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More results on generalized closed sets

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Abstract
In this paper discuss some properties of generalized closed sets and introduce some basic concepts in generalized topology.

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Generalized Topology, generalized neighborhoods, generalized neighborhood system, generalized dense, generalized accumulation points, generalized derived set.

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Contents

1. Introduction

In 2002 Császár [2] Introduced the notion of generalized topological spaces (GTS) and generalized continuity in his paper named Generalized topology, generalized continuity. The purpose of present paper is to discuss some basic properties of the generalized closed sets. In section 2, collect all preliminaries and basic definitions useful for subsequent sections. In section 3 discuss some theorems relating generalized closed sets. In section 4 introduce the concepts, generalized neighborhoods and generalized accumulation points and generalized derived set also discuss some results relating to these concepts.

2. Preliminaries

Recall some basic definitions and notions of most essential concepts needed in the following.

Definition 2.1. *([2]) Let X be a set and $exp(X)$ its power set. According to Császár, a subset $\mathscr{G}$, of $exp(X)$ is called generalized topology (GT) on X and $(X,\mathscr{G})$ is called a generalized topological space (GTS) if $\mathscr{G}$ has the following properties.*

(1) $\phi \in \mathscr{G}$

(2) Any union of elements of $\mathscr{G}$ belongs to $\mathscr{G}$

Definition 2.2. *([2]) A GT $\mathscr{G}$ is called strong if $X \in \mathscr{G}$.*

Definition 2.3. *([2]) A subset A is called $\mathscr{G}$-open if $A \in$ mathcalG. A subset B is called $\mathscr{G}$-closed if $X \backslash B$ is $\mathscr{G}$-open. The generalized topology is denoted by $\mathscr{G}$-topology.*

Definition 2.4. *([1]) The $\mathscr{G}$-closure of A is denoted by $C_{\mathscr{G}}(A)$, is the intersection of all $\mathscr{G}$-closed set containing A.*

Theorem 2.5. *([1]) Let $(X,\mathscr{G})$ be a GTS and $A,B \subseteq X$. Then the following statements are hold.*

(1) $x \in C_{\mathscr{G}}(A)$ if and only if $x \in U \in \mathscr{G}$ implies $U \cap A \neq \phi$

(2) If $U,V \in \mathscr{G}$ and $U \cap V = \phi$ then $C_{\mathscr{G}}(U) \cap V = \phi$ and $U \cap C_{\mathscr{G}}(V) = \phi$

Definition 2.6. *([3]) Let $(X,\mathscr{G})$ be a $\mathscr{G}$-topological space and $\mathscr{B}$ be a sub collection of $\mathscr{G}$ is called a base for $\mathscr{G}$-topological space, if every $\mathscr{G}$-open set can be expressed as union of some members of $\mathscr{B}$.*

3. More results on generalized closed sets

Theorem 3.1. *Let $\mathscr{C}$ be the family of all generalized closed sets in a GT $(X,\mathscr{G})$. Then $\mathscr{C}$ has the following properties:*

(i) $X \in \mathscr{C}$

(ii) *$\mathscr{C}$ is closed under arbitrary intersections.*

Conversely, given any set X and a family $\mathscr{C}$ of its subsets which satisfies these two properties, there exist a unique GT $\mathscr{G}$ on X such that $\mathscr{C}$ coincides with the family of generalized closed subsets of $(X,\mathscr{G})$.

Proof. The first part follows from the definition of $\mathscr{G}$-closed sets and De Morgan's laws. Define $\mathscr{G} = \{B \subseteq X : X - B \in \mathscr{C}\}$.
Claim: $\mathscr{G}$ is a GT on X

(1) $\emptyset \in \mathscr{G}$.

(2) Let collection $\{V_\alpha\}$ be an arbitrary collection of members of $\mathscr{G}$. Then for each α, $(X - V_\alpha) \in \mathscr{C}$. To prove that $(U_\alpha V_\alpha)^c \in \mathscr{C}$. We have by DeMorgan's law $(U_\alpha V_\alpha)^c$ $= \cap_\alpha V_\alpha^c \in \mathscr{C}$. Thus $\mathscr{G}$ is closed under arbitrary union. Hence $\mathscr{G}$ is a GT on X. The $\mathscr{G}$-open subsets of X are precisely the complements of $\mathscr{C}$. Also $\mathscr{G}$ is unique.

□

Remark 3.2. *In ordinary topological spaces finite union of closed set is closed. But in GTS finite union of $\mathscr{G}$-closed set need not be $\mathscr{G}$-closed.*

Example 3.3. *$X = \{a,b,c,d\}$ and $\mathscr{G} = \{\emptyset,\{c,d\},\{a,d\},\{a,c,d\}\}$. Then $\mathscr{G}$ is GT on X. Note that $\{a,b\}$ and $\{b,c\}$ are $\mathscr{G}$-closed but their union $\{a,b,c\}$ is not $\mathscr{G}$-closed.*

Theorem 3.4. *Let A, B be subsets of a GTS $(X,\mathscr{G})$. Then the following conditions are hold.*

(i) *$C_\mathscr{G}(A)$ is $\mathscr{G}$-closed in X. More over it is the smallest $\mathscr{G}$-closed subset of X containing A.*

(ii) *A is $\mathscr{G}$-closed in X if and only if $C_\mathscr{G}(A) = A$.*

(iii) $C_\mathscr{G}(C_\mathscr{G}(A)) = C_\mathscr{G}(A)$.

(iv) $C_\mathscr{G}(A) \cup C_\mathscr{G}(B) \subseteq C_\mathscr{G}(A \cup B)$

Proof. (i) Using above theorem $C_\mathscr{G}(A)$ is $\mathscr{G}$-closed in X. By definition of generalized closure $C_\mathscr{G}(A)$ is the smallest $\mathscr{G}$-closed subset of X containing A.

(ii) Suppose A is $\mathscr{G}$-closed. Since $C_\mathscr{G}(A)$ is the smallest $\mathscr{G}$-closed subset of X containing A, we get $C_\mathscr{G}(A) = A$. Converse is trivial.

(iii) Since $C_\mathscr{G}(A)$ is $\mathscr{G}$-closed in X, from second part $C_\mathscr{G}(C_\mathscr{G}(A)) = C_\mathscr{G}(A)$.

(iv) If $A_1 \subseteq A_2$ then $C_\mathscr{G}(A_1) \subseteq C_\mathscr{G}(A_2)$. Since $A \subseteq A \cup B \Rightarrow C_\mathscr{G}(A) \subseteq C_\mathscr{G}(A \cup B)$. Similarly $C_\mathscr{G}(B) \subseteq C_\mathscr{G}(A \cup B)$. Hence $C_\mathscr{G}(A) \cup C_\mathscr{G}(A) \subseteq C_\mathscr{G}(A \cup B)$.

□

Remark 3.5. *Let A and B are subsets of a GTS then $C_\mathscr{G}(A \cup B)$ need not be contained in $C_\mathscr{G}(A) \cup C_\mathscr{G}(B)$.*

Example 3.6. *$X = \{a,b,c,d\}$ and $\mathscr{G} = \{\emptyset,\{c,d\},\{a,d\},\{a,c,d\}\}$. Then $\mathscr{G}$ is GT on X. Let $A = \{a,b\}$ and $B = \{b,c\}$. Then $C_\mathscr{G}(A) = \{a,b\}$ and $C_\mathscr{G}(B) = \{b,c\}$ but $C_\mathscr{G}(A \cup B) = X$.*

Definition 3.7. *Let $(X,\mathscr{G})$ be a GTS. The generalized closure operator associated with it is defined as the function $c : P(X) \to P(X)$ such that $c(A) = C_\mathscr{G}(A)$, $A \in P(X)$.*

Remark 3.8. *The generalized closure operator has the following properties.*

(i) *The fixed points of c are precisely the $\mathscr{G}$-closed subset of X.*

(ii) *c is idempotent .i.e $coc = c$ or in other words for any $A \in P(X)$, $c(c(A)) = c(A)$.*

Theorem 3.9. *Let X be a set, $\theta : P(X) \to P(X)$ a function such that*

(i) *For any $A \in P(X)$, $A \subseteq \theta(A)$.*

(ii) *θ is idempotent.*

(iii) $\theta(A) \cup \theta(B) \subseteq \theta(A \cup B)$.

Then there exist a unique generalized topology on X such that θ coincides with the generalized closure operator associated with . Conversely any generalized closure operator satisfies these properties.

Proof. Let $\mathscr{C} = \{A \subseteq X : \theta(A) = A\}$.
Claim: $\mathscr{C}$ satisfies hypothesis in Theorem 3.1.

(i) Since $X \subseteq \theta(X)$. Hence $\theta(X) = X$. So $X \in \mathscr{C}$.

(ii) Let $\{A_\alpha$ be any collection of members in $\mathscr{C}$. θ is monotonic because, if $A \subseteq B$, then $B = A \cup (B - A)$ implies $\theta(B) = \theta(A \cup (B - A)) \subseteq \theta(A) \cup \theta(B - A) \Rightarrow \theta(A) \subseteq \theta(B)$. Since $\cap_\alpha \alpha A_\alpha \subseteq A_\alpha$, for each α. Then $\theta(\cap_\alpha \alpha A_\alpha) \subseteq \theta(A_\alpha)$, for each α. That is $\theta(\cap_\alpha \alpha A_\alpha) \subseteq A_\alpha$, for each α. So $\theta(\cap_\alpha \alpha A_\alpha) \subseteq \cap_\alpha \alpha A_\alpha$. Using first property of θ, we have $\cap_\alpha \alpha A_\alpha \subseteq \theta(\cap_\alpha \alpha A_\alpha)$. Hence $\theta(\cap_\alpha \alpha A_\alpha) = \cap_\alpha \alpha A_\alpha$. So $\cap_\alpha A_\alpha \in \mathscr{C}$. Then by Theorem 3.1,there is a unique GT $\mathscr{G}$ on X such that $\mathscr{C}$ coincides with the family of $\mathscr{G}$-closed subsets of $(X,\mathscr{G})$. It remains to verified that the generalized closure operator associated with $\mathscr{G}$ coincides with θ. Let $A \subseteq X$, to prove that $\theta(A) = C_\mathscr{G}(A)$. We have $C_\mathscr{G}(A)$ is the intersection of all $\mathscr{G}$-closed subsets of X containing A. But $\mathscr{G}$-closed subsets of X with respect to $\mathscr{G}$ are the members of $\mathscr{C}$. Therefore $C_\mathscr{G}(A) = \cap\{B \subseteq X : A \subseteq B, \theta(B) = B\}$. Now whenever $A \subseteq B$, $\theta(A) \subseteq \theta(B)$. So if $A \subseteq B$ and

$\theta(B) = B$ implies $\theta(A) \subseteq B$. But $C_{\mathscr{G}}(A)$ is the intersection of such B's and so $C_{\mathscr{G}}(A) \supseteq \theta(A)$. Using condition (ii), $\theta(A) \in \mathscr{C}$. From (i) $A \subseteq \theta(A)$. That is $\theta(A)$ is a $\mathscr{G}$-closed set containing A, but $C_{\mathscr{G}}(A)$ is the smallest $\mathscr{G}$-closed set containing A. Hence $C_{\mathscr{G}}(A) \subseteq \theta(A)$. Hence $C_{\mathscr{G}}(A) = \theta(A)$.

□

Definition 3.10. *Let A be a subset of a GTS. Then A is said to generalized dense (or $\mathscr{G}$-dense) in X if $C_{\mathscr{G}}(A) = X$.*

Example 3.11. *Let $X = \{a,b,c,d\}$ and $\mathscr{G} = \{\emptyset, \{a,b\}\}$, and $A = \{b,c\}$ then $C_{\mathscr{G}}(A) = X$. So A is $\mathscr{G}$-dense in X.*

Theorem 3.12. *A subset A of a GTS X is $\mathscr{G}$-dense in X if and only if A intersect every nonempty $\mathscr{G}$-open subsets of X.*

Proof. Suppose A is $\mathscr{G}$-dense in X and B is a nonempty $\mathscr{G}$-open subset of X. If $A \cap B = \phi \Rightarrow A \subseteq X - B \Rightarrow C_{\mathscr{G}}(A) \subseteq X - B$. But $X - B$ is a proper nonempty subset of X. Contradicting $C_{\mathscr{G}}(A) = X$.
Conversely suppose A intersect every nonempty $\mathscr{G}$-open subsets of X. So the only $\mathscr{G}$-closed set containing A is X. So $C_{\mathscr{G}}(A) = X$. □

4. Generalized neighborhoods and generalized accumulation points

Definition 4.1. *Let $(X, \mathscr{G})$ be a GTS, $x_0 \in X$ and $N \subseteq X$. Then N is said to be a generalized neighborhood ($\mathscr{G}$- neighborhood) of x_0, if there is a $\mathscr{G}$-open set V such that $x_0 \in V$ and $V \subseteq N$.*

Example 4.2. *Let $X = \{a,b,c,d\}$ and $\mathscr{G} = \{\emptyset, \{a,b\}\}$. Let $N = \{a,b,c\}$. Then N is a generalized neighborhood of both a and b.*

Remark 4.3. *In ordinary topological space every element has a neighborhood because the whole set is a neighborhood is a neighborhood of each of its points. But in a GTS every point need not have a $\mathscr{G}$-neighborhood, in a strong GTS every point has a $\mathscr{G}$-neighborhood.*

Theorem 4.4. *A subset A of a GTS X is $\mathscr{G}$-open if and only if it is a $\mathscr{G}$-neighborhood of each of its points.*

Proof. Let X be a GTS and $G \subseteq X$. Suppose G is $\mathscr{G}$-open. By definition of $\mathscr{G}$-neighborhood, G is a $\mathscr{G}$-neighborhood of each of its points.
Conversely suppose G is a $\mathscr{G}$-neighborhood of each of its points. So for each $x \in G$ there exist $\mathscr{G}$-open set V_x such that $x \in V_x \subseteq G$. Hence $G = \cup_{x \in G} V_x \Rightarrow G$ $\mathscr{G}$-open. □

Definition 4.5. *Let $(X, \mathscr{G})$ be a GTS. Let $\mathscr{N}_x$ be the set of all $\mathscr{G}$- neighborhoods of x in X. The family $\mathscr{N}_x$ is called the generalized neighborhood system at x.*

Example 4.6. *Let $X = \{a,b,c,d\}$ and $\mathscr{G} = \{\emptyset, \{a,b\}\}$. Here $\mathscr{N}_c = \emptyset$.*

Remark 4.7. *In ordinary topological spaces the neighborhood system must be nonempty, but the generalized neighborhood system may be empty.*

Theorem 4.8. *Let X be a GTS and for $x \in X$ such that $\mathscr{N}_x$ is nonempty. Then*

(1) If $U \in \mathscr{N}_x$ then $x \in U$.

(2) If $V \in \mathscr{N}_x$ and $U \supseteq V$ then $U \in \mathscr{N}_x$.

(3) A set G is $\mathscr{G}$-open in X if and only if $G \in \mathscr{N}_x$ for all $x \in G$.

(4) If $U \in \mathscr{N}_x$ then there exist $V \in \mathscr{N}_x$ such that $V \subseteq U$ and $V \in \mathscr{N}_y$ for all $y \in V$

Proof. (1) It is trivial.

(2) $V \in \mathscr{N}_x \Rightarrow V$ is a $\mathscr{G}$-neighborhood of x. So there exist a $\mathscr{G}$-open set W such that $x \in W \subseteq V$. Since $U \supseteq V \Rightarrow x \in W \subseteq V \subseteq U \Rightarrow U \in \mathscr{N}_x$.

(3) It is clear from Theorem 4.4.

(4) If $U \in \mathscr{N}_x$ then U is a $\mathscr{G}$-neighborhood of x. So there exist $\mathscr{G}$-open set V such that $x \in V \subseteq U$. By Theorem 4.4 V is a $\mathscr{G}$-neighborhood of each of its points, in particular it is a $\mathscr{G}$-neighborhood of x. That is $V \in \mathscr{N}_x$ such that $V \subseteq U$ and $V \in \mathscr{N}_y$ for all $y \in V$.

□

Theorem 4.9. *Let X be a set and suppose for each $x \in X$, $\mathscr{N}_x$ be a family of subsets of X satisfying following properties*

(1) If $U \in \mathscr{N}_x$ then $x \in U$.

(2) If $V \in \mathscr{N}_x$ and $U \supseteq V$ then $U \in \mathscr{N}_x$.

(3) If $U \in \mathscr{N}_x$ then there exist $V \in \mathscr{N}_x$ such that $V \subseteq U$ and $V \in \mathscr{N}_y$ for all $y \in V$.

Then there exist a unique GTS $\mathscr{G}$ on X such that for each $x \in X$, $\mathscr{N}_x$ coincides with the family of all $\mathscr{G}$-neighborhoods of x with respect to $\mathscr{G}$.

Proof. Let $\mathscr{G} = \{U \subseteq X : U \in \mathscr{N}_x, \forall x \in U\}$.
Claim: $\mathscr{G}$ is GT on X.
Clearly $\phi \in \mathscr{G}$. Let $\{A_\alpha\}$ be an arbitrary collection of members of $\mathscr{G}$. By definition of $\mathscr{G}$, each A_α is a member of $\mathscr{N}_x$. By condition (2) $U_\alpha A_\alpha \in \mathscr{N}_x$, $\forall x \in U_\alpha A_\alpha$. Hence $\mathscr{G}$ is a GT on X.
Claim: For any $x \in X$, the generalized neighborhood system of x with respect to is coincides with $\mathscr{N}_x$.
Let $U \in \mathscr{N}_x$, by (3) there exist $V \in \mathscr{N}_x$ such that $V \subseteq U$ and $V \in \mathscr{N}_y$, $\forall y \in V$. So $V \in \mathscr{G} \Rightarrow U$ is a $\mathscr{G}$-neighborhood of x. Conversely le U is a $\mathscr{G}$-neighborhood of x. So there exist $V \in \mathscr{G}$, such that $x \in V \subseteq U$. Hence $V \in \mathscr{N}_x$. From (2) $U \in \mathscr{N}_x$. □

Definition 4.10. *Let A be a subset of a GTS X and $y \in X$. Then y is said to be a generalized accumulation point($\mathscr{G}$-accumulation point) of A if every $\mathscr{G}$-open set containing y contains at least one point of A other than y.*

Example 4.11. *Let $\mathscr{G}$ be the GT on the set of real numbers generated by $\{(a,\infty),(-\infty,b) : a,b \in \mathbb{R}\}$. Then every real number is a generalized accumulation point of the set of natural numbers $\mathbb{N}$.*

Definition 4.12. *Let A be a subset of a GTS X, then the generalized derived set of A is the set of all a generalized accumulation point of A. It is denoted by $A^{'\mathscr{G}}$.*

Theorem 4.13. *Let A be a subset of a GTS X, then $C_{\mathscr{G}}(A) = A \cup A^{'\mathscr{G}}$.*

Proof. **Claim:** $A \cup A^{'\mathscr{G}}$ is $\mathscr{G}$-closed. It is enough to prove that $X - (A \cup A^{'\mathscr{G}})$ is $\mathscr{G}$-open.
Let $y \in X - (A \cup A^{'\mathscr{G}})$. Then y is not a $\mathscr{G}$-accumulation point of A. So there is a $\mathscr{G}$-open set V containing y such that V contains no point of A except possibly y. But $y \notin A$. So $A \cap V = \emptyset$. Next claim that $A^{'\mathscr{G}} \cap V = \emptyset$. For let $z \in A^{'\mathscr{G}} \cap V$. So z is a $\mathscr{G}$-accumulation point of A and V is a $\mathscr{G}$-open set containing y. Hence $A \cap V \neq \emptyset$. Which is a contradiction. So $A^{'\mathscr{G}} \cap V = \emptyset$ and $V \subseteq X(A \cup A^{'\mathscr{G}})$. Hence $A \cup A^{'\mathscr{G}}$ is $\mathscr{G}$-closed. There for $C_{\mathscr{G}}(A) \subseteq A \cup A^{'\mathscr{G}}$. For the reverse inclusion claim that $A^{'\mathscr{G}} \subseteq C_{\mathscr{G}}(A)$. Let $y \in A^{\mathscr{G}}$. If $y \notin C_{\mathscr{G}}(A)$ then $X - C_{\mathscr{G}}(A)$ is a $\mathscr{G}$-open set containing y. But y is $\mathscr{G}$-accumulation point of A, so $A \cap (X - C_{\mathscr{G}}(A)) \notin \emptyset$. Which is a contradiction since $(X - C_{\mathscr{G}}(A)) \subseteq (X - A)$. Hence $y \in C_{\mathscr{G}}(A)$. □

5. Conclusion

In this paper we found that some results in ordinary topological space do not hold in generalized topological space and also proved some basic results in generalized topological spaces.

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