Malaya Journal of Matematik, Vol. 8, No.3, 1162-1170, 2020

https://doi.org/10.26637/MJM0803/0075

Planarity of a unit graph: Part -II $|Max(R)| = 2$ case

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Abstract

The rings considered in this article are commutative with identity $1 \neq 0$. Recall that the unit graph of a ring R is a simple undirected graph whose vertex set is the set of all elements of the ring R and two distinct vertices x, y are adjacent in this graph if and only if $x+y \in U(R)$ where $U(R)$ is the set of all unit elements of ring R. We denote this graph by $UG(R)$. In this article we classified rings R with $|Max(R)| = 2$ such that $UG(R)$ is planar.

Keywords

Planar graph, (Ku_1^*) and (Ku_2^*).

2010 Mathematics Subject Classification

13A15, 05C25.

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Article History: Received **24** March **2020**; Accepted **12** July **2020**

Contents

1. Introduction

We first recall the following definitions and results from graph theory. A graph G=(V,E) is said to be complete if every pair of distinct vertices of G are adjacent in G. A complete graph on n vertices is denoted by K_n [4, Definition 1.1.11]. A graph G=(V,E) is said to be bipartite if the vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and other in Y. The pair (X,Y) is called a bipartition of G. A bipartite graph G with bipartition (X,Y) is denoted by G(X,Y). A bipartite graph G(X,Y) is said to be complete if each vertex of X is adjacent to all the vertices of Y. If G(X,Y) is a complete bipartite graph with $|X| = m$ and $|Y| = n$, then it is denoted by $K_{m,n}$ [4, Definition 1.1.12]. Let G=(V,E) be a graph.By a clique of G, we mean a complete subgraph of G [4, Definition 1.2.2]. We say that the clique number of G equals n if n is the largest positive integer such that K_n is a subgraph of G [4, p.185]. The clique number of a graph G is denoted by the notation $\omega(G)$. If G contains K_n as a subgraph for all $n \geq 1$, then we set $\omega(G) = \infty$.

A graph G is said to be planar if it can be drawn in a plane in such a way that no two edges of G intersect in a point other than a vertex of G [4, Definition 8.1.1]. Two adjacent edges of a graph G are said to be in series if their common vertex is of degree two [5, p.9]. Two graphs are said to be homeomorphic if one graph can be obtained from the other graph by the creation of edges in series (i.e by insertion of vertices of degree two) or by the merger of edges in series[5, p.100]. Recall from [5, p.93] that K_5 is referred to as Kuratowski's first graph and $K_{3,3}$ is referred to as Kuratowski's second graph. A celebrated theorem of Kuratowski says that a necessary and sufficient condition for a graph G to be planar is that G does not contain either of Kuratowski's two graphs or any graph homeomorphic to either of them [5, Theorem 5.9].

In view of Kuratowski's Theorem, [5, Theorem 5.9] we introduce the following definitions. We say that a graph G=(V,E) satisfies Ku_1 if G does not contain K_5 as a subgraph and we say that graph G=(V,E) satisfies Ku_2 if G does not contain $K_{3,3}$ as a subgraph. We say that a graph G = (V,E) satisfies Ku_1^* if G satisfies Ku_1 and moreover, G does not contain any subgraph homeomorphic to K_5. We say that a graph G = (V,E) satisfies Ku_2^* if G satisfies Ku_2 and moreover, G does not contain any subgraph homeomorphic to $K_{3,3}$.

If a graph G is planar, then it follows from Kuratowski's theorem [5, Theorem 5.9] that G satisfies both Ku_1^* and Ku_2^* . Hence G satisfies both Ku_1 and Ku_2. It is interesting to note that a graph G may be nonplanar even if it satisfies both Ku_1 and Ku_2. For example of this type refer [5, Figure 5.9(a), p.101] and the graph G in this example does not satisfies Ku_2^*. We do not know an example of a graph G such that G satisfies Ku_1 but G does not satisfy Ku_1^*.

Let R be a ring. In Section 2 of this article we proved some important results regarding planarity of $UG(R)$ with the assumption that R is semiquasilocal ring. In Remark 2.4 we have proved that if R is semiquasilocal ring and if $UG(R)$ satisfies Ku_2, then R must be finite. In Example 2.5 we gave an Example to show that Remark 2.4 can fail to hold if the hypothesis semiquasilocal is omitted. It is natural to know whether $UG(R)$ satisfies (Ku_1) implies that R is finite. We showed in Corollary 2.9 that if $2 \in U(R)$ and if $UG(R)$ satisfies (Ku_1), then R is finite.

Let R be a ring. With the hypothesis that R is a finite ring, a classification of finite rings R such that $UG(R)$ is planar was given in [2, Theorem 5.14]. In Section 3, we assume that R is semiquasilocal and we show that if $UG(R)$ is planar, then R is necessarily finite. Indeed, In Theorem 3.25 we proved some stronger condition that $UG(R)$ is planar if and only if it satisfies Ku_2^*.

The rings considered in this article are commutative with identity and are nonzero. A ring R which has a unique maximal ideal is referred to as a quasilocal ring. A ring R which has only a finite number of maximal ideals is referred to as a semiquasilocal ring. A Noetherian quasilocal (respctively, semiquasilocal) ring is referred to as a local (respectively, semilocal) ring. We denote the set of all maximal ideals of a ring R by $Max(R)$. We used $J(R)$ to denote Jacobson radical of ring R.

2. Some preliminary results

Let R be a ring. In this section we proved some basic results regarding planarity with the assumption that R is semiquasilocal ring.

Let R be a semiquasilocal ring such that $|Max(R)| \geq 2$. We next try to classify such rings R in order that $UG(R)$ is planar.

Lemma 2.1. *Let R be a semiquasilocal ring with $|Max(R)| = n \geq 2$. If $UG(R)$ satisfies (Ku_2), then there exist nonzero rings R_1 and R_2 such that $R \cong R_1 \times R_2$ as rings.*

Proof. Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ denote the set of all maximal ideals of R. It follows from [3, Proposition 1.11(ii)] that $\prod_{i=2}^{n} \mathfrak{m}_i \not\subseteq \mathfrak{m}_1$. Hence, for each $i \in \{2, \ldots, n\}$, there exists $x_i \in \mathfrak{m}_i$ such that $\prod_{i=2}^{n} x_i \notin \mathfrak{m}_1$. Therefore, $\mathfrak{m}_1 + R(\prod_{i=2}^{n} x_i) = R$. Thus for each $i \in \{1, 2, \ldots, n\}$, there exists $a_i \in \mathfrak{m}_i$ such that $a_1 + \prod_{i=2}^{n} a_i = 1$. For convenience, let us denote a_1 by a and $\prod_{i=2}^{n} a_i$ by b. Observe that for any $r, s \in \mathbb{N}$, $a^r + b^s \notin \mathfrak{m}_i$ for any $i \in \{1, 2, \ldots, n\}$. Hence, $a^r + b^s \in U(R)$ for any $r, s \in \mathbb{N}$. We claim that R admits a nontrivial idempotent. If either $a^i = a^j$ for some distinct $i, j \in \{1, 2, 3\}$ or $b^i = b^j$ for some distinct $i, j \in \{1, 2.3\}$, then it follows that R admits a nontrivial idempotent. Suppose that $a^i \neq a^j$ and $b^i \neq b^j$ for all distinct $i, j \in \{1, 2, 3\}$. Let $V_1 = \{a, a^2, a^3\}$ and let $V_2 = \{b, b^2, b^3\}$. Note that $V_1 \cup V_2 \subseteq V(UG(R))$ and $V_1 \cap V_2 = \emptyset$. For any $x \in V_1$ and for any $y \in V_2$, $x + y \in U(R)$ and so, x and y are adjacent in $UG(R)$. It is clear that V_i is an independent set of $UG(R)$ for each $i \in \{1, 2\}$ and so, the subgraph of $UG(R)$ induced on $V_1 \cup V_2$ is a $K_{3,3}$. This is in contradiction to the assumption that $UG(R)$ satisfies (Ku_2). Therefore, there exists an idempotent element $e \in R \backslash \{0, 1\}$. The mapping $f : R \to Re \times R(1-e)$ defined by $f(r) = (re, r(1-e))$ is an isomorphism of rings. Let us denote the ring Re by R_1 and $R(1-e)$ by R_2. It is clear that R_1 and R_2 are nonzero rings and $R \cong R_1 \times R_2$ as rings. □

Lemma 2.2. *Let T_1, T_2 be nonzero rings and let $T = T_1 \times T_2$. If $UG(T)$ satisfies (Ku_2), then $UG(T_i)$ satisfies (Ku_2) for each $i \in \{1, 2\}$.*

Proof. We first verify that $UG(T_1)$ satisfies (Ku_2). Suppose that $UG(T_1)$ does not satisfy (Ku_2). Then there exist distinct elements $a_1, a_2, a_3, b_1, b_2, b_3 \in T_1$ such that $a_i + b_j \in U(T_1)$ for all $i, j \in \{1, 2, 3\}$. Let $V_1 = \{(a_1, 0), (a_2, 0), (a_3, 0)\}$ and let $V_2 = \{(b_1, 1), (b_2, 1), (b_3, 1)\}$. Note that $V_1 \cup V_2 \subseteq V(UG(T))$ and $V_1 \cap V_2 = \emptyset$. As $a_i + b_j \in U(T_1)$ for all $i, j \in \{1, 2, 3\}$ and $0 + 1 = 1 \in U(T_2)$, we get that for any $x \in V_1$ and $y \in V_2$, $x + y \in U(T)$ and so, the subgraph of $UG(T)$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. This contradicts the assumption that $UG(T)$ satisfies (Ku_2). Therefore, $UG(T_1)$ satisfies (Ku_2). Similarly, it can be shown that $UG(T_2)$ satisfies (Ku_2). □

Proposition 2.3. *Let R be a semiquasilocal ring such that $|Max(R)| = n \geq 2$. If $UG(R)$ satisfies (Ku_2), then there exists a quasilocal ring $(R_i, \mathfrak{m}_i)$ for each $i \in \{1, 2, \ldots, n\}$ such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings*

Proof. We prove this proposition using induction on $|Max(R)| = n \geq 2$. Suppose that $|Max(R)| = 2$. As $UG(R)$ satisfies (Ku_2), we obtain from Lemma 2.1 that there exist nonzero rings R_1 and R_2 such that $R \cong R_1 \times R_2$ as rings. Since $|Max(R)| = 2$, it follows that R_i is a quasilocal ring for each $i \in \{1, 2\}$. Suppose that $|Max(R)| = n \geq 3$. We know from Lemma 2.1 that there exist nonzero rings T_1 and T_2 such that $R \cong T_1 \times T_2$ as rings. It is clear that both T_1 and T_2 are semiquasilocal rings. We know from Lemma 2.2 that $UG(T_i)$ satisfies (Ku_2) for each $i \in \{1, 2\}$. Let $|Max(T_i)| = n_i$ for each $i \in \{1, 2\}$. Observe that $1 \leq n_i < n$ for each $i \in \{1, 2\}$ and $n_1 + n_2 = n$. It follows from the induction hypothesis that there exist quasilocal rings $R_{11}, \ldots, R_{1n_1}, R_{21}, \ldots, R_{2n_2}$ such that $T_1 \cong R_{11} \times \cdots \times R_{1n_1}$ as rings and $T_2 \cong R_{21} \times \cdots \times R_{2n_2}$ as rings. Therefore, $R \cong R_{11} \times \cdots \times R_{1n_1} \times R_{21} \times \cdots \times R_{2n_2}$ as rings. After a change of notation, we arrive at the conclusion that for each $i \in \{1, 2, \ldots, n\}$, there exists a quasilocal ring $(R_i, \mathfrak{m}_i)$ such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings. □

Remark 2.4. *Let R be a semiquasilocal ring. If $UG(R)$ satisfies (Ku_2), then R is finite.*

Proof. We consider the following cases.
Case(i) R is quasilocal

Since $UG(R)$ satisfies (Ku_2) by assumption, we obtain from $(iv) \Rightarrow (v)$ of [9, Theorem 2.5] that R is finite. Indeed, $|R| \in \{2,3,4,5\}$.
Case(ii) R is not quasilocal

Let $n \geq 2$ be the number of maximal ideals of R. Since $UG(R)$ satisfies (Ku_2) by assumption, it follows from Proposition 2.3 that for each $i \in \{1,2,\ldots,n\}$, there exists a quasilocal ring $(R_i, \mathfrak{m}_i)$ such that $R \cong R_1 \times R_2 \times \cdots \times R_n$ as rings. We know from Lemma 2.2 that $UG(R_i)$ satisfies (Ku_2) for each $i \in \{1,2,\ldots,n\}$. Therefore, we obtain from Case(i) that $|R_i| \in \{2,3,4,5\}$ for each $i \in \{1,2,\ldots,n\}$. Therefore, we get that R is finite. □

We provide an example in Example 2.5 to illustrate that Remark 2.4 can fail to hold if the hypothesis that R is semiquasilocal is omitted.

Example 2.5. *$UG(\mathbb{Z})$ satisfies (Ku_2).*

Proof. Let $a \in \mathbb{Z}$. If $b \in \mathbb{Z}$ is such that a and b are adjacent in $UG(\mathbb{Z})$, then $a + b \in U(\mathbb{Z}) = \{1,-1\}$. This implies that the set of all neighbors of a in $UG(\mathbb{Z})$ equals $\{1-a, -1-a\}$. Hence, we get that $UG(\mathbb{Z})$ satisfies (Ku_2). □

Let R be a semiquasilocal ring such that $UG(R)$ satisfies (Ku_1). It is natural to know whether $UG(R)$ satisfies (Ku_1) implies that R is finite. We prove in Corollary 2.9 that if $2 \in U(R)$ and if $UG(R)$ satisfies (Ku_1), then R is finite. We provide in Example 2.10 an example of an infinite local ring $(R, \mathfrak{m})$ such that $\omega(UG(R)) = 2$.

Lemma 2.6. *Let F be a field. Then $\omega(UG(F)) < \infty$ if and only if F is finite.*

Proof. Assume that $\omega(UG(F)) < \infty$. If $char(F) = 2$, then we know from [2, Theorem 3.4] that $UG(F)$ is complete. As $\omega(UG(F)) < \infty$ and $V(UG(F)) = F$, we obtain that F is finite. Hence, we can assume that $char(F) \neq 2$. Let $\omega(UG(F)) = t$. Let $A = \{\alpha_i | i \in \{1,2,\ldots,t\}\} \subseteq F$ be such that the subgraph of $UG(F)$ induced on A is a clique. We can assume without loss of generality that $\alpha_1 = 0$. Let $\beta \in F \backslash A$. Then the subgraph of $UG(F)$ induced on $A \cup \{\beta\}$ is not a clique. Hence, $\beta + \alpha_i = 0$ for some $i \in \{2,\ldots,t\}$. Therefore, $F = \{\alpha_1, \alpha_2, \ldots, \alpha_t\} \cup \{-\alpha_i | i \in \{2,\ldots,t\}\}$. This proves that F is finite.

Conversely, if F is finite, then it is clear that $\omega(UG(F)) < \infty$. □

Lemma 2.7. *Let R_1, R_2 be nonzero rings and let $R = R_1 \times R_2$. If $2 \in U(R)$ and if $\omega(UG(R)) < \infty$, then $\omega(UG(R_i)) < \infty$ for each $i \in \{1,2\}$.*

Proof. As $2 \in U(R)$, it follows that $2 \in U(R_i)$ for each $i \in \{1,2\}$. Let $A_1 \subseteq R_1$ be such that the subgraph of $UG(R_1)$ induced on A_1 is a clique. Let $A = \{(x,1) | x \in A_1\}$. Since $2 \in U(R_2)$, it follows that the subgraph of $UG(R)$ induced on A is a clique. Therefore, $|A_1| = |A| \leq \omega(UG(R))$. This proves that $\omega(UG(R_1)) \leq \omega(UG(R)) < \infty$. Similarly, it follows that $\omega(UG(R_2)) \leq \omega(UG(R)) < \infty$. □

Proposition 2.8. *Let R be a semiquasilocal ring such that $2 \in U(R)$. If $\omega(UG(R)) < \infty$, then R is finite.*

Proof. Since $2 \in U(R)$, we obtain from [2, Lemma 2.7(c)] that the subgraph of $UG(R)$ induced on $\{1 + x | x \in J(R)\}$ is a clique. As we are assuming that $\omega(UG(R)) < \infty$, it follows that $J(R)$ is finite. We are assuming that R is semiquasilocal. Let $|Max(R)| = n$ and let $\{\mathfrak{m}_i | i \in \{1,\ldots,n\}\}$ denote the set of all maximal ideals of R. First, we assert that $\omega(UG(\frac{R}{J(R)})) < \infty$. If $x, y \in R$ are such that $x + J(R)$ and $y + J(R)$ are adjacent in $UG(\frac{R}{J(R)})$, then it is known that x and y are adjacent in $UG(R)$ [2, Lemma 2.7(a)]. Hence, we obtain that $\omega(UG(\frac{R}{J(R)})) < \infty$. Suppose that $n = 1$. In such a case, $\frac{R}{J(R)} = \frac{R}{\mathfrak{m}_1}$ is a field and so, we obtain from Lemma 2.6 that $\frac{R}{J(R)}$ is finite. Hence, R is finite. Suppose that $n \geq 2$. As $\mathfrak{m}_i + \mathfrak{m}_j = R$ for all distinct $i, j \in \{1,2,\ldots,n\}$, we obtain from the Chinese remainder theorem [3, Proposition 1.10(ii) and (iii)] that the mapping $f : R \to \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2} \times \cdots \times \frac{R}{\mathfrak{m}_n}$ defined by $f(r) = (r + \mathfrak{m}_1, r + \mathfrak{m}_2, \ldots, r + \mathfrak{m}_n)$ is a surjective homomorphism of rings with $Kerf = \cap_{i=1}^{n} \mathfrak{m}_i = J(R)$. Therefore, we obtain from the fundamental theorem of homomorphism of rings that $\frac{R}{J(R)} \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2} \times \cdots \times \frac{R}{\mathfrak{m}_n}$ as rings. Let us denote the field $\frac{R}{\mathfrak{m}_i}$ by F_i for each $i \in \{1,2,\ldots,n\}$. Now, $\frac{R}{J(R)} \cong F_1 \times F_2 \times \cdots \times F_n$ as rings. As $2 \in U(\frac{R}{J(R)})$ and $\omega(UG(\frac{R}{J(R)})) < \infty$, we obtain from Lemma 2.7 that $\omega(UG(F_i)) < \infty$ for each $i \in \{1,2,\ldots,n\}$. Therefore, we obtain from Lemma 2.6 that F_i is finite for each $i \in \{1,2,\ldots,n\}$. Hence, $\frac{R}{J(R)}$ is finite and so, R is finite. □

Corollary 2.9. *Let R be a semiquasilocal ring such that $2 \in U(R)$. If $UG(R)$ satisfies (Ku_1), then R is finite.*

Proof. If $UG(R)$ satisfies (Ku_1), then $\omega(UG(R)) \leq 4$ and so, we obtain from Proposition 2.8 that R is finite. □

Example 2.10. *Let $R = \mathbb{Z}_2[[X]]$ be the power series ring in one variable X over $\mathbb{Z}_2$. Then $\omega(UG(R)) = 2$.*

Proof. It is well-known that $R = \mathbb{Z}_2[[X]]$ is a discrete valuation ring. We know from [3, Exercise 5(i), page 11] that $U(R) = \{1 + Xf(X) | f(X) \in R\}$. Observe that the subgraph of $UG(R)$ induced on $\{0,1\}$ is a clique. Hence, $\omega(UG(R)) \geq 2$. We claim that $\omega(UG(R)) \leq 2$. Suppose that $\omega(UG(R)) \geq 3$. Then there exist $r_1, r_2, r_3 \in R$ such that the subgraph of $UG(R)$ induced on $\{r_1, r_2, r_3\}$ is a clique. Note that $r_1 + r_2 \in U(R)$ and $r_1 + r_3 \in U(R)$. Hence, $r_1 + r_2 = 1 + Xf(X)$ and $r_1 + r_3 = 1 + Xg(X)$ for some $f(X), g(X) \in R$. Since $char(R) = 2$, we obtain that $r_2 + r_3 = X(f(X) + g(X))$. This implies that $r_2 + r_3 \notin U(R)$. This is a contradiction. Therefore, $\omega(UG(R)) \leq 2$ and so, $\omega(UG(R)) = 2$. It is clear that R is an infinite local domain with $\mathfrak{m} = RX$ as its unique maximal ideal. □

3. Classification of rings R with $|Max(R)| = 2$ in order that $UG(R)$ is planar

Let R be a ring such that $|Max(R)| = 2$. The aim of this section is to classify such rings in order that $UG(R)$ is planar.

Remark 3.1. *Let R be a ring such that $|Max(R)| = 2$. We try to classify R in order that $UG(R)$ is planar. Suppose that $UG(R)$ is planar. Then we know from [5, Theorem 5.9] that $UG(R)$ satisfies (Ku_2). Therefore, we obtain from Proposition 2.3 and Remark 2.4 that there exist finite local rings $(R_1, \mathfrak{m}_1)$ and $(R_2, \mathfrak{m}_2)$ such that $R \cong R_1 \times R_2$ as rings. We know from Lemma 2.2 that $UG(R_i)$ satisfies (Ku_2) for each $i \in \{1, 2\}$. It now follows from $(iv) \Rightarrow (v)$ of [9, Theorem 2.5] that R_i is isomorphic to one of the rings from the collection $\mathscr{B} = \{\mathbb{Z}_2, \mathbb{F}_4, \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$ for each $i \in \{1, 2\}$.*

Lemma 3.2. *Let R_1, R_2 be rings and let $R = R_1 \times R_2$. If $|U(R_i)| \geq 3$ for each $i \in \{1, 2\}$, then $UG(R)$ does not satisfy (Ku_2).*

Proof. We are assuming that $|U(R_i)| \geq 3$ for each $i \in \{1, 2\}$. Let $u_1, u_2 \in U(R_1)\backslash\{1\}$ and let $v_1, v_2 \in U(R_2)\backslash\{1\}$. Let $V_1 = \{(1, 0), (u_1, 0), (u_2, 0)\}$ and let $V_2 = \{(0, 1), (0, v_1), (0, v_2)\}$. It is clear that $V_1 \cup V_2 \subseteq V(UG(R))$ and $V_1 \cap V_2 = \emptyset$. For any $u \in U(R_1)$ and for any $v \in U(R_2)$, $(u, 0) + (0, v) = (u, v) \in U(R)$. Thus for any $x \in V_1$ and $y \in V_2$, $x + y \in U(R)$ and so, x and y are adjacent in $UG(R)$. Note that V_i is an independent set of $UG(R)$ for each $i \in \{1, 2\}$. Therefore, the subgraph of $UG(R)$ induced on $V_1 \cup V_2$ is a $K_{3,3}$. Hence, we obtain that $UG(R)$ does not satisfy (Ku_2). □

Lemma 3.3. *Let R_1, R_2 be rings and let $R = R_1 \times R_2$. Suppose that there exist $a \in R_1, b \in R_2$ such that $2a = 0, 2b = 0, 1 + a \in U(R_1)$, and $1 + b \in U(R_2)$. Then $UG(R)$ does not satisfy (Ku_2).*

Proof. Let $V_1 = \{(0, 0), (0, b), (a, b)\}$ and let $V_2 = \{(1, 1), (1, 1 + b), (1 + a, 1 + b)\}$. Note that $V_1 \cup V_2 \subseteq V(UG(R))$ and $V_1 \cap V_2 = \emptyset$. From the assumption $2a = 0, 2b = 0, 1 + a \in U(R_1)$, and $1 + b \in U(R_2)$, it follows that for any $x \in V_1$ and $y \in V_2$, $x + y \in U(R)$. Hence, x and y are adjacent in $UG(R)$. This shows that the subgraph of $UG(R)$ induced on $V_1 \cup V_2$ contains $K_{3,3}$ as a subgraph. Therefore, we get that $UG(R)$ does not satisfy (Ku_2). □

We make use of [2, Proposition 2.4] in some of the results to follow in this Section. For the sake of convenient reference, we state it here as Proposition 3.4.

Proposition 3.4 (2, Proposition 2.4)**.** *Let R be a finite ring. Then the following hold.*

(i) If $2 \notin U(R)$, then $deg_{UG(R)}x = |U(R)|$ for any $x \in R$.

(ii) If $2 \in U(R)$, then $deg_{UG(R)}x = |U(R)| - 1$ if $x \in U(R)$ and for any $x \in R\backslash U(R)$, $deg_{UG(R)}x = |U(R)|$.

Proposition 3.5. *Let $n \geq 2$. Let $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors). Then $UG(R)$ is planar.*

Proof. Note that $|R| = 2^n$, $2 \notin U(R)$, and $(1, 1, \ldots, 1)$ is the only unit of R. Hence, we obtain from Proposition 3.4 (i) that $deg_{UG(R)}r = 1$ for any $r \in R$. Let $i \in \{1, 2, \ldots, n\}$ and let us denote the element of R whose i-th coordinate equals 1 and whose j-th coordinate equals 0 for all $j \in \{1, 2, \ldots, n\}\backslash\{i\}$ by e_i. Note that $\sum_{i=1}^{n} e_i$ is the only unit of R. For any $r \in R$, the component of $UG(R)$ containing r is the complete graph on two vertices $\{r, r + \sum_{i=1}^{n} e_i\}$. It is clear that $UG(R)$ has exactly $\frac{|R|}{2} = \frac{2^n}{2} = 2^{n-1}$ components and each component is a complete graph on two vertices and so, it follows that $UG(R)$ is planar. □

Remark 3.6. *Let $T_1 = \mathbb{Z}_4$. Note that $UG(T_1)$ is the cycle of length four given by $0 - 1 - 2 - 3 - 0$. Let us denote the ring $\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}$ by T_2. It is convenient to denote the ring $\mathbb{Z}_2[X]$ by R and the ideal X^2R by I. For any element $r \in R$, we denote $r + I$ by $\overline{r}$. Observe that $UG(T_2)$ is the cycle of length four given by $\overline{0} - \overline{1} - \overline{X} - \overline{1 + X} - \overline{0}$.*

Proposition 3.7. *Let $n \geq 1$ and let $R_1 = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors). Let $R = R_1 \times R_2$, where $R_2 \in \{\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. Then $UG(R)$ is planar.*

Proof. Note that $|U(R)| = |U(R_1)||U(R_2)| = 2$, and $2 \notin U(R)$. Hence, we obtain from Proposition 3.4 (i) that $deg_{UG(R)}r = 2$ for any $r \in R$. Note that $R_1 = \{(x_1, \ldots, x_n) | x_i \in \{0, 1\}$ for each $i \in \{1, \ldots, n\}\}$. Suppose that $R_2 = \mathbb{Z}_4$. It is easy to verify that for any $(x_1, \ldots, x_n) \in R_1$, the component of $UG(R)$ containing $(x_1, \ldots, x_n, 0)$ is the cycle of length four given by $(x_1, \ldots, x_n, 0) - (1 + x_1, \ldots, 1 + x_n, 1) - (x_1, \ldots, x_n, 2) - (1 + x_1, \ldots, 1 + x_n, 3) - (x_1, \ldots, x_n, 0)$ and it is also the component of $(x_1, \ldots, x_n, i)$ for any $i \in \mathbb{Z}_4$. Suppose that $R_2 = \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}$. With the same use of notation as in Remark 3.6 note that $R_2 = \{\overline{0}, \overline{1}, \overline{X}, \overline{1 + X}\}$. For any $(x_1, \ldots, x_n) \in R_1$, the component of $UG(R)$ containing $(x_1, \ldots, x_n, \overline{0})$ is the cycle of length four given by $(x_1, \ldots, x_n, \overline{0}) - (1 + x_1, \ldots, 1 + x_n, \overline{1}) - (x_1, \ldots, x_n, \overline{X}) - (1 + x_1, \ldots, 1 + x_n, \overline{1 + X}) - (x_1, \ldots, x_n, \overline{0})$ and it is also the component of $(x_1, \ldots, x_n, y)$ for any $y \in R_2$. In both the cases, it follows that $UG(R)$ has exactly $\frac{|R|}{4} = \frac{2^{n+2}}{4} = 2^n$ components and each component is a cycle of length four. As any cycle of length four is planar, we obtain that $UG(R)$ is planar. □

Remark 3.8. *Note that $\mathbb{F}_4$ can be expressed as $\mathbb{F}_4 = \{0, 1, \alpha, \alpha^2\}$, where $\alpha \in \mathbb{F}_4\backslash\{0\}$ is such that $1 + \alpha + \alpha^2 = 0$. Observe that $UG(\mathbb{F}_4)$ is a complete graph on four vertices.*

Proposition 3.9. *Let $n \geq 1$. Let $R_1 = \mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$ (n factors). Let $R = R_1 \times \mathbb{F}_4$. Then $UG(R)$ is planar.*

Proof. Note that $|R| = 2^{n+2}, |U(R)| = |U(R_1)||U(\mathbb{F}_4)| = 3$, and $2 \notin U(R)$. Hence, we obtain from Proposition 3.4 (i) that $deg_{UG(R)} r = 3$ for any $r \in R$. Observe that $R_1 = \{(x_1, \ldots, x_n) | x_i \in \{0,1\}$ for each $i \in \{1, \ldots, n\}\}$. As in Remark 3.8, let us denote $\mathbb{F}_4$ by $\mathbb{F}_4 = \{0, 1, \alpha, 1+\alpha\}$, where $\alpha \in \mathbb{F}_4 \backslash \{0\}$ is such that $1+\alpha+\alpha^2 = 0$. For any $(x_1, \ldots, x_n) \in R_1$, let us denote the component of $UG(R)$ containing $(x_1, \ldots, x_n, 0)$ by H. It is not hard to verify that H is the union of a cycle Γ given by $\Gamma : v_1 = (x_1, \ldots, x_n, 0) - v_2 = (1+x_1, \ldots, 1+x_n, 1) - v_3 = (x_1, \ldots, x_n, \alpha) - v_4 = (1+x_1, \ldots, 1+x_n, 1+\alpha) - v_5 = (x_1, \ldots, x_n, 1) - v_6 = (1+x_1, \ldots, 1+x_n, 0) - v_7 = (x_1, \ldots, x_n, 1+\alpha) - v_8 = (1+x_1, \ldots, 1+x_n, \alpha) - v_1 = (x_1, \ldots, x_n, 0)$ and the edges $e_1 : v_1 - v_4, e_2 : v_5 - v_8, e_3 : v_2 - v_7$, and $e_4 : v_3 - v_6$. It is clear that H is also the component of $(x_1, \ldots, x_n, \beta)$ for any $\beta \in \mathbb{F}_4$. The graph H is shown in Figure 1. It follows from the figure of H that it is planar. Observe that if r is any element of R, then the component of $UG(R)$ containing r contains exactly 8 vertices and is isomorphic to H. Note that the number of components of $UG(R)$ equals $\frac{|R|}{8} = \frac{2^{n+2}}{8} = 2^{n-1}$. Since H is planar, it follows that $UG(R)$ is planar.

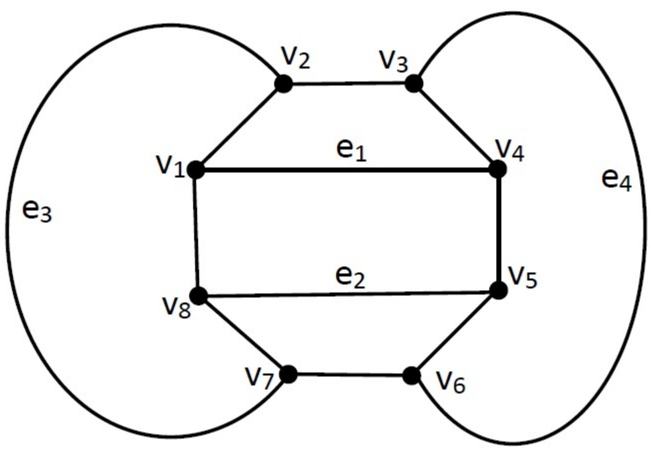

Figure 1. H

□

Lemma 3.10. *Let $R = \mathbb{Z}_2 \times \mathbb{Z}_3$. Then $UG(R)$ is planar*

Proof. Observe that $R \cong \mathbb{Z}_6$ as rings. Let us denote the ring $\mathbb{Z}_6$ by T. Observe that $UG(T)$ is the cycle of length 6 given by $0-1-4-3-2-5-0$. Hence, $UG(T)$ is planar and so, $UG(R)$ is planar. □

Remark 3.11. *Let $R = T \times \mathbb{Z}_5$, where T is a nonzero ring such that $U(T) = \{1\}$. Then $UG(R)$ satisfies (Ku_2).*

Proof. Suppose that $UG(R)$ does not satisfy (Ku_2). Then there exist subsets V_1, V_2 of R such that $|V_i| = 3$ for each $i \in \{1,2\}$, $V_1 \cap V_2 = \emptyset$, and for any $x \in V_1$ and $y \in V_2$, x and y are adjacent in $UG(R)$. Let $V_1 = \{(a_1,b_1),(a_2,b_2),(a_3,b_3)\}$ and let $V_2 = \{(a_4,b_4),(a_5,b_5),(a_6,b_6)\}$. Observe that $a_i \in T$ and $b_i \in \mathbb{Z}_5$ for each $i \in \{1,2,3,4,5,6\}$. Now, $a_1 + a_j \in U(T) = \{1\}$ for each $j \in \{4,5,6\}$ and so, $a_j = 1 + a_1$. Similarly, it follows that $a_j = 1 + a_2 = 1 + a_3$ for each $j \in \{4,5,6\}$. Therefore, $a_1 = a_2 = a_3$ and $a_4 = a_5 = a_6 = 1 + a_1$. Thus $V_1 = \{(a_1,b_1),(a_1,b_2),(a_1,b_3)\}$ and $V_2 = \{(1+a_1,b_4),(1+a_1,b_5),(1+a_1,b_6)\}$. It is clear that b_1,b_2,b_3 are distinct elements of $\mathbb{Z}_5$ and b_4,b_5,b_6 are distinct elements of $\mathbb{Z}_5$ and $b_k + b_t \in U(\mathbb{Z}_5)$ for all $k \in \{1,2,3\}$ and $t \in \{4,5,6\}$. That is, $b_k + b_t \neq 0$ for all $k \in \{1,2,3\}$ and $t \in \{4,5,6\}$. Note that either $b_i \neq 0$ for each $i \in \{1,2,3\}$ or $b_j \neq 0$ for each $j \in \{4,5,6\}$. Without loss of generality we can assume that $b_i \neq 0$ for each $i \in \{1,2,3\}$. Note that at least two among b_4,b_5,b_6 are nonzero elements of $\mathbb{Z}_5$. Without loss of generality, we can assume that b_4 and b_5 are nonzero. Since $|U(\mathbb{Z}_5)| = 4$, it follows that at least one between $b_4, b_5 \in \{b_1,b_2,b_3\}$. We can assume without loss of generality that $b_4 = b_1$. Now, both $b_1 + b_2$ and $b_1 + b_3$ are nonzero. Therefore, $b_2, b_3 \in \{2b_1, 3b_1\}$. From $b_5 + b_i \neq 0$ for each $i \in \{2,3\}$, we get that $b_5 = 4b_1$. In such a case, it follows that $b_5 + b_1 = 5b_1 = 0$. This is in contradiction to the assumption that $b_5 + b_1 \neq 0$. Therefore, we obtain that $UG(T \times \mathbb{Z}_5)$ satisfies (Ku_2). □

Proposition 3.12. *Let $R = \mathbb{Z}_5 \times T$, where T is a ring with $char(T) = 2$. Then $UG(R)$ does not satisfy (Ku_2^*).*

Proof. Let $V_1 = \{(0,0),(4,0),(3,0)\}$ and let $V_2 = \{(3,1),(2,1),(4,1)\}$. It is clear that V_1 is an independent set of $UG(R)$. It follows from $char(T) = 2$ that V_2 is an independent set of $UG(R)$. Note that both $(0,0)$ and $(4,0)$ are adjacent to each element of V_2 in $UG(R)$. Observe that $(3,0)$ is adjacent to both $(3,1)$ and $(4,1)$ in $UG(R)$, whereas $(3,0)$ is not adjacent to $(2,1)$ in $UG(R)$. It is obvious to verify that $(3,0)-(1,1)-(1,0)-(2,1)$ is a path of length 3 in $UG(R)$. Let H be the subgraph of $UG(R)$ induced on $V_1 \cup V_2 \cup \{(1,1),(1,0)\}$. Let us denote the edges $(0,0)-(1,1)$ and $(1,0)-(3,1)$ of H by e_1 and e_2. Let H_1 be the subgraph of H defined by $H - \{e_1, e_2\}$. The subgraph H_1 is shown in Figure 2. It is clear that H_1 is homeomorphic to $K_{3,3}$. This shows that $UG(R)$ contains a subgraph homeomorphic to $K_{3,3}$ and so, $UG(R)$ does not satisfy (Ku_2^*).

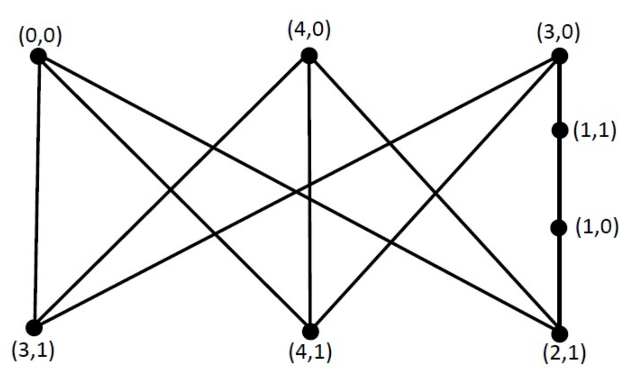

Figure 2. H_1

□

Proposition 3.13. *Let $R = \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $UG(R)$ is planar.*

Proof. Note that $V(UG(R)) = \{v_1 = (0,0), v_2 = (0,2), v_3 = (1,2), v_4 = (1,0), v_5 = (1,1), v_6 = (0,1), v_7 = (2,1), v_8 = (2,0), v_9 = (2,2)\}$. It is not hard to verify that $UG(R)$ is the union of a cycle Γ of length 8 given by $\Gamma : v_2 - v_3 - v_4 - v_5 - v_6 - v_7 - v_8 - v_9 - v_2$ and the edges $e_1 : v_1 - v_3, e_2 : v_1 - v_5, e_3 : v_1 - v_7, e_4 : v_1 - v_9$, $e_5 : v_2 - v_4, e_6 : v_4 - v_6, e_7 : v_6 - v_8$, and $e_8 : v_8 - v_2$. The cycle Γ can be represented by means of a polygon of size 8. The vertex v_1 can be plotted inside the polygon representing Γ and it can be joined to v_3, v_5, v_7, v_9 by means of line segments representing the edges e_1, e_2, e_3, e_4 without any crossing over of the edges. The edges e_5, e_6, e_7, e_8 are chords of the polygon representing Γ and they can be drawn outside the polygon representing Γ in such a way that there are no crossing over of the edges. It is clear from the above description of $UG(R)$ that $UG(R)$ is planar. The graph $UG(\mathbb{Z}_3 \times \mathbb{Z}_3)$ is shown in Figure 3. One can also refer [2, Figure 4, page 2869].

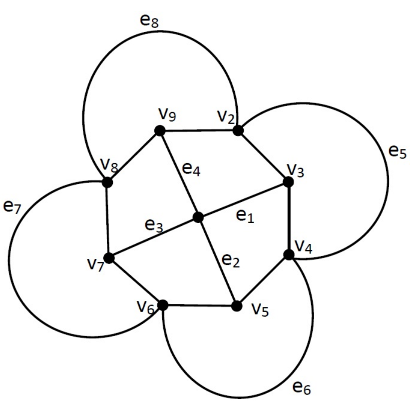

Figure 3. $UG(\mathbb{Z}_3 \times \mathbb{Z}_3)$

□

Proposition 3.14. *Let $R = S \times F$, where S is a quasilocal ring which is not a field and F is a field. If $|F| \geq 3$, then $UG(R)$ does not satisfy (Ku_2^*).*

Proof. Let $\mathfrak{m}$ denote the unique maximal ideal of S. Let $x \in \mathfrak{m}, x \neq 0$. Since F is a field with $|F| \geq 3$, there exist $\alpha, \beta \in F \backslash \{0\}$ such that $\alpha \neq \beta$. If $UG(S)$ does not satisfy (Ku_2), then we know from Lemma 2.2 that $UG(R)$ does not satisfy (Ku_2) and so, $UG(R)$ does not satisfy (Ku_2^*). Hence, we can assume that $UG(S)$ satisfies (Ku_2). In such a case, we know from $(iii) \Rightarrow (iv)$ of Lemma [9, 2.4] that S is isomorphic to one of the rings from the collection $\{\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. Let $V_1 = \{(1+x,0), (1+x,\alpha), (1,\alpha)\}$ and let $V_2 = \{(x,0), (x,-\beta), (0,-\beta)\}$. It is clear that both $(1+x,\alpha)$ and $(1,\alpha)$ are adjacent to each element of V_2 in $UG(R)$. Observe that $(1+x,0)$ is adjacent to both $(x,-\beta)$ and $(0,-\beta)$ in $UG(R)$, whereas $(1+x,0)$ is not adjacent to $(x,0)$ in $UG(R)$. Note that $(1+x,0) - (x,\alpha) - (1+x,-\beta) - (x,0)$ is a path of length 3 in $UG(R)$. Let H be the subgraph of $UG(R)$ induced on $V_1 \cup V_2 \cup \{(x,\alpha), (1+x,-\beta)\}$. It is not hard to verify that H contains a subgraph H_1 such that H_1 is homeomorphic to $K_{3,3}$. The subgraph H_1 is shown in Figure 4. As H_1 is homeomorphic to $K_{3,3}$, we obtain that $UG(R)$ does not satisfy (Ku_2^*).

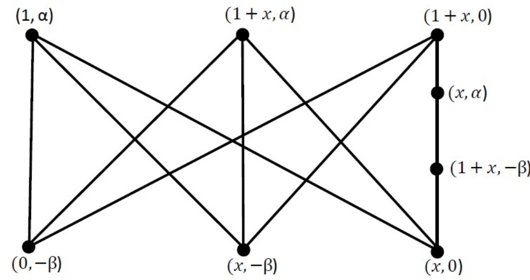

Figure 4. H_1

□

Proposition 3.15. *Let $R = S \times F \times T$, where S is a quasilocal ring which is not a field, F is a field with $|F| \geq 3$, and T is a nonzero ring. Then $UG(R)$ does not satisfy (Ku_2^*).*

Proof. We use the same notations that are used in the proof of Proposition 3.14. Let $x \in \mathfrak{m} \backslash \{0\}$ and let $\alpha, \beta \in F \backslash \{0\}$ be such that $\alpha \neq \beta$. Using the same reasoning as in the proof of Proposition 3.14, it can be assumed that S is isomorphic to one of the rings from the collection $\{\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. Let $W_1 = \{(1+x,0,0), (1+x,\alpha,0), (1,\alpha,0)\}$ and let $W_2 = \{(x,0,1), ((x,-\beta,1), (0,-\beta,1)\}$. Observe that $(1+x,0,0) - (x,\alpha,1) - (1+x,-\beta,0) - (x,0,1)$ is a path of length 3 in $UG(R)$. It can be shown as in the proof of Proposition 3.14 that $UG(R)$ contains a subgraph g such that g is homeomorphic to $K_{3,3}$. The subgraph g is shown in Figure 5. This proves that $UG(R)$ does not satisfy (Ku_2^*).

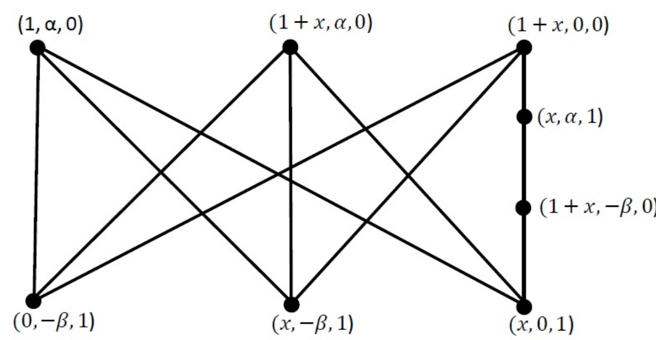

Figure 5. H_1

□

Corollary 3.16. *Let $S \in \{\mathbb{Z}_4, \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. Let $R = \mathbb{Z}_3 \times S$. Then $UG(R)$ does not satisfy (Ku_2^*).*

Proof. It follows from Proposition 3.14 that $UG(R)$ does not satisfy (Ku_2^*). □

Proposition 3.17. *Let $R = \mathbb{Z}_3 \times \mathbb{F}_4$. Then $UG(R)$ does not satisfy (Ku_2^*).*

Proof. Note that $\mathbb{F}_4 = \{0,1,\alpha,1+\alpha\}$, where $\alpha \in \mathbb{F}_4$ is such that $\alpha^2+\alpha+1 = 0$. Let $V_1 = \{(1,1),(0,1+\alpha),(1,1+\alpha)\}$ and let $V_2 = \{(1,0),(1,\alpha),(0,\alpha)\}$. Note that $V_1 \cup V_2 \subseteq R = V(UG(R))$. Let H be the subgraph of $UG(R)$ induced on $V_1 \cup V_2 \cup \{(2,1)\}$. Observe that $(1,1)$ (respectively, $(1,1+\alpha)$) is adjacent to all the elements of V_2 in $UG(R)$ and $(0,1+\alpha)$ is adjacent to both $(1,0)$ and $(1,\alpha)$ in $UG(R)$ and $(0,1+\alpha) - (2,1) - (0,\alpha)$ is a path of length 2 in $UG(R)$. Consider the subgraph H_1 of H shown in Figure 6. It is clear that $(2,1)$ is of degree 2 in H_1 and H_1 is homeomorphic to $K_{3,3}$. Therefore, we obtain that $UG(R)$ does not satisfy (Ku_2^*).

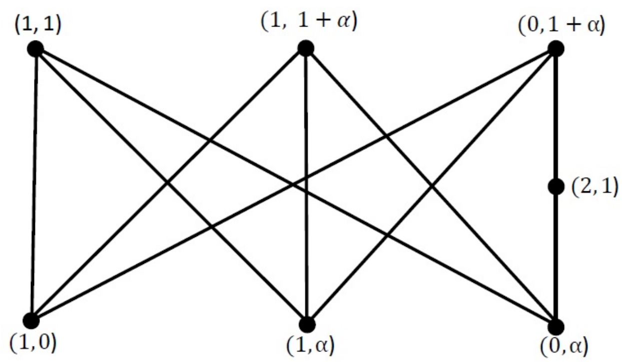

Figure 6. H_1

□

Proposition 3.18. *Let $R = \mathbb{Z}_3 \times \mathbb{F}_4 \times T$, where T is a ring with $char(T) = 2$. Then $UG(R)$ does not satisfy (Ku_2^*).*

Proof. We use the same notations as in the proof of Proposition 3.17 and proceed as in the proof of Proposition 3.17. Let $V_1 = \{(1,1,0),(0,1+\alpha,0),(1,1+\alpha,0)\}$ and let $V_2 = \{(1,0,1),(1,\alpha,1),(0,\alpha,1)\}$. It is clear that both $(1,1,0)$ and $(1,1+\alpha,0)$ are adjacent to each element of V_2 in $UG(R)$. Observe that $(0,1+\alpha,0)-(2,1,1)-(2,1+\alpha,0)-(0,\alpha,1)$ is a path of length 3 in $UG(R)$. Note that $(0,1+\alpha,0)$ is adjacent to both $(1,0,1)$ and $(1,\alpha,1)$ in $UG(R)$. Let H be the subgraph of $UG(R)$ induced on $V_1 \cup V_2 \cup \{2,1,1),(2,1+\alpha,0)\}$. It is clear that V_1, V_2 are independent sets of $UG(R)$ and $(2,1,1)$ and $(2,1+\alpha,0)$ are vertices of degree 2 in H. The graph H is shown in Figure 7. From the above given arguments, it follows that H is homeomorphic to $K_{3,3}$. Therefore, we get that $UG(R)$ does not satisfy (Ku_2^*).

□

Proposition 3.19. *Let $R = \mathbb{Z}_3 \times \mathbb{Z}_5$. Then $UG(R)$ does not satisfy (Ku_2^*).*

Proof. Let $V_1 = \{(0,0),(0,4),(0,3)\}$ and let $V_2 = \{(2,2),(1,4),(2,3)\}$. Let H be the subgraph of $UG(R)$ induced on $V_1 \cup V_2 \cup \{(2,0)\}$. Observe that $(0,0)$ (respectively, $(0,4)$) is adjacent to each element of V_2 in $UG(R)$. It is clear that $(0,3)$ is adjacent to

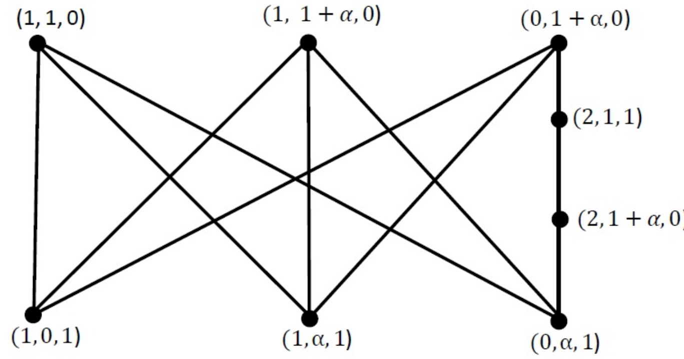

Figure 7. H

$(1,4)$ and $(2,3)$ in $UG(R)$ and $(0,3)-(2,0)-(2,2)$ is a path of length two in $UG(R)$. Note that V_i is an independent set of $UG(R)$ for each $i \in \{1,2\}$ and consider the subgraph H_1 of H shown in Figure 8. It is clear that H_1 is homeomorphic to $K_{3,3}$. Therefore, we obtain that $UG(R)$ does not satisfy (Ku_2^*).

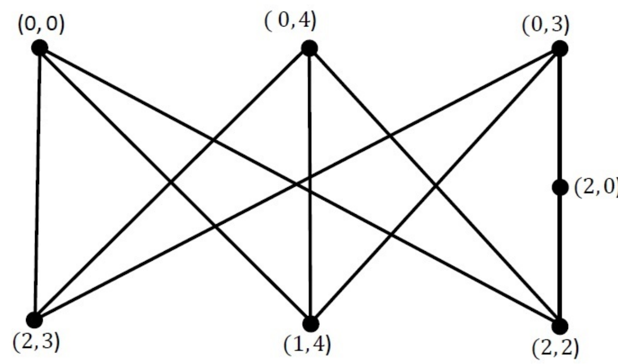

Figure 8. H_1

□

Corollary 3.20. *Let $R \in \{\mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5\}$. Then $UG(R)$ does not satisfy (Ku_2).*

Proof. Let $R \in \{\mathbb{F}_4 \times \mathbb{F}_4, \mathbb{F}_4 \times \mathbb{Z}_5\}$. Note that $|U(\mathbb{F}_4)| = 3$ and $|U(\mathbb{Z}_5)| = 4$. Hence, we obtain from Lemma 3.2 that $UG(R)$ does not satisfy (Ku_2). □

Corollary 3.21. *Let $R \in \{\mathbb{F}_4 \times \mathbb{Z}_4, \mathbb{F}_4 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$. Then $UG(R)$ does not satisfy (Ku_2)*

Proof. Now, $\mathbb{F}_4 = \{0,1,\alpha,1+\alpha\}$, $\mathbb{Z}_4 = \{0,1,2,3\}$, and $\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]} = \{\overline{0},\overline{1},\overline{X},\overline{1+X}\}$, where for an element $f(X) \in \mathbb{Z}_2[X]$, we denote $f(X)+X^2\mathbb{Z}_2[X]$ by $\overline{f(X)}$. Let $R = \mathbb{F}_4 \times \mathbb{Z}_4$. Let $a = \alpha$ and $b = 2$. Note that $2a = 0, 2b = 0$, $1+a \in U(\mathbb{F}_4)$, and $1+b \in U(\mathbb{Z}_4)$. Therefore, we obtain from Lemma 3.2 that $UG(R)$ does not satisfy (Ku_2). Let $R = \mathbb{F}_4 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}$. Let $a = \alpha$ and $b = \overline{X}$. Then $2a = 0, 2b = \overline{0}$, $1+a \in U(\mathbb{F}_4)$, and $\overline{1+X} \in U(\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]})$. Hence, we obtain from Lemma 3.2 that $UG(R)$ does not satisfy (Ku_2). Thus if $R \in \{\mathbb{F}_4 \times \mathbb{Z}_4, \mathbb{F}_4 \times \frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$, then $UG(R)$ does not satisfy (Ku_2). □

Lemma 3.22. *Let $(R_1,\mathfrak{m}_1)$ and $(R_2,\mathfrak{m}_2)$ be quasilocal rings such that $\mathfrak{m}_i\neq(0)$ for each $i\in\{1,2\}$. Let $R=R_1\times R_2$. Then $UG(R)$ does not satisfy (Ku_2).*

Proof. Let $x_i\in\mathfrak{m}_i\backslash\{0\}$ for each $i\in\{1,2\}$. Let $V_1=\{(0,0),(x_1,0),(0,x_2)\}$ and let $V_2=\{(1,1),(1,1+x_2),(1+x_1,1+x_2)\}$. Observe that $V_1\cup V_2\subseteq V(UG(R))$ and $V_1\cap V_2=\emptyset$. For any $x\in V_1$ and $y\in V_2$, $x+y\in U(R)$ and so, x and y are adjacent in $UG(R)$. Hence, the subgraph of $UG(R)$ induced on $V_1\cup V_2$ contains $K_{3,3}$ as a subgraph. This proves that $UG(R)$ does not satisfy (Ku_2). □

Corollary 3.23. *Let $R=R_1\times R_2$, where $R_i\in\{\mathbb{Z}_4,\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$ for each $i\in\{1,2\}$. Then $UG(R)$ does not satisfy (Ku_2).*

Proof. It is clear that both $\mathbb{Z}_4$ and $\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}$ are finite local rings which are not fields. Hence, we obtain from Lemma 3.22 that $UG(R)$ does not satisfy (Ku_2). □

Lemma 3.24. *Let $R=R_1\times R_2$, where $R_1\in\{\mathbb{Z}_4,\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$ and $R_2=\mathbb{Z}_5$. Then $UG(R)$ does not satisfy (Ku_2).*

Proof. Let $R_1=\mathbb{Z}_4$. Let $V_1=\{(0,0),(2,0),(0,2)\}$ and let $V_2=\{(1,1),(3,1),$
$(3,2)\}$. Note that $V_1\cup V_2\subseteq V(UG(R))$ and $V_1\cap V_2=\emptyset$. It is clear that V_i is an independent set of $UG(R)$ for each $i\in\{1,2\}$ and for any $x\in V_1$ and $y\in V_2$, $x+y\in U(R)$ and so, x and y are adjacent in $UG(R)$. Hence, the subgraph of $UG(R)$ induced on $V_1\cup V_2$ is a $K_{3,3}$. Therefore, $UG(R)$ does not satisfy (Ku_2).

Let $R_1=\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}$. Let $V_1=\{(\overline{0},0),(\overline{X},0),(\overline{0},2)\}$ and let $V_2=\{(\overline{1},1),(\overline{1+X},1),(\overline{1+X},2)\}$, where for any $f(X)\in\mathbb{Z}_2[X]$, we denote $f(X)+X^2\mathbb{Z}_2[X]$ by $\overline{f(X)}$. Now, it follows as in the previous paragraph that the subgraph of $UG(R)$ induced on $V_1\cup V_2$ is a $K_{3,3}$. Therefore, $UG(R)$ does not satisfy (Ku_2). □

In Theorem 3.25, we classify rings R with $|Max(R)|=2$ such that $UG(R)$ is planar.

Theorem 3.25. *Let R be a ring such that $|Max(R)|=2$. The following statements are equivalent:*

(i) $UG(R)$ is planar.

(ii) $UG(R)$ satisfies both (Ku_1^*) and (Ku_2^*).

(iii) $UG(R)$ satisfies (Ku_2^*).

(iv) R is isomorphic to one of the rings from the collection $\mathscr{A}=\{\mathbb{Z}_2\times\mathbb{Z}_2,\mathbb{Z}_2\times\mathbb{Z}_3,\mathbb{Z}_2\times\mathbb{F}_4,\mathbb{Z}_2\times\mathbb{Z}_4,\mathbb{Z}_2\times\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]},\mathbb{Z}_3\times\mathbb{Z}_3\}$.

Proof. $(i)\Rightarrow(ii)$ This follows from Kuratowski's theorem [5, Theorem 5.9].

$(ii)\Rightarrow(iii)$ This is clear.

$(iii)\Rightarrow(iv)$ We are assuming that $|Max(R)|=2$ and $UG(R)$ satisfies (Ku_2^*). Hence, $UG(R)$ satisfies (Ku_2). Therefore, we obtain from Remark 3.1 that there exist finite local rings $(R_1,\mathfrak{m}_1)$ and $(R_2,\mathfrak{m}_2)$ such that $R\cong R_1\times R_2$ as rings, where $R_i\in\{\mathbb{Z}_2,\mathbb{F}_4,\mathbb{Z}_3,\mathbb{Z}_5,\mathbb{Z}_4,\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$ for each $i\in\{1,2\}$. If $R_1=\mathbb{Z}_2$ and $R_2=\mathbb{Z}_5$, then we know from Proposition 3.12 that $UG(R_1\times R_2)$ does not satisfy (Ku_2^*). If $R_1=\mathbb{Z}_3$ and $R_2\in\{\mathbb{Z}_4,\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]},\mathbb{F}_4,\mathbb{Z}_5\}$, then we know from Corollary 3.16 and Propositions 3.17 and 3.19 that $UG(R_1\times R_2)$ does not satisfy (Ku_2^*). If $R_1=\mathbb{F}_4$ and $R_2\in\{\mathbb{F}_4,\mathbb{Z}_5,\mathbb{Z}_4,\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$, then we know from Corollaries 3.20 and 3.21 that $UG(R_1\times R_2)$ does not satisfy (Ku_2). If $R_i\in\{\mathbb{Z}_4,\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$ for each $i\in\{1,2\}$, then we know from Corollary 3.23 that $UG(R_1\times R_2)$ does not satisfy (Ku_2). If $R_1\in\{\mathbb{Z}_4,\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]}\}$ and $R_2=\mathbb{Z}_5$, then we know from Lemma 3.24 that $UG(R_1\times R_2)$ does not satisfy (Ku_2). From the above given arguments, it is clear that if $UG(R)$ satisfies (Ku_2^*), then R is isomorphic to one of the rings from the collection $\mathscr{A}$, where $\mathscr{A}$ is as in the statement (iv) of this theorem.

$(iv)\Rightarrow(i)$ We are assuming that R is isomorphic to one of the rings from the collection $\mathscr{A}$, where $\mathscr{A}$ is as in the statement(iv) of this theorem. Let $T\in\mathscr{A}$. If $T=\mathbb{Z}_2\times\mathbb{Z}_2$, then $UG(T)$ is a simple graph on four vertices and so, $UG(T)$ is planar. If $T=\mathbb{Z}_2\times\mathbb{Z}_3$, then it is noted in Lemma 3.10 that $UG(T)$ is planar. If $T\in\{\mathbb{Z}_2\times\mathbb{Z}_4,\mathbb{Z}_2\times\frac{\mathbb{Z}_2[X]}{X^2\mathbb{Z}_2[X]},\mathbb{Z}_2\times\mathbb{F}_4\}$, then we know from Propositions 3.7 and 3.9 that $UG(T)$ is planar. If $T=\mathbb{Z}_3\times\mathbb{Z}_3$, then we know from Proposition 3.13 that $UG(T)$ is planar. This proves that if R is isomorphic to one of the rings from the collection $\mathscr{A}$, then $UG(R)$ is planar. □

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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