Malaya Journal of Matematik, Vol. 8, No. 3, 1155-1157, 2020
https://doi.org/10.26637/MJM0803/0073

Planarity of a unit graph: Part-I local case

Jaydeep Parejiya[1]*, Patat Sarman [2] and Pravin Vadhel[3]

Abstract
The rings considered in this article are commutative with identity $1 \neq 0$. Recall that the unit graph of a ring R is a simple undirected graph whose vertex set is the set of all elements of the ring R and two distinct vertices x, y are adjacent in this graph if and only if $x + y \in U(R)$ where $U(R)$ is the set of unit elements of ring R. We denote this graph by $UG(R)$. In this article we classified local ring R such that $UG(R)$ is planar.

Keywords
Planar graph, (Ku_1^*) and (Ku_2^*).

2010 Mathematics Subject Classification
13A15, 05C25.

[1] *Department of Mathematics, Government Polytechnic, Rajkot, Rajkot-360005, India.*
[2] *Department of Mathematics, Sir B.P.T.I, Bhavnagar, Bhavnagar-364002, India.*
[3] *Department of Mathematics, V.V.P. Engineering College, Rajkot, Rajkot-360005, India.*
***Corresponding author**: [1] parejiyajay@gmail.com; [2]patat.sarman03@gmail.com; [3]vadhelpravin@gmail.com
Article History: Received **24** March **2020**; Accepted **11** July **2020**

Contents

1. Introduction

We first recall the following definitions and results from graph theory. A graph G=(V,E) is said to be complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n [4, Definition 1.1.11]. A graph G=(V,E) is said to be bipartite if the vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and other in Y. The pair (X,Y) is called a bipartition of G. A bipartite graph G with bipartition (X,Y) is denoted by G(X,Y). A bipartite graph G(X,Y) is said to be complete if each vertex of X is adjacent to all the vertices of Y. If G(X,Y) is a complete bipartite graph with $|X| = m$ and $|Y| = n$, then it is denoted by $K_{m,n}$ [4, Definition 1.1.12]. Let G=(V,E) be a graph.By a clique of G, we mean a complete subgraph of G [4, Definition 1.2.2]. We say that the clique number of G equals n if n is the largest positive integer such that K_n is a subgraph of G [4, p.185]. The clique number of a graph G is denoted by the notation $\omega(G)$. If G contains K_n as a subgraph for all $n \geq 1$, then we set $\omega(G) = \infty$.

A graph G is said to be planar if it can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G [4, Definition 8.1.1]. Two adjacent edges of a graph G are said to be in series if their common vertex is of degree two [5, p.9]. Two graphs are said to be homeomorphic if one graph can be obtained from the other graph by the creation of edges in series (i.e by insertion of vertices of degree two) or by the merger of edges in series[5, p.100]. Recall from [5, p.93] that K_5 is referred to as Kuratowski's first graph and $K_{3,3}$ is referred to as Kuratowski's second graph. A celebrated theorem of Kuratowski says that a necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [5, Theorem 5.9].

In view of Kuratowski's Theorem, [5, Theorem 5.9] we introduce the following definitions. We say that a graph G=(V,E) satisfies Ku_1 if G does not contain K_5 as a subgraph and we say that graph G=(V,E) satisfies Ku_2 if G does not contain $K_{3,3}$ as a subgraph. We say that a graph G = (V,E) satisfies Ku_1^* if G satisfies Ku_1 and moreover, G does not contain any subgraph homeomorphic to K_5. We say that a graph G = (V,E) satisfies Ku_2^* if G satisfies Ku_2 and moreover, G does not contain any subgraph homeomorphic to $K_{3,3}$.

If a graph G is planar, then it follows from Kuratowski's theorem [5, Theorem 5.9] that G satisfies both Ku_1^* and Ku_2^* . Hence G satisfies both Ku_1 and Ku_2. It is interesting to note that a graph G may be nonplanar even if it satisfies both

Ku_1 and Ku_2. For example of this type refer [5, Figure 5.9(a), p.101] and the graph G in this example does not satisfies Ku_2^*. We do not know an example of a graph G such that G satisfies Ku_1 but G does not satisfy Ku_1^*.

Let R be a ring. With the hypothesis that R is a finite ring, a classification of finite rings R such that $UG(R)$ is planar was given in [2, Theorem 5.14]. In section 2, we assume that R is local and we show that if $UG(R)$ is planar, then R is necessarily finite. Indeed, we show in Theorem 2.5 that if $UG(R)$ satisfies (Ku_2) if and only if it is planar if and only if R is isomorphic to one of the rings from the collection $\mathscr{B} = \{\mathbb{Z}_2, \mathbb{F}_4, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$.

The rings considered in this article are commutative with identity and are nonzero. A ring R which has a unique maximal ideal is referred to as a quasilocal ring. A ring R which has only a finite number of maximal ideals is referred to as a semiquasilocal ring. A Noetherian quasilocal (respctively, semiquasilocal) ring is referred to as a local (respectively, semilocal) ring. We denote the set of all maximal ideals of a ring R by $Max(R)$. We used $J(R)$ to denote Jacobson radical of ring R.

2. Planarity of $UG(R)$, where R is quasilocal ring

Lemma 2.1. *Let $(R,\mathfrak{m})$ be a quasilocal ring. If $UG(R)$ satisfies (Ku_2), then $|\mathfrak{m}| \leq 2$.*

Proof. First, we verify that if a ring T is such that $UG(T)$ satisfies (Ku_2), then $|J(T)| \leq 2$. This fact was already verified in [2, See Definitions and Remarks 5.13]. For the sake of convenience, we include a proof of it here. Assume that $UG(T)$ satisfies (Ku_2). We assert that $|J(T)| \leq 2$. Suppose that $|J(T)| \geq 3$. Let $\{0, x_1, x_2\} \subseteq J(T)$. Observe that $\{1, 1+x_1, 1+x_2\} \subseteq U(T)$. Let $V_1 = \{0, x_1, x_2\}$ and let $V_2 = \{1, 1+x_1, 1+x_2\}$. It is clear that $V_1 \cup V_2 \subseteq V(UG(T))$ and $V_1 \cap V_2 = \emptyset$. Note that for any $a \in J(T)$ and for any $b \in U(T)$, $a+b \in U(T)$ and hence, a and b are adjacent in $UG(T)$. Therefore, we obtain that the subgraph of $UG(T)$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $UG(T)$ satisfies (Ku_2). Therefore, $|J(T)| \leq 2$.

Note that if $(R,\mathfrak{m})$ is a quasilocal ring, then $J(R) = \mathfrak{m}$. Thus if $UG(R)$ satisfies (Ku_2), then $|\mathfrak{m}| \leq 2$. □

If $(R,\mathfrak{m})$ is a quasilocal ring with $|\mathfrak{m}| = 1$, then R is necessarily a field. Let F be a field. We next proceed to characterize fields F such that $UG(F)$ is planar.

Lemma 2.2. *Let F be a field with $char(F) = 2$. Then the following statements are equivalent:*

(i) $UG(F)$ is planar.
(ii) $UG(F)$ satisfies both (Ku_1^*) and (Ku_2^*).
(iii) $UG(F)$ satisfies (Ku_2).
(iv) $|F| \in \{2,4\}$.

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [5, Theorem 5.9].

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (iv)$ As $char(F) = 2$ by assumption, we obtain from [2, Theorem 3.4] that $UG(F)$ is complete. It is clear that if $\omega(UG(F)) \geq 6$, then $UG(F)$ does not satisfy (Ku_2). Thus if $UG(F)$ satisfies (Ku_2), then $|F| \leq 5$. Since $char(F) = 2$, we obtain that $|F| = 2^n$ for some $n \geq 1$ and so, it follows that $|F| \in \{2,4\}$.

$(iv) \Rightarrow (i)$ If $|F| \in \{2,4\}$, then $|V(UG(F))| \in \{2,4\}$. Since any simple graph on at most four vertices is planar, we obtain that $UG(F)$ is planar. □

Lemma 2.3. *Let F be a field with $char(F) \neq 2$. Then the following statements are equivalent:*

(i) $UG(F)$ is planar.
(ii) $UG(F)$ satisfies both (Ku_1^*) and (Ku_2^*).
(iii) $UG(F)$ satisfies (Ku_2).
(iv) $|F| \in \{3,5\}$.

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [5, Theorem 5.9].

$(ii(\Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (iv)$ We claim that $|F^*| \leq 4$. Suppose that $|F^*| \geq 5$. Let $\alpha_1 \in F^*$. As we are assuming that $char(F) \neq 2$, we get that $\alpha_1 \neq -\alpha_1$. Since we are assuming that $|F^*| \geq 5$, it is possible to find distinct $\alpha_2, \alpha_3, \alpha_4 \in F^* \backslash \{\alpha_1, -\alpha_1\}$. Let $V_1 = \{0, \alpha_1, -\alpha_1\}$ and let $V_2 = \{\alpha_2, \alpha_3, \alpha_4\}$. Note that $V_1 \cup V_2 \subseteq V(UG(F))$ and $V_1 \cap V_2 = \emptyset$. It is clear from the choice of the elements α_i, where $i \in \{1,2,3,4\}$ that for any $a \in V_1$ and for any $b \in V_2$, $a+b \in F^*$, and so, a and b are adjacent in $UG(F)$. Hence, the subgraph of $UG(F)$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This is in contradiction to the assumption that $UG(F)$ satisfies (Ku_2). Therefore, $|F^*| \leq 4$. As $char(F) \neq 2$, it follows that $|F| \in \{3,5\}$.

$(iv) \Rightarrow (i)$ Suppose that $|F| = 3$. Then $F \cong \mathbb{Z}_3$ as fields and $UG(F)$ is a simple graph on three vertices and so, $UG(F)$ is planar. Suppose that $|F| = 5$. Then $F \cong \mathbb{Z}_5$ as fields. We can assume without loss of generality that $F = \mathbb{Z}_5 = \{0,1,2,3,4\}$. Note that $UG(\mathbb{Z}_5)$ is the union of the cycle Γ of length 5 given by $\Gamma : 0-1-2-4-3-0$ and the edges $e_1 : 0-2, e_2 : 0-4$, and $e_3 : 1-3$. Observe that e_1, e_2, and e_3 are chords of Γ. It is clear that Γ can be represented by means of a pentagon and the edges e_1, e_2 can be drawn inside the pentagon representing Γ and e_3 can be drawn outside the pentagon representing Γ in such a way that there are no crossing over of the edges. This proves that $UG(\mathbb{Z}_5)$ is planar. The graph $UG(\mathbb{Z}_5)$ is shown in Figure 1.

□

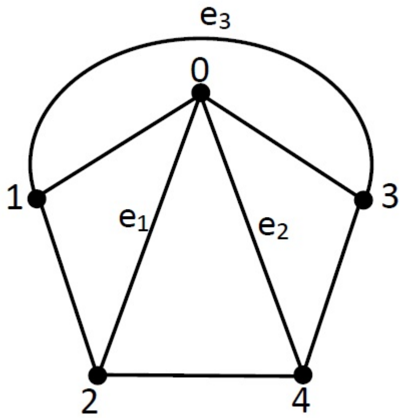

Figure 1. $UG(\mathbb{Z}_5)$

Lemma 2.4. *Let $(R,\mathfrak{m})$ be a quasilocal ring which is not a field. The following statements are equivalent:*

(i) *$UG(R)$ is planar.*

(ii) *$UG(R)$ satisfies both (Ku_1^*) and (Ku_2^*).*

(iii) *$UG(R)$ satisfies (Ku_2).*

(iv) *R is isomorphic to one of the following rings from the collection $\mathscr{A} = \{\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$.*

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [5, Theorem 5.9].

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (iv)$ We know from Lemma 2.1 that $|\mathfrak{m}| \leq 2$. We are assuming that R is not a field. Therefore, we obtain that $|\mathfrak{m}| = 2$. We know from [11, Lemma 6]that $|\frac{R}{\mathfrak{m}}| = 2$. Hence, $|R| = |\mathfrak{m}||\frac{R}{\mathfrak{m}}| = 4$. If $char(R) = 4$, then $R \cong \mathbb{Z}_4$ as rings and if $char(R) = 2$, then $R \cong \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}$ as rings. This proves that R is isomorphic to one of the rings from the collection $\mathscr{A} = \{\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$.

$(iv) \Rightarrow (i)$ For any ring $T \in \mathscr{A}$, $|T| = 4$ and since R is isomorphic one of the rings from the collection $\mathscr{A}$, we get that $|R| = 4$. Hence, $|V(UG(R))| = 4$. Since any simple graph on four vertices is planar, we obtain that $UG(R)$ is planar. □

Theorem 2.5. *Let $(R,\mathfrak{m})$ be a quasilocal ring. The following statements are equivalent:*

(i) *$UG(R)$ is planar.*

(ii) *$UG(R)$ satisfies both (Ku_1^*) and (Ku_2^*).*

(iii) *$UG(R)$ satisfies both (Ku_1) and (Ku_2).*

(iv) *$UG(R)$ satisfies (Ku_2).*

(v) *R is isomorphic to one of the rings from the collection $\mathscr{B} = \{\mathbb{Z}_2, \mathbb{F}_4, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$.*

Proof. $(i) \Rightarrow (ii)$ This follows from Kuratowski's theorem [5, Theorem 5.9].

$(ii) \Rightarrow (iii)$ This is clear.

$(iii) \Rightarrow (iv)$ This is obvious.

$(iv) \Rightarrow (v)$ Assume that $UG(R)$ satisfies (Ku_2). We know from Lemma 2.1 that $|\mathfrak{m}| \leq 2$. If $|\mathfrak{m}| = 1$, then we get from $(iii) \Rightarrow (iv)$ of Lemmas 2.1 and 2.3 that R is isomorphic to one of the rings from the collection $\{\mathbb{Z}_2, \mathbb{F}_4, \mathbb{Z}_3, \mathbb{Z}_5\}$. If $|\mathfrak{m}| = 2$, then we obtain from $(iii) \Rightarrow (iv)$ of Lemma 2.4 that R is isomorphic to one of the rings from the collection $\mathscr{A}$, where $\mathscr{A}$ is as in the statement (iv) of Lemma 2.4. Therefore, R is isomorphic to one of the rings from the collection $\mathscr{B}$, where $\mathscr{B}$ is as in the statement (v) of this Theorem.

$(v) \Rightarrow (i)$ Assume that R is isomorphic to one of the rings from the collection $\mathscr{B}$. Then we obtain from $(iv) \Rightarrow (i)$ of Lemmas 2.2, 2.3, and 2.4 that $UG(R)$ is planar. □

References

[1] S. Akbari, B. Miraftar and R. Nikandish, A note on comaximal ideal graph of commutative rings, arXiv:1307.5401 [math.AC], 2013.

[2] N. Ashrafi, H.R. Mainmani, M.R. Pournaki and S. Yassemi, Unit graph associated with rings, *Comm. Algebra,* 38(2010), 2851–2871.

[3] M.F. Atiyah and I.G.Macdonald, *Introduction to Commutative Algebra*, Addison-Wesley publishing Company, 1969.

[4] R. Balakrishnan and K. Ranganathan, *A Textbook of Graph Theory*, Springer-Verlag, New York, 2000.

[5] N. Deo, *Graph Theory with Applications to Engineering and Computer Science*, Prentice Hall of India Private Limited, New Delhi, 1994.

[6] M.I. Jinnah and S.C. Mathew, When is the comaximal graph split?, *Comm. Algebra* 40 (7)(2012), 2400–2404.

[7] H.R. Maimani, M. Salimi, A. Sattari, and S. Yassemi, Comaximal graph of commutative rings, *J. Algebra* 319(2008), 1801–1808.

[8] S.M. Moconja and Z.Z. Petrovic, On the structure of comaximal graphs of commutative rings with identity, *Bull. Aust. Math. Soc.,* 83(2011), 11–21.

[9] K. Samei, On the comaximal graph of a commutative ring, *Canad. Math. Bull.,* 57(2)(2014), 413–423.

[10] P.K. Sharma and S.M. Bhatwadekar, A note on graphical representation of rings, *J. Algebra,* 176(1995), 124–127.

[11] S. Visweswaran and Jaydeep Parejiya, When is the complement of the comaximal graph of a commutative ring planar?, *ISRN Algebra,* 2014(2014), 8 pages.

[12] M. Ye and T. Wu, Comaximal ideal Graphs of commutative rings, *J. Algebra Appl.,* 6(2012), 1–09.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆