\{\beta_k\}$ also converge to the same point $\mu \in X$.
Suppose the pair (G,I) is G- WRC then

$$lim_{k\to\infty} S(GI\alpha_{2k}, GI\alpha_{2k}, G\mu) \to 0. \quad (3.4)$$

Also the pair (G,I) is semi compatible then

$$lim_{k\to\infty} S(GI\alpha_{2k}, GI\alpha_{2k}, I\mu) \to 0. \quad (3.5)$$

From (3.4) and (3.5) we get

$$G\mu = I\mu. \quad (3.6)$$

Since $G(X) \subseteq J(X)$ implies $\exists u \in X$ such that $Ju = G\alpha_{2k}$ and since $G\alpha_{2k} \to \mu$ as $k \to \infty$.
Which implies

$$Ju = \mu. \quad (3.7)$$

Now we prove $Ju = Hu = \mu$.
Putting $\alpha = \alpha_{2k}$ and $\beta = u$ in contraction condition $(C-2)$ we have

$$S(G\alpha_{2k}, G\alpha_{2k} Hu) \leq \lambda max\{S(I\alpha_{2k}, I\alpha_{2k}, Ju), \frac{S(G\alpha_{2k}, G\alpha_{2k}, I\alpha_{2k})S(Hu, Hu, Ju)}{S(I\alpha_{2k}, I\alpha_{2k}, Ju)}, S(Hu, Hu, G\alpha_{2k})\}$$

this gives

$$S(\mu, \mu, Hu) \leq \lambda max\{S(\mu, \mu, Ju), \frac{S(\mu, \mu, \mu)S(Hu, Hu, Ju)}{S(\mu, \mu, Ju)}, S(Hu, Hu, \mu)\}$$

and this gives
$S(\mu, \mu, Hu) \leq \lambda \ \max(S(\mu, \mu, Ju), 1, S(Hu, Hu, \mu))$
which implies
$S(\mu, \mu, Hu) \leq \lambda S(\mu, \mu, Hu)$
which gives
$Hu = \mu$.
Therefore $Ju = Hu = \mu$.
Since the pair $(H.J)$ is weakly compatible and u is a coincidence point then $HJu = JHu$ which gives

$$H\mu = J\mu. \quad (3.8)$$

From (3.7) and ((3.8)

$$J\mu = H\mu = \mu. \quad (3.9)$$

Putting $\alpha=\mu$ and $\beta=\mu$ in contraction condition $(C-2)$ we have

$$S(G\mu,G\mu,J\mu) \leq \lambda max\{S(I\mu,I\mu,J\mu), \frac{S(G\mu,G\mu,I\mu)S(J\mu,J\mu,J\mu)}{S(I\mu,I\mu,J\mu)}, S(J\mu,J\mu,G\mu)\}$$

which gives

$$S(G\mu,G\mu,\mu) \leq \lambda max\{S(G\mu,G\mu,\mu), \frac{S(G\mu,G\mu G\mu)S(\mu,\mu,\mu)}{S(G\mu,G\mu,\mu)}, S(\mu,\mu,G\mu)\}$$

this gives

$$S(G\mu,G\mu,\mu) \leq \lambda max\{S(G\mu,G\mu,\mu), \frac{1}{(S(G\mu,G\mu,\mu)}, S(G\mu,G\mu,\mu)\}$$

and this gives $S(G\mu,G\mu,\mu)\leq\lambda S(G\mu,G\mu,\mu)$ which implies $G\mu=\mu$.
Therefore

$$G\mu=J\mu=\mu. \tag{3.10}$$

From (3.9) and(3.10) we get

$$G\mu=J\mu=H\mu=I\mu=\mu. \tag{3.11}$$

This implies μ is a common fixed point of G,H,I and J.
For Uniqueness Consider $\phi(\mu\neq\phi)$ is another common fixed point of G,I,H,and J.
Then $G\phi=J\phi=H\phi=I\phi=\phi$ and substitute $\alpha=\phi$ and $\beta=\mu$ in the inequality $(C-2)$ we have

$$S(G\phi,G\phi,H\mu) \leq \lambda max\{S(I\phi,I\phi,J\mu), \frac{S(G\phi,G\phi,I\mu)S(H\phi,H\phi,J\mu)}{S(I\phi,I\phi,J\mu)}, S(H\mu,H\mu,G\phi)\}$$

this gives

$$S(\phi,\phi,\mu) \leq \lambda max\{S(\phi,\phi,\mu), \frac{S(\phi,\phi,\mu)S(\phi,\phi,\mu)}{S(\phi,\phi,\mu)}, S(\mu,\mu,\phi)\}$$

and which implies
$S(\phi,\phi,\mu)\leq\lambda \max\{S(\phi,\phi,\mu),(S(\phi,\phi,\mu),(S(\phi,\phi,\mu)\}$
this gives
$S(\phi,\phi,\mu)\leq\lambda S(\phi,\phi,\mu)$
thus $\phi=\mu$.
This assures the uniqueness of the common fixed point. □

Now we support our theorem with a suitable Example.

Example 3.2. *Suppose $X=(0,\infty)$ in S-metric space, δ_1 and δ_2 are two metrics on X such that* $S(\alpha,\beta,\gamma)=\delta_1(\alpha,\gamma)+\delta_2(\beta,\gamma)$ *for all $\alpha,\beta,\gamma\in X$.*
We define self maps G,H,I and J as follows

$$G(\alpha)=H(\alpha)=\begin{cases} \alpha & if\, 0<\alpha\leq\frac{1}{2} \\ \frac{3\alpha-1}{4} & if\, \frac{1}{2}<\alpha<1 \end{cases}$$

and

$$I(\alpha)=J(\alpha)=\begin{cases} \frac{2\alpha+1}{4} & if\, 0<\alpha\leq\frac{1}{2} \\ \frac{2\alpha-1}{2} & if\, \frac{1}{2}<\alpha<1. \end{cases}$$

Now G(X)=H(X)=$(0,\frac{1}{2}]\cup(\frac{1}{8},\frac{1}{2})$ and
I(X)=J(X)= $(\frac{1}{4},\frac{1}{2}]\cup(0,\frac{1}{2})$.
This gives $G(X)\subseteq J(X)$and $H(X)\subseteq I(X)$
so that the condition$(C-1)$ is satisfied.
Now take a sequence $\{\alpha_k\}$ as $\alpha_k=1-\frac{1}{k}$ for k≥ 0.
Now $G(\alpha_k)=G(1-\frac{1}{k})=\frac{3(1-\frac{1}{k})-1}{4}=(\frac{1}{2}-\frac{3}{4k})=\frac{1}{2}$ *as* $k\to\infty$
and $I(\alpha_k)=I(1-\frac{1}{k})=\frac{2(1-\frac{1}{k})-1}{2}=(\frac{1}{2}-\frac{1}{k})=\frac{1}{2}$ *as* $k\to\infty$.
Therefore $G(\alpha_k)=I(\alpha_k)=\frac{1}{2}$ *as* $k\to\infty$.
Further $GI(\alpha_k)=GI(1-\frac{1}{k})=G(\frac{2(1-\frac{1}{k})-1}{2})=G(\frac{1}{2}-\frac{1}{k})=(\frac{1}{2}-\frac{1}{k})=\frac{1}{2}$ *as* $k\to\infty$
and $IG(\alpha_k)=IG(1-\frac{1}{k})=I(\frac{3(1-\frac{1}{k})-1}{4})=I(\frac{1}{2}-\frac{3}{4k})=\frac{2(\frac{1}{2}+\frac{3}{4k})-1}{2}=0$ *as* $k\to\infty$.
Thus $S(GI\alpha_k,GI\alpha_k,IG\alpha_k)=S(\frac{1}{2},\frac{1}{2},0)\neq 0$ *and showing that the pair (G,I) is not compatible.*
Further $G(\frac{1}{2})=H(\frac{1}{2})=\frac{1}{2}$ *and* $I(\frac{1}{2})=J(\frac{1}{2})=\frac{1}{2}$ *this gives* $H(\frac{1}{2})=J(\frac{1}{2})$.
This assures $HJ(\frac{1}{2})=H(\frac{1}{2})=\frac{1}{2}$ *and* $JH(\frac{1}{2})=J(\frac{1}{2})=\frac{1}{2}$ *and this gives* $HJ(\frac{1}{2})=JH(\frac{1}{2})$.
Showing that the pair (H,J) is weakly compatible mapping.
Hence the condition (C-4) is satisfied.
Further $S(GI\alpha_k,GI\alpha_k,G(\frac{1}{2}))=S(\frac{1}{2},\frac{1}{2},\frac{1}{2})=0$ *as* $k\to\infty$.
Showing that the pair (G,I) is G-WRC.
Also $S(GI\alpha_k,GI\alpha_k,I(\frac{1}{2}))=S(\frac{1}{2},\frac{1}{2},\frac{1}{2})=0$ *as* $k\to\infty$ *proving that the pair (G,I) is semi compatible.*
Hence condition (C-3) is satisfied.
We now establish that the mappings G,H,I and J satisfy the condition (C-2) .
Case I
If $\alpha,\beta\in(0,\frac{1}{2}]$ *and* $S(\alpha,\beta,\gamma)=|\alpha-\gamma|+|\beta-\gamma|$.
Putting $\alpha=\frac{1}{4},\beta=\frac{1}{3}$ *the inequality **(C-2)** gives*

$$S(\frac{1}{4},\frac{1}{4},\frac{1}{3})\leq\lambda max\{S(\frac{3}{8},\frac{3}{8},\frac{5}{12}), \frac{S(\frac{1}{4},\frac{1}{4},\frac{3}{8})S(\frac{1}{2},\frac{1}{2},\frac{5}{12})}{S(\frac{3}{8},\frac{3}{8},\frac{5}{12})}, S(\frac{1}{2},\frac{1}{2},\frac{1}{4})$$

$0.166\leq\lambda max\big(0.08,\frac{0.25X0.16}{0.08},0.5\big)$
$0.166\leq\lambda(0.5)$,*thus* λ*=0.33*$\in(0,1)$ *so that the inequality **(C-2)** holds.*
Case II
If $\alpha,\beta\in(\frac{1}{2},1)$ *and* $S(\alpha,\beta,\gamma)=|\alpha-\gamma|+|\beta-\gamma|$.

Putting $\alpha=\frac{3}{4},\beta=\frac{4}{5}$ *the inequality **(C-2)** gives*

$$S(\frac{5}{16},\frac{5}{16},\frac{7}{20})\leq\lambda max\{S(\frac{1}{4},\frac{1}{4},\frac{3}{10}),\frac{S(\frac{5}{16},\frac{5}{16},\frac{1}{4})S(\frac{7}{20},\frac{7}{20},\frac{3}{10})}{S(\frac{1}{4},\frac{1}{4},\frac{3}{10})},S(\frac{7}{20},\frac{7}{20},\frac{5}{16})\}$$

$0.075\leq\lambda\ max\big(0.1,\frac{0.125X0.1}{0.1},0.075\big)$
$0.075\leq\lambda(0.125)$,*thus* λ=0.6 $\in(0,1)$ *so that the inequality **(C-2)** holds.*
Similarly we can prove the other cases. It is also the fact to note that $\frac{1}{2}$ *is the unique common fixed point of the four mappings H,G,I and J.*
Now we prove another theorem on OWC mappings.

Theorem 3.3. *Let* (X,S) *be a complete S-metric space and there are four mappings G, H, I and J holding the conditions*
(D-1) $G(X)\subseteq J(X)$ *and* $H(X)\subseteq I(X)$
(D-2) $S(G\alpha,G\alpha,H\beta)\leq\lambda\ max\ \{S(I\alpha,I\alpha,J\beta),\frac{S(G\alpha,G\alpha,I\alpha)S(H\beta,H\beta,J\beta)}{S(I\alpha,I\alpha,J\beta)},S(H\beta,H\beta,G\alpha)\}$
for all $\alpha,\beta\in X$*,where* $\lambda\in(0,1)$
(D-3) *the pairs* (G,I) *and* (H,J) *are having coincidence points*
(D-4) *the pairs* (G,I) *and* (H,J) *are OWC mappings .*
Then the above mappings will be having the unique common fixed point.

Proof. Since the pair (G,I) is OWC then there exists $u\in C(G,I)$ such that $Gu=Iu=\mu$ (say) and $GIu=IGu=\mu^1$ (say). Hence we have

$$G\mu=I\mu=\mu^1. \tag{3.12}$$

Again since the pair (H,J) is OWC there exists $v\in C(G,I)$ such that $Hv=Jv=\delta$(say) and $HJv=JHv=\delta^1$(say). Hence we have

$$H\delta=J\delta=\delta^1. \tag{3.13}$$

Now we claim $\mu=\delta$.
Substitute $\alpha=u$ and $\beta=v$ in the inequality **(C-2)** we get

$$S(Gu,Gu,Hv)\leq\lambda max\{S(Iu,Iu,Jv),\frac{S(Gu,Gu,Iu)S(Hv,Hv,Jv)}{S(Iu,Iu,Jv)},S(Hv,Hv,Gu)\}$$

this gives

$$S(\mu,\mu,\delta)\leq\lambda max\{S(\mu,\mu,\delta),\frac{S(\mu,\mu,\mu)S(\delta,\delta,\delta)}{S(\mu,\mu,\delta)},S(\delta,\delta,\mu)\}$$

and this implies

$$S(\mu,\mu,\delta)\leq\lambda max\{S(\mu,\mu,\delta),\frac{1}{S(\mu,\mu,\delta)},S(\mu,\mu,\delta)\}$$

and this gives
$S(\mu,\mu,\delta)\leq\lambda S(\mu,\mu,\delta)$
which gives

$$\mu=\delta. \tag{3.14}$$

Substitute $\alpha=\mu$ and $\beta=\delta$ in the inequality **(C-2)**

$$S(G\mu,G\mu,H\delta)\leq\lambda max\{S(I\mu,I\mu,J\delta),\frac{S(G\mu,G\mu,I\mu)S(H\delta,H\delta,J\delta)}{S(I\mu,I\mu,J\delta)},S(H\delta,H\delta,G\mu)\}$$

this gives

$$S(\mu^1,\mu^1,\delta^1)\leq\lambda max\{S(\mu^1,\mu^1,\delta^1),\frac{S(\mu^1,\mu^1,\mu^1)S(\delta^1,\delta^1,\delta^1)}{S(\mu^1,\mu^1,\delta^1)},S(\delta^1,\delta^1,\mu^1)\}$$

and this implies

$$S(\mu^1,\mu^1,\delta^1)\leq\lambda max\{S(\mu^1,\mu^1,\delta^1),\frac{1}{S(\mu^1,\mu^1,\delta^1)},S(\mu^1,\mu^1,\delta^1)\}$$

this results
$S(\mu^1,\mu^1,\delta^1)\leq\lambda S(\mu^1,\mu^1,\delta^1)$
which implies

$$\mu^1=\delta^1. \tag{3.15}$$

Substitute $\alpha=u$ and $\beta=\delta$ in the inequality **(C-2)** we obtain

$$S(Gu,Gu,H\delta)\leq\lambda max\{S(Iu,Iu,J\delta),\frac{S(Gu,Gu,Iu)S(H\delta,H\delta,J\delta)}{S(Iu,Iu,J\delta)},S(H\delta,H\delta,Gu)\}$$

this implies

$$S(\mu,\mu,\delta^1)\leq\lambda max\{S(\mu,\mu,\delta^1),\frac{S(\mu,\mu,\mu)S(\delta^1,\delta\cdot\delta^1)}{S(\mu,\mu,\delta^1)},S(\mu,\mu,\delta^1)\}$$

and this gives
$S(\mu,\mu,\delta^1)\leq\lambda\ \max\{S(\mu,\mu,\delta^1),\frac{1}{S(\mu,\mu,\delta^1)},S(\mu,\mu,\delta^1)\}$
this results
$S(\mu,\mu,\delta^1)\leq\lambda(S(\mu,\mu,\delta^1)$
which gives

$$\mu=\delta^1 \tag{3.16}$$

Thus from (3.14),(3.15) and (3.16) we have

$$\delta=\mu=\delta^1=\mu^1. \tag{3.17}$$

From (3.14),(3.15) and (3.17) we have

$$G\mu = I\mu = J\mu = H\mu = \mu. \tag{3.18}$$

The Uniqueness can be proved easily.

□

Now we present another illustration to discuss the validity of the above theorem.

Example 3.4. *Suppose $X = (0,\infty)$ in S-metric space with δ_1 and δ_2 are two metrics on X with*
$S(\alpha,\beta,\gamma)$=$\delta_1(\alpha,\gamma)+\delta_2(\beta,\gamma)$ for all $\alpha,\beta,\gamma\in X$.
We define self maps G,H,I and J as follows

$$G(\alpha) = H(\alpha) = \begin{cases} \alpha & \text{if } 0<\alpha\leq\frac{1}{3} \\ \frac{3-\alpha}{4} & \text{if } \frac{1}{3}<\alpha<1 \end{cases}$$

and

$$I(\alpha) = J(\alpha) = \begin{cases} 1-2\alpha & \text{if } 0<\alpha\leq\frac{1}{3} \\ \frac{3\alpha+1}{4} & \text{if } \frac{1}{3}<\alpha<1. \end{cases}$$

Now G(X)=H(X)=$(0,\frac{1}{3}]\cup(\frac{2}{3},\frac{1}{2})$ and
I(X)=J(X)= $(\frac{1}{3},1]\cup(0,\frac{2}{5})$.
This gives $G(X)\subseteq J(X)$and $H(X)\subseteq I(X)$ so that the condition $(D-1)$ is satisfied.
Now take a sequence $\{\alpha_k\}$ as $\alpha_k = \frac{1}{3}-\frac{1}{k}$ for k≥ 0.
Now $G(\alpha_k) = G(\frac{1}{3}-\frac{1}{k})$=$(\frac{1}{3}-\frac{1}{k})$=$\frac{1}{3}$ as $k\to\infty$
and $I(\alpha_k)$=$I(\frac{1}{3}-\frac{1}{k})$= $1-2(\frac{1}{3}-\frac{1}{k})$=$(\frac{1}{3}+\frac{1}{k})$=$\frac{1}{3}$ as $k\to\infty$.
Therefore $G(\alpha_k) = I(\alpha_k) = \frac{1}{3}$ as $k\to\infty$.
Further $GI(\alpha_k) = GI(\frac{1}{3}-\frac{1}{k})$=$G(1-2(\frac{1}{3}-\frac{1}{k}))$=
$G(\frac{1}{3}+\frac{1}{k}) = \frac{3-(\frac{1}{3}+\frac{1}{k})}{4} = (\frac{2}{3}+\frac{1}{k})$ =$\frac{2}{3}$ as $k\to\infty$
and $IG(\alpha_k) = IG(\frac{1}{3}-\frac{1}{k})$=$I(\frac{1}{3}-\frac{1}{k})$=$1-2(\frac{1}{3}-\frac{1}{k}) =$
$(\frac{1}{3}+\frac{1}{k}) = \frac{1}{3}$ as $k\to\infty$.
Thus $S(GI\alpha_k, GI\alpha_k, IG\alpha_k)$=$S(\frac{2}{3},\frac{2}{3},\frac{1}{3})\neq 0$.
Showing that the pair(G , I) is not compatible.
Further $G(\frac{1}{3}) = I(\frac{1}{3}) = \frac{1}{3}$ and $G(\frac{1}{2}) = I(\frac{1}{2}) = \frac{5}{8}$.
Therefore $\frac{1}{3}$and$\frac{5}{8}$ are the coincidence points of G and I.
But $GI(\frac{1}{3})$ = $\frac{1}{3}$,$IG(\frac{1}{3})$=0 implies $GI(\frac{1}{3})\neq IG(\frac{1}{3})$
and $GI(\frac{1}{2})$ = $\frac{5}{8}$,$IG(\frac{1}{2})$= $\frac{5}{8}$ implies $GI(\frac{1}{2}) = IG(\frac{1}{2})$, showing that the pair (G,I) is OWC but not weakly compatible.
Hence the condition (D-4) is satisfied.
We now establish that the mappings G,H,I and J satisfy the Condition (D-2).

Case I

If $\alpha,\beta\in(0,\frac{1}{3}]$ and $S(\alpha,\beta,\gamma) = |\alpha-\gamma|+|\beta-\gamma|$.
*Putting $\alpha = \frac{1}{5}$ and $\beta = \frac{1}{4}$ the inequality **(D-2)** gives*

$$S(\tfrac{1}{5},\tfrac{1}{5},\tfrac{1}{4})\leq\lambda\, max\{S(\tfrac{3}{5},\tfrac{3}{5},\tfrac{1}{2}),\frac{S(\frac{1}{5},\frac{1}{5},\frac{3}{5})S(\frac{1}{4},\frac{1}{4},\frac{1}{2})}{S(\frac{3}{5},\frac{3}{5},\frac{1}{2})},S(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{5})\}$$

$$0.1\leq\lambda max\big(0.2,\frac{0.8X0.5}{0.2},0.1\big)$$

$0.1\leq\lambda(0.02)$*thus* λ*=0.05*$\in(0,1)$ *so that the inequality **(D-2)** holds.*

Case II

If $\alpha,\beta\in(\frac{1}{3},1)$ and $S(\alpha,\beta,\gamma) = |\alpha-\gamma|+|\beta-\gamma|$.
*Putting $\alpha = \frac{1}{2}$ and $\beta = \frac{2}{3}$ the inequality **(D-2)** gives*

$$S(\tfrac{5}{8},\tfrac{5}{8},\tfrac{7}{12})\leq\lambda\, max\{S(\tfrac{5}{8},\tfrac{5}{8},\tfrac{3}{4}),\frac{S(\frac{5}{8},\frac{5}{8},\frac{5}{8})S(\frac{7}{12},\frac{7}{12},\frac{3}{4})}{S(\frac{5}{8},\frac{5}{8},\frac{3}{4})},S(\tfrac{7}{12},\tfrac{7}{12},\tfrac{5}{8})\}$$

$$0.083\leq\lambda\ max\big(0.25,\frac{0X0.83}{0.25},0.083\big)$$

$0.083\leq\lambda(0.25)$*thus* λ=0.332 $\in(0,1)$ *so that the inequality **(D-2)** holds.*

Similarly we can prove the other cases.
It is observed that $\frac{1}{3}$ is the unique common fixed point of the four mappings H,G,I and J.

4. Conclusion

In this paper we established two results on S- metric space using the new contraction condition along with WRC, semi compatible, OWC and weakly compatible mappings .It is observed that the mappings in the two theorems are neither continuous nor compatible mappings.Further these two results are justified with suitable examples.Thus we assert that our theorems stand as generalizations of many existing theorems in S-metric space.

References

[1] S. Sedghi, N. Shobe and A. Aliouche, A generalization of fixed point theorems in S-metric space, *Mat. Vesn.,* 64(2012), 258–266.

[2] W. Shatanawi, Coupled fixed point theorems in generalized metric space, *Hacettepe J. Math.,* 40(3)(2011), 441–447.

[3] S. Sedghi, A. Gholidahneh, T. Dosenavic, J. Estahani and S. Rsde-novic, Common fixed point of four maps in S-metric spaces, *J. Linear Topol. Algebra,* 05(02)(2016), 93–104.

[4] Y.J. Cho, B.K. Sharma and R.D. Sahu, Semi-compatibility and fixed points, *Math. Japon.,* 42(1)(1995), 91–98.

[5] S. Ravi and V. Srinivas, A fixed point theorem using weakly semi compatible mappings in metric space, *International Journal of Mathematical Analysis*, 11(15)(2017), 695–706.

[6] S. Sedghi and N.V. Dung, Fixed point theorems on S-metric spaces, *Mat.Vesn.,* 66(2014), 113–124.

[7] G. Junck, Compatible mappings and common fixed points, *Internat. J. Math. Math. Sci.,* 9(4)(1986), 771–779.

[8] M.A. Al-thagafi and N.A. Shahzad, A note on occasionally weakly compatible mappings , *Int. J. Math. Anal.,* 3(2009), 55–58.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆