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Stress regular graphs

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Abstract
The stress of any vertex of a graph G is defined as the number of shortest path passing through that vertex as an internal vertex. If the stress of every vertex in a graph is equal, such graphs are called stress regular graphs. This paper investigates the stress of different families of graphs and study the properties of stress regular graphs.

Keywords
Stress of a graph, stress regular graphs, distance regular graphs.

AMS Subject Classification
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1. Introduction

The analysis of complex networks that represent relations between various objects involve the tasks of the identification of objects which has a prominent role within the network. The concept of stress is one of such parameter and is introduced by Shimbel [9].

Let $G(V(G),E(G))$ be a simple connected undirected graph with vertex set $V(G)$ and edge set $E(G)$, n and m denote the number of its vertices and edges respectively. For every vertex $v \in V(G)$, the open neighbourhood of v is the set $N(v) = \{u \in V : uv \in E(G)\}$ and the closed neighbourhood of v is the set $N[v] = N(v) \cup \{v\}$. The degree of a vertex $v \in V(G)$ is $deg(v) =| N(v) |$. A graph is regular with valency k if the degree of each of its vertices is k. For any two vertices $u,v \in V(G)$, the distance $d(u,v)$ between u and v is the length of a shortest path between u and v in G. The eccentricity of a vertex u is the number $e(u) = max\ \{d(u,v) : v \in V\}$. The maximum eccentricity of the vertices of G is called the diameter of G and is denoted by D. The i-neighbourhood $N_i(v)$ of a vertex $v \in V(G)$ is the set of vertices at distance i from v [8]. Also we denote $N_i^j(x)$ is the set of all unordered pairs of elements in $N_i(x)$ which are at distance j. A graph automorphism is a permutation φ of the vertex set $V(G)$ with the property that uv is an edge if and only if $\varphi(u)\varphi(v)$ is an edge. The automorphism group of a graph G is transitive if there exists an automorphism φ to any pair u,v of vertices in G such that $\varphi(u) = v$. In this case G is called vertex transitive [1].

A simple connected graph G is called distance regular if it is regular, and if for any two vertices $x,y \in G$ at distance i, there are constant number of neighbors c_i and b_i of y at distance $i-1$ and $i+1$ from x respectively, that is there are precisely c_i neighbors of y in $N_{i-1}(x)$ and b_i neighbors of y in $N_{i+1}(x)$ [6].

A graph G of order n is said to be strongly regular with parameters (n,k,λ,μ), if G is a regular graph of valency k such that every pair of distinct vertices in the graph have λ or μ common neighbors according as they are adjacent or not [6].

A graph G is distance transitive if, for vertices u,v,x,y in $V(G)$, with $d(u,v) = d(x,y)$ there exists some $g \in aut(G)$ satisfying $g(u) = x$ and $g(v) = y$ [7].

If node x is used as an intermediate station for contacting node u to node v, then in such a network node x has a certain responsibility to nodes u and v . If we count all of the minimum paths which pass through node x, then we have a measure of the stress which node x must undergo during the activity of the network. A vector giving this number for each number of the network would give us a good idea of stress conditions throughout the system [9] .

2. Stress of a graph

In this section we calculate the stress of vertices of certain graph classes .

Definition 2.1. [9] : *For a simple connected undirected graph $G(V,E)$, the stress of a vertex $x \in V$ is defined as*

$$S(x) = \sum_{u \neq v \neq x} \sigma_{uv}(x)$$

where $\sigma_{uv}(x)$ is the number of shortest paths with vertices u and v as their end vertices and include the vertex x.

Definition 2.2. : *The average stress of a graph $G(V,E)$ of order n is defined as*

$$\bar{S}(G) = \frac{1}{n} \sum_{x \in V} S(x)$$

In a complete graph K_n, the length of any shortest path is one and therefore there is no shortest path passing through any of the vertices as an internal vertex . Hence for a vertex v in a complete graph K_n , the stress of v is zero. In a star graph $K_{1,n}$ the stress of the central vertex is $\frac{n(n-1)}{2}$ and since the vertex of degree one is not an internal vertex of any shortest path in the graph, the stress of each vertex of degree one is zero .

Theorem 2.3. *The stress of a vertex v_i in a path graph P_n is $S(v_i) = (i-1)(n-i)$ and the average stress $\bar{S}(P_n) = \frac{1}{n}\sum_{i=1}^{n}(i-1)(n-i)$*

Proof. Consider a path graph P_n of n vertices $v_1, v_2, \ldots v_n$. Since v_1 and v_n are pendant vertices, $S(v_1) = S(v_n) = 0$. Also for any $v_i \in V(P_n)$ there are $(i-1)$ vertices on one side and $(n-i)$ vertices on the other side of v_i. Hence there are $(i-1)(n-i)$ number of shortest path passing through v_i. □

Theorem 2.4. *The stress of any vertex v in a cycle graph C_{2n} is $S(v) = \frac{n(n-1)}{2}$.*

Proof. Let $v_1, v_2, \ldots, v_{2n}$ be the vertices of C_{2n}. Consider a vertex v_1. Then for each pair of antipodal vertices v_i, v_{n+i}, $i = 2,3,\ldots,n$, there exists a shortest path passing through v_1. There are $(n-1)$ such paths. Now , let us take all paths of length less than n containing v_1. There are $n-i$ paths joining v_i to the vertex v_{n+1+i} passing through v_1 for $i = 2,3,\ldots,n-1$. Hence $S(v_1) = (n-1) + \sum_{i=2}^{n-1}(n-i) = \frac{n(n-1)}{2}$.
C_{2n} is a vertex transitive graph. Hence the stress of any vertex $v \in V(C_{2n})$ is $S(v) = \frac{n(n-1)}{2}$. □

Theorem 2.5. *The stress of any vertex v in a cycle graph C_{2n+1} is $S(v) = \frac{n(n-1)}{2}$.*

Theorem 2.6. *The stress of a vertex in a friendship graph F_{2n+1} is given by $S(v) = 2n(n-1)$, if v is the central vertex and is zero for all other vertices.*

Proof. Let $v, v_1, v_2, \ldots, v_{2n}$ be the vertices of F_{2n+1} having v as the central vertex. There is no shortest path passing through v_i for $i = 1,2,\ldots,2n$. Hence $S(v_i) = 0$ for $i = 1,2,\ldots,2n$. Now consider the central vertex v. For each pair $\{v_i, v_j\}$ of nonadjacent vertices , there is a shortest path passing through v. Since there are n pairs of adjacent vertices $\{v_i, v_j\}$, the total number of shortest path which passes through v is $\frac{2n(2n-1)}{2} - n$. Hence $S(v) = 2n(2n-1)$. □

Theorem 2.7. *For any graph G of diameter 2, the stress of a vertex $x \in V(G)$ is given by $S(x) = \mid N_1^2(x) \mid$.*

Proof. Consider a graph G of order n and diameter 2. take two vertices u, v in $V(G)$. If $uv \in E(G)$, then the shortest $u-v$ path doesnot contain the vertex x. If $u, v \notin N_1(x)$, then the shortest $u-v$ path doesnot contain x , otherwise $d(u,v) > 2$, a contradiction. If $u \in N_1(x)$ and $v \notin N_1(x)$, then also the shortest $u-v$ path doesnot contain the vertex x, otherwise $d(u,v) > 2$, a contradiction. If $u, v \in N_1(x)$, then , since the diameter of G is 2, the shortest path having end points u and v must pass through the vertex x. Hence the result. □

Theorem 2.8. *The stress of a vertex x in a complete bipartite graph $K_{m,n}$ is , $S(x) = \frac{n(n-1)}{2}$ if degree of x is n and $S(x) = \frac{m(m-1)}{2}$ if degree of x is m.*

Proof. Consider the complete bipartite graph $K_{m,n}$ having vertices $u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n$, where degree of u_i is n for $i = 1,2,\ldots,m$ and degree of v_j is m for $j = 1,2,\ldots,n$. $N_1(u_i) = \{v_1, v_2, \ldots, v_n\}$. Also $d(v_i, v_j) = 2$ for all $i \neq j$. Hence by Theorem 2.7. $S(u_i) = \mid N_1^2(u_i) \mid = \frac{n(n-1)}{2}$, for $i = 1,2,\ldots,m$. Similarly we can prove that $S(v_j) = \mid N_1^2(v_j) \mid = \frac{m(m-1)}{2}$, for $j = 1,2,\ldots,n$. Hence the result. □

Theorem 2.9. *The stress of any vertex in $K_{1,n} \wedge a$, the join of a star graph and any vertex a is given by $S(u) = \frac{n(n-1)}{2}$, if u is the central vertex or $u = a$ and is zero for all other vertices.*

Proof. Let $K_{1,n}$ be the star graph with vertex set $\{u, u_1, u_2, \ldots, u_n\}$. having u as the central vertex Consider the join $K_{1,n} \wedge a$ where a is any vertex. Then diameter of $K_{1,n} \wedge a$ is 2. We have
$N_1(u) = \{u, u_1, u_2, \ldots, u_n\}$ and $N_1(a) = \{a, u_1, u_2, \ldots, u_n\}$.
Since $uu_i, au_i \in E(K_{1,n} \wedge a)$ for $i = 1,2,\ldots,n$, we have $S(u) = \mid N_1^2(u) \mid = \frac{n(n-1)}{2}$ and $S(a) = \mid N_1^2(a) \mid = \frac{n(n-1)}{2}$. Also $N_1(u_i) = \{a, u\}$ for $i = 1,2,\ldots,n$ and $au \in E(K_{1,n} \wedge a)$ we get $S(u_i) = 0$ for $i = 1,2,\ldots,n$. □

3. Some properties of Stress Regular Graphs

In this section we define stress regular graphs and investigate the properties of stress regular graphs.

Definition 3.1. : *Graphs with vertices having the same stress are called stress regular graphs .*

There exists regular graphs which is not stress regular. Complete graphs are stress regular graphs having zero stress. Clearly every vertex transitive graphs are stress regular graphs.

Theorem 3.2. *Complete bipartite graph $K_{n,n}$ is stress regular.*

Proof. If $m = n$ in Theorem 2.8. we get the result. □

Theorem 3.3. *Cycle C_n is stress regular.*

The cocktail party graph $CP(n)$ [5] is a unique regular graph of degree $2n-2$ on $2n$ vertices. It is obtained from K_{2n} by deleting a perfect matching.

Theorem 3.4. *Cocktail party graph $CP(n)$ is stress regular.*

Proof. Consider $CP(n)$ with vertex set $\{u_1, u_2, ..., u_n, v_1, v_2, ..., v_n\}$ having $N_1(u_i) = \{u_1, u_2, ..., u_{i-1}, u_{i+1}, ..., u_n, v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_n\}$ and $N_1(v_j) = \{u_1, u_2, ..., u_{j-1}, u_{j+1}, ..., u_n, v_1, v_2, ..., v_{j-1}, v_{j+1}, ..., v_n\}$. Here $N_1^2(u_i) = \{(u_j, v_j) : j \neq i\}$ and $N_1^2(v_j) = \{(u_i, v_i) : i \neq j\}$.
Since diameter of $CP(n)$ is 2, by Theorem 2.7. $S(u_i) =| N_1^2(u_i) |= n-1$, and $S(v_j) =| N_1^2(v_j) |= n-1$ for $i, j = 1, 2, ..., n$. □

Theorem 3.5. *Petersen graph is stress regular.*

Proof. Consider the petersen graph $KG_{5,2}$. For any vertex $v \in KG_{5,2}$, $N_1(v)$ consists of three nonadjacent vertices. Since diameter of $KG_{5,2}$ is 2, $S(v) =| N_1^2(v) |= 3$. Hence $KG_{5,2}$ is stress regular. □

Theorem 3.6. *Every Distance regular graph is stress regular.*

Proof. Let G be a distance regular graph and x be an arbitrary vertex in $V(G)$. Let $k_i =| N_i(v) |$. Since G is distance regular, k_i vertices in $N_i(x)$ are adjacent to c_i vertices in $N_{i-1}(x)$. Now consider any two vertices u, v in $V(G)$ such that $d(u, v) = s$. If $s = 1$, then $\sigma_{uv} = 1 = c_1$. If $s = 2$, then vertices in $N_2(v)$ are adjacent to c_2 vertices in $N_1(v)$. Hence $\sigma_{uv} = c_2$. Thus by induction on s we get $\sigma_{uv} = \prod_{i=1}^{s} c_i$. Now let u, v be any two vertices in $V(G)$ such that the shortest path having end points u and v passing through the vertex x and $d(u, v) = s$. If $d(u, x) = t$, then $\sigma_{uv}(x) = \sigma_{ux}\sigma_{xv} = \prod_{i=1}^{t} c_i \prod_{i=1}^{s-t} c_i$. That is the number of shortest path having end points u and v doesnot depend on the selection of the vertices u and v and depends only on the distance between u and v. Also x is an arbitrary vertex of G. Hence $S(x)$ is same for every vertices of G. That is G is stress regular. □

Corollary 3.7. *Every Strongly regular graph is stress regular.*

Proof. Since every strongly regular graph is a distance regular graph of diameter 2 [6], it is stress regular. □

Converse need not be true. For example cycle C_6 is not strongly regular but stress regular.

Corollary 3.8. *Every distance transitve graph is stress regular.*

Proof. Since every distance transitive graph is distance regular , it is stress regular. □

Converse need not be true. For example Shrikande graph is stress regular but not distance transitive.

4. Conclusion

In this paper stress of various graph classes are computed and prove that every distance regular graphs are stress regular. It would be interesting to investigate which among the following class of graphs ie,distance balanced, distance degree regular and walk regular are stress regular.

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