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Cartesian magicness of 3-dimensional boards

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Abstract

A (p,q,r)-board that has $pq+pr+qr$ squares consists of a (p,q)-, a (p,r)-, and a (q,r)-rectangle. Let S be the set of the squares. Consider a bijection $f : S \to [1, pq+pr+qr]$. Firstly, for $1 \le i \le p$, let x_i be the sum of all the $q+r$ integers in the i-th row of the $(p,q+r)$-rectangle. Secondly, for $1 \le j \le q$, let y_j be the sum of all the $p+r$ integers in the j-th row of the $(q,p+r)$-rectangle. Finally, for $1 \le k \le r$, let z_k be the the sum of all the $p+q$ integers in the k-th row of the $(r,p+q)$-rectangle. Such an assignment is called a (p,q,r)-design if $\{x_i : 1 \le i \le p\} = \{c_1\}$ for some constant c_1, $\{y_j : 1 \le j \le q\} = \{c_2\}$ for some constant c_2, and $\{z_k : 1 \le k \le r\} = \{c_3\}$ for some constant c_3. A (p,q,r)-board that admits a (p,q,r)-design is called (1) Cartesian tri-magic if c_1, c_2 and c_3 are all distinct; (2) Cartesian bi-magic if c_1, c_2 and c_3 assume exactly 2 distinct values; (3) Cartesian magic if $c_1 = c_2 = c_3$ (which is equivalent to supermagic labeling of $K(p,q,r)$). Thus, Cartesian magicness is a generalization of magic rectangles into 3-dimensional space. In this paper, we study the Cartesian magicness of various (p,q,r)-board by matrix approach involving magic squares or rectangles. In Section 2, we obtained various sufficient conditions for (p,q,r)-boards to admit a Cartesian tri-magic design. In Sections 3 and 4, we obtained many necessary and (or) sufficient conditions for various (p,q,r)-boards to admit (or not admit) a Cartesian bi-magic and magic design. In particular, it is known that $K(p,p,p)$ is supermagic and thus every (p,p,p)-board is Cartesian magic. We gave a short and simpler proof that every (p,p,p)-board is Cartesian magic.

Keywords

3-dimensional boards, Cartesian tri-magic, Cartesian bi-magic, Cartesian magic.

AMS Subject Classification

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Contents

1. Introduction

For positive integers p_i, $1 \le i \le k$, $k \ge 2$, the k-tuple $(p_1, p_2, \ldots, p_k)$ is called a $(p_1, p_2, \ldots, p_k)$-*board*, or a *generalized plane*, in k-space that is formed by $\binom{k}{2}$ rectangles P_iP_j $(1 \le i < j \le k)$ of size $p_i \times p_j$. Abusing the notation, we also let P_iP_j denote a matrix of size $p_i \times p_j$, where P_iP_j is an entry of a block matrix B as shown below

$$\begin{pmatrix} \star & P_1P_2 & P_1P_3 & \cdots & \cdots & P_1P_k \\ (P_1P_2)^T & \star & P_2P_3 & \cdots & \cdots & P_2P_k \\ (P_1P_3)^T & (P_2P_3)^T & \star & \cdots & \cdots & P_3P_k \\ \vdots & \vdots & \vdots & \ddots & \cdots & \vdots \\ (P_1P_{k-1})^T & (P_2P_{k-1})^T & (P_3P_{k-1})^T & \cdots & \ddots & P_{k-1}P_k \\ (P_1P_k)^T & (P_2P_k)^T & (P_3P_k)^T & \cdots & (P_{k-1}P_k)^T & \star \end{pmatrix} \tag{1.1}$$

such that the r-th row of B, denoted B_r $(1 \le r \le k)$, is a submatrix of size $p_r \times (p_1 + \cdots + p_k)$. For $1 \le d \le k$, we say a $(p_1, \ldots, p_k)$-board is (k,d)-*magic* if

(i) the row sum of all the entries of each row of M_r is a constant c_r, and

(ii) $\{c_1, c_2, \ldots, c_k\}$ has exactly d distinct elements.

We also say that a (k,d)-magic $(p_1,\ldots,p_k)$-board admits a (k,d)-*design*. Thus, a (2,2)-magic (p,q)-board is what has been known as a magic rectangle while a (2,1)-magic (p,q)-board is what has been known as a magic square. We shall say a (3,3)-magic, a (3,2)-magic and a (3,1)-magic (p,q,r)-board is *Cartesian tri-magic*, *Cartesian bi-magic* and *Cartesian magic* respectively. In this paper, we determine Cartesian magicness of (p,q,r)-boards by matrix approach involving magic squares or rectangles.

For $a,b\in\mathbb{Z}$ and $a\le b$, we use $[a,b]$ to denote the set of integers from a to b. Let S be the set of the $pq+pr+qr$ squares of a (p,q,r)-board. Consider a bijection $f: S\to[1,pq+pr+qr]$. For convenience of presentation, throughout this paper, we let PQ, PR, and QR be the images of (p,q)-, (p,r)-, and (q,r)-rectangles under f in matrix form, respectively. Hence, PQ, PR and QR are matrices of size $p\times q$, $p\times r$ and $q\times r$, respectively.

Let $G=(V,E)$ be a graph containing p vertices and q edges. If there exists a bijection $f: E\to[1,q]$ such that the map $f^+(u)=\sum\limits_{uv\in E} f(uv)$ induces a constant map from V to $\mathbb{Z}$, then G is called *supermagic* and f is called a *supermagic labeling* of G [13, 14].

A *labeling matrix* for a labeling f of G is a matrix whose rows and columns are named by the vertices of G and the (u,v)-entry is $f(uv)$ if $uv\in E$, and is $*$ otherwise. Sometimes, we call this matrix a *labeling matrix of G*. In other words, suppose A is an adjacency matrix of G and f is a labeling of G, then a labeling matrix for f is obtained from $A=(a_{u,v})$ by replacing $a_{u,v}$ by $f(uv)$ if $a_{u,v}=1$ and by $*$ if $a_{u,v}=0$. This concept was first introduced by Shiu, *et al.* in [11]. Moreover, if f is a supermagic labeling, then a labeling matrix of f is called a *supermagic labeling matrix* of G [12]. Thus, a simple (p,q)-graph $G=(V,E)$ is supermagic if and only if there exists a bijection $f: E\to[1,q]$ such that the row sums (as well as the column sums) of the labeling matrix for f are the same. For purposes of these sums, entries labeled with $*$ will be treated as 0. It is easy to see that $K(p,q,r)$ is supermagic if and only if the (p,q,r)-board is Cartesian magic.

Note that the block matrix B in (1.1) is a labeling matrix of the complete k-partite graph $K(p_1,\ldots,p_k)$. In particular, consider the complete tripartite graph $K(p,q,r)$. Suppose f is an edge-labeling of $K(p,q,r)$. According to the vertex-list $\{x_1,\ldots,x_p,y_1,\ldots,y_q,z_1,\ldots,z_r\}$, the labeling matrix of f is

$$B=\begin{pmatrix}\bigstar & PQ & PR\\ (PQ)^T & \bigstar & QR\\ (PR)^T & (QR)^T & \bigstar\end{pmatrix},$$

where PQ, PR and QR are defined before, each $\bigstar$ is a certain size matrix whose entries are $*$. For convenience, we use QP, RP and RQ to denote $(PQ)^T$, $(PR)^T$ and $(QR)^T$, respectively.

Throughout this paper, we will use $s(PQ)$, $s(PR)$ and $s(QR)$ to denote the sum of integers in PQ, PR and QR, respectively.

2. Cartesian tri-magic

In this section, we will make use of the existence of magic rectangles. From [3, 4], we know that a $h\times k$ magic rectangle exists when $h,k\ge 2$, $h\equiv k \pmod 2$ and $(h,k)\ne(2,2)$.

Theorem 2.1. *Suppose $3\le p<q<r$, where p, q are odd and r is even. The (p,q,r)-board is Cartesian tri-magic.*

Proof. Fill the $(p+q)\times r$ rectangle with integers in $[1,(p+q)r]$ and the $p\times q$ rectangle with integers in $[(p+q)r+1, pq+pr+qr]$ to form two magic rectangles. Thus, $c_1=(pr^2+qr^2+pq^2+q+r)/2+pqr+q^2r$, $c_2=(p^2q+pr^2+qr^2+p+r)/2+pqr+p^2r$, and $c_3=(p^2r+q^2r+p+q)/2+pqr$. Observe that

$$\begin{aligned}2(c_1-c_2)&=(q-p)(pq+1+2(p+q)r),\\ 2(c_2-c_3)&=pr(r+p)+(qr+1)(r-q)+p^2q.\end{aligned}$$

Clearly, $c_1>c_2>c_3$. Hence, the theorem holds. □

Theorem 2.2. *If $3\le p<q<r$, where q is even, and p and r are odd, then the (p,q,r)-board is Cartesian tri-magic.*

Proof. Fill the $p\times r$ rectangle with integers in $[1,pr]$ and $(p+r)\times q$ rectangle with integers in $[pr+1,pq+pr+qr]$ to form two magic rectangles. We have

$$\begin{aligned}2(c_2-c_1)&=p^2q-pq^2-q^2r+qr^2+pr^2+2p^2r+p-q\\ &=p(r^2-p^2)+qr(r-q)+2p^2r+(p^2-1)q+p\\ &>0,\end{aligned}$$

$2(c_1-c_3)=pr^2-p^2r+r-p=(r-p)(pr+1)>0.$
Thus $c_2>c_1>c_3$. Hence the theorem holds. □

Theorem 2.3. *If $2\le p<q<r$, where p is even, and q and r are odd, then the (p,q,r)-board is Cartesian tri-magic.*

Proof. Fill the $q\times r$ rectangle with integers in $[1,qr]$ and the $p\times(q+r)$ rectangle with integers in $[qr+1,pq+pr+qr]$ to form two magic rectangles. Thus, $2c_1=2q^2r+2qr^2+pq^2+pr^2+q+r+2pqr$, $2c_2=qr^2+p^2q+p^2r+r+p+2pqr$, and $2c_3=p^2q+p^2r+q^2r+q+p+2pqr$. Now

$$\begin{aligned}2(c_2-c_3)&=(r-q)(qr+1),\\ 2(c_1-c_2)&=pr(r-p)+(pq+1)(q-p)+2q^2r+qr^2.\end{aligned}$$

Clearly, $c_1>c_2>c_3$. Hence, the theorem holds. □

Theorem 2.4. *Suppose $2\le p\le q\le r$, where p, q and r have the same parity and $(p,q)\ne(2,2)$. Then (p,q,r)-board is Cartesian tri-magic.*

Proof. Fill the $p\times r$ rectangle with integers in $[1,pr]$, the $p\times q$ rectangle with integers in $[pr+1,pr+pq]$, and the $q\times r$ rectangle with integers in $[pr+pq+1,pr+pq+qr]$ to form three magic rectangles. Now, $2c_1=2pqr+pq^2+pr^2+r+q$,

$2c_2 = 2pqr + 2p^2r + 2pr^2 + p^2q + qr^2 + r + p$, and $2c_3 = 2pqr + 2pq^2 + p^2r + q^2r + q + p$. Therefore,

$$\begin{aligned}
2(c_2 - c_3) &= p^2r + 2pr^2 + p^2q + qr^2 \\
&\quad - 2pq^2 - q^2r + r - q \\
&= 2p(r^2 - q^2) + p^2(r+q) + (r-q)(qr+1) \\
&> 0, \\
2(c_2 - c_1) &= 2p^2r + p^2q + qr^2 + pr^2 - pq^2 + p - q \\
&= 2p^2r + p^2q + pr^2 + p + q(r^2 - pq - 1) \\
&> 0, \\
2(c_3 - c_1) &= pq^2 + p^2r + q^2r - pr^2 - r + p. \qquad (2.1)
\end{aligned}$$

Fill the $p \times r$ rectangle with integers in $[1, pr]$, the $q \times r$ rectangle with integers in $[pr+1, pr+qr]$, and the $p \times q$ rectangle with integers in $[pr+qr+1, pr+qr+pq]$ to form three magic rectangles. Now, $2c_1 = 2pqr + pq^2 + pr^2 + 2q^2r + r + q$, $2c_2 = 2pqr + 2p^2r + 2pr^2 + p^2q + qr^2 + r + p$, and $2c_3 = 2pqr + p^2r + q^2r + q + p$. Therefore,

$$\begin{aligned}
2(c_1 - c_3) &= pr^2 - p^2r + q^2r + pq^2 + r - p \\
&= (r-p)(pr+1) + q^2(p+r) > 0, \\
2(c_2 - c_3) &= p^2r + 2pr^2 + p^2q + qr^2 - q^2r + r - q \\
&= p^2(r+q) + 2pr^2 + (r-q)(qr+1) > 0, \\
2(c_2 - c_1) &= pr^2 + qr^2 + 2p^2r - 2q^2r \\
&\quad + p^2q - pq^2 - q + p. \qquad (2.2)
\end{aligned}$$

The sum of (2.1) and (2.2) is

$$\begin{aligned}
&3p^2r - q^2r + qr^2 + p^2q - r - q + 2p \\
&= qr(r-q) + q(p^2-1) + r(3p^2-1) + 2p > 0.
\end{aligned}$$

So at least one of (2.1) and (2.2) is positive. Hence we have the theorem. □

For $p = q = 2$, we have the following.

Theorem 2.5. *For all $r \geq 1$, the $(2,2,r)$-board is Cartesian tri-magic.*

Proof. For $r = 1$, a labeling matrix for $(1,2,2)$ is:

$$\left(\begin{array}{c|cc|cc}
* & 6 & 8 & 1 & 5 \\ \hline
6 & * & * & 7 & 2 \\
8 & * & * & 3 & 4 \\ \hline
1 & 7 & 3 & * & * \\
5 & 2 & 4 & * & *
\end{array}\right)
\left(\begin{array}{c}
20 \\ \hline
15 \\
15 \\ \hline
11 \\
11
\end{array}\right)$$

The right column contains the row sums of the left matrix.

For $r = 2$, consider

$$PQ = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array} \qquad PR = \begin{array}{|c|c|}\hline 7 & 5 \\ \hline 6 & 8 \\ \hline\end{array} \qquad QR = \begin{array}{|c|c|}\hline 11 & 12 \\ \hline 10 & 9 \\ \hline\end{array}$$

Clearly, we get a Cartesian tri-magic design with $c_1 = 18$, $c_2 = 26$, and $c_3 = 34$.

Now assume $r \geq 3$. For $r \equiv 0 \pmod 4$, consider

$$PQ = \begin{array}{|c|c|}\hline 4r+1 & 4r+4 \\ \hline 4r+3 & 4r+2 \\ \hline\end{array}$$

$$PR = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2r-1 & 2r-2 & 4 & \cdots & r-7 & r+7 & r+6 & r-4 & r-3 & r+3 & r+2 & r \\ \hline 2r & 2 & 3 & 2r-3 & \cdots & r+8 & r-6 & r-5 & r+5 & r+4 & r-2 & r-1 & r+1 \\ \hline\end{array}$$

$$QR = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 2r+1 & 4r-1 & 4r-2 & 2r+4 & \cdots & 3r-7 & 3r+7 & 3r+6 & 3r-4 & 3r-3 & 3r+3 & 3r+2 & 3r+1 \\ \hline 4r & 2r+2 & 2r+3 & 4r-3 & \cdots & 3r+8 & 3r-6 & 3r-5 & 3r+5 & 3r+4 & 3r-2 & 3r-1 & 3r \\ \hline\end{array}$$

Clearly, we get a Cartesian tri-magic design with $c_1 = r^2 + 17r/2 + 5$, $c_2 = 3r^2 + 17r/2 + 5$, and $c_3 = 8r + 2$.

For $r \equiv 1 \pmod 4$, consider

$$PQ = \begin{array}{|c|c|}\hline 4r+3 & 4r+2 \\ \hline 4r+1 & 4r+4 \\ \hline\end{array}$$

$$PR = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2r-1 & 2r-2 & 4 & \cdots & r-8 & r+8 & r+7 & r-5 & r-4 & r+4 & r-2 & r+2 & 3r \\ \hline 2r & 2 & 3 & 2r-3 & \cdots & r+9 & r-7 & r-6 & r+6 & r+5 & r-3 & 3r-2 & r-1 & r+1 \\ \hline\end{array}$$

$$QR = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 2r+1 & 4r-1 & 4r-2 & 2r+4 & \cdots & 3r-8 & 3r+8 & 3r+7 & 3r-5 & 3r-4 & 3r+4 & r+3 & 3r+2 & 3r+1 \\ \hline 4r & 2r+2 & 2r+3 & 4r-3 & \cdots & 3r+9 & 3r-7 & 3r-6 & 3r+6 & 3r+5 & 3r-3 & 3r+3 & 3r-1 & r \\ \hline\end{array}$$

Clearly, we get a Cartesian tri-magic design with $c_1 = r^2 + (21r+5)/2$, $c_2 = 3r^2 + (13r+15)/2$, and $c_3 = 8r + 2$.

For $r \equiv 2 \pmod 4$, consider

$$PQ = \begin{array}{|c|c|}\hline 4r+2 & 4r+4 \\ \hline 4r+1 & 4r+3 \\ \hline\end{array}$$

$$PR = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2r-1 & 2r-2 & 4 & \cdots & r-5 & r+5 & r+4 & r-2 & r-1 & r+1 \\ \hline 2r & 2 & 3 & 2r-3 & \cdots & r+6 & r-4 & r-3 & r+3 & r+2 & r \\ \hline\end{array}$$

$$QR = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|}\hline 2r+1 & 4r-1 & 4r-2 & 2r+4 & \cdots & 3r-5 & 3r+5 & 3r+4 & 3r-2 & 3r+2 & 3r+1 \\ \hline 4r & 2r+2 & 2r+3 & 4r-3 & \cdots & 3r+6 & 3r-4 & 3r-3 & 3r+3 & 3r-1 & 3r \\ \hline\end{array}$$

Clearly, we get a Cartesian tri-magic design with $c_1 = r^2 + 17r/2 + 5$, $c_2 = 3r^2 + 17r/2 + 5$, and $c_3 = 8r + 2$.

Finally for $r \equiv 3 \pmod 4$, consider

$$PQ = \begin{array}{|c|c|}\hline 4r+1 & 4r+3 \\ \hline 4r+2 & 4r+4 \\ \hline\end{array}$$

$$PR = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 1 & 2r-1 & 2r-2 & 4 & \cdots & r-6 & r+6 & r+5 & r-3 & r-2 & r+2 & 3r \\ \hline 2r & 2 & 3 & 2r-3 & \cdots & r+7 & r-5 & r-4 & r+4 & 3r-2 & r-1 & r+1 \\ \hline\end{array}$$

$$QR = \begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 2r+1 & 4r-1 & 4r-2 & 2r+4 & \cdots & 3r-6 & 3r+6 & 3r+5 & 3r-3 & r+3 & 3r+2 & 3r+1 \\ \hline 4r & 2r+2 & 2r+3 & 4r-3 & \cdots & 3r+7 & 3r-5 & 3r-4 & 3r+4 & 3r+3 & 3r-1 & r \\ \hline\end{array}$$

Clearly, we get a Cartesian tri-magic design with $c_1 = r^2 + (21r+5)/2$, $c_2 = 3r^2 + (13r+15)/2$, and $c_3 = 8r + 2$. □

Corollary 2.6. *The (p,p,p)-board is Cartesian tri-magic for all $p \geq 1$.*

We now consider the case $p=1$. We first introduce some notation about matrices.

Let m,n be two positive integers. For convenience, we use $M_{m,n}$ to denote the set of $m\times n$ matrices over $\mathbb{Z}$. For any matrix $M\in M_{m,n}$, $r_i(M)$ and $c_j(M)$ denote the i-th row sum and the j-th column sum of M, respectively.

We want to assign the integers in $[1,q+r+qr]$ to matrices $PR\in M_{1,r}$, $QR\in M_{q,r}$ and $QP=(PQ)^T\in M_{q,1}$ such that the matrix

$$M=\begin{pmatrix} * & PR\\ QP & QR\end{pmatrix}$$

has the following properties:

P.1 Each integers in $[1,q+r+qr]$ appears once.

P.2 $r_i(M)$ is a constant not equal to $r_1(M)+c_1(M)$, $2\le i\le q+1$.

P.3 $c_j(M)$ is a constant not equal to $r_i(M)$ or $r_1(M)+c_1(M)$, $2\le j\le r+1$.

Such a matrix M is called a *Cartesian labeling matrix* of the $(1,q,r)$-board (or the graph $K(1,q,r)$.)

Theorem 2.7. *The $(1,1,r)$-board is Cartesian tri-magic.*

Proof. A Cartesian labeling matrix of the $(1,1,r)$-board is

$$\left(\begin{array}{c|cccc} * & 1 & 2 & \cdots & r\\ \hline 2r+1 & 2r & 2r-1 & \cdots & r+1\end{array}\right)$$

Clearly, we get a Cartesian tri-magic design with $c_1=(r^2+5r+2)/2, c_2=(3r^2+5r+2)/2$ and $c_3=2r+1$. □

Note that the $(1,1,2)$-board also admits a different Cartesian labeling matrix

$$\left(\begin{array}{c|cc} * & 2 & 5\\ \hline 3 & 4 & 1\end{array}\right)$$

with $c_1=10$, $c_2=8$, and $c_3=6$ respectively.

Theorem 2.8. *Suppose $q\equiv r \pmod 2$ and $q\ge 2$. The $(1,q,r)$-board is Cartesian tri-magic if $q<r$.*

Proof. Let A be a $(q+1)\times(r+1)$ magic rectangle. Exchanging columns and exchanging rows if necessary, we may assume that $(q+1)(r+1)$ is put at the $(1,1)$-entry of A. Now let PR be the $1\times r$ matrix obtained from the first row of A by deleting the $(1,1)$-entry; let QP be the $q\times 1$ matrix obtained from the first column of A by deleting the $(1,1)$-entry; let QR be the $q\times r$ matrix obtained from A by deleting the first row and the first column.

It is easy to check that
$c_1=\frac{(q+r+2)[(q+1)(r+1)+1]}{2}-2(q+1)(r+1)$,
$c_2=\frac{(r+1)[(q+1)(r+1)+1]}{2}$, and
$c_3=\frac{(q+1)[(q+1)(r+1)+1]}{2}$ with $c_1>c_2>c_3$ if $q<r$. □

Suppose $q=2s+1$ and $r=2k$, where $k>s\ge 1$. We assign the integers in $[1,4sk+4k+2s+1]$ to form a matrix M satisfying the properties P.1-P.3.

Let $\alpha=\begin{pmatrix}1 & 2 & \cdots & k\end{pmatrix}$ and $\beta=\begin{pmatrix}k & k-1 & \cdots & 1\end{pmatrix}$ be row vectors in $M_{1,k}$. Let $J_{m,n}$ be the $m\times n$ matrix whose entries are 1.

Let A be the following $(2s+1)\times(2k)$ matrix:

$$\left(\begin{array}{c|c} \alpha+[2s+1]J_{1,k} & \alpha+[2s+1+k]J_{1,k}\\ \beta+[2s+1+3k]J_{1,k} & \beta+[2s+1+2k]J_{1,k}\\ \alpha+[2s+1+4k]J_{1,k} & \alpha+[2s+1+5k]J_{1,k}\\ \vdots & \vdots\\ \beta+[2s+1+(4s-1)k]J_{1,k} & \beta+[2s+1+(4s-2)k]J_{1,k}\\ \alpha+[2s+1+4sk]J_{1,k} & \alpha+[2s+1+(4s+1)k]J_{1,k}\end{array}\right).$$

We separate A into two parts, left and right. Now reverse the rows of the right part of A from top to bottom:

$$B=\left(\begin{array}{c|c} \alpha+[2s+1]J_{1,k} & \alpha+[2s+1+(4s+1)k]J_{1,k}\\ \beta+[2s+1+3k]J_{1,k} & \beta+[2s+1+(4s-2)k]J_{1,k}\\ \alpha+[2s+1+4k]J_{1,k} & \alpha+[2s+1+(4s-3)k]J_{1,k}\\ \vdots & \vdots\\ \beta+[2s+1+(4s-1)k]J_{1,k} & \beta+[2s+1+2k]J_{1,k}\\ \alpha+[2s+1+4sk]J_{1,k} & \alpha+[2s+1+k]J_{1,k}\end{array}\right).$$

We insert

$$\left(\beta+[2s+1+(4s+3)k]J_{1,k} \;\middle|\; \beta+[2s+1+(4s+2)k]J_{1,k}\right)$$

to B as the first row. So we get a $(2s+2)\times(2k)$ matrix

$$C=\left(\begin{array}{c|c} \beta+[2s+1+(4s+3)k]J_{1,k} & \beta+[2s+1+(4s+2)k]J_{1,k}\\ \hline \alpha+[2s+1]J_{1,k} & \alpha+[2s+1+(4s+1)k]J_{1,k}\\ \beta+[2s+1+3k]J_{1,k} & \beta+[2s+1+(4s-2)k]J_{1,k}\\ \alpha+[2s+1+4k]J_{1,k} & \alpha+[2s+1+(4s-3)k]J_{1,k}\\ \vdots & \vdots\\ \beta+[2s+1+(4s-1)k]J_{1,k} & \beta+[2s+1+2k]J_{1,k}\\ \alpha+[2s+1+4sk]J_{1,k} & \alpha+[2s+1+k]J_{1,k}\end{array}\right).$$

Each column sum of C is $(s+1)(4sk+4s+4k+3)$. Each row sum (except the 1st row) of C is $k(4sk+4s+2k+3)$ and $r_1(C)=k(8sk+4s+6k+3)$. The set of remaining integers is $[1,2s+1]$ which will form the column matrix QP.

It is easy to see that the difference between the $(2i+1)$-st and the $(2i+2)$-nd rows of C is

$$\begin{pmatrix}-1 & -3 & \cdots & -(2k-3) & -(2k-1) & 2k-1 & 2k-3 & \cdots & 3 & 1\end{pmatrix}$$

for $1\le i\le s$. We let QP be

$$\left(s+1 \mid s+2 \quad s \mid s+3 \quad s-1 \mid \cdots \quad \cdots \mid 2s+1 \quad 1\right)^T\in M_{2s+1,1}.$$

Now let

$$N=\left(\begin{array}{c|c|c} * & \beta+[2s+1+(4s+3)k]J_{1,k} & \beta+[2s+1+(4s+2)k]J_{1,k}\\ \hline s+1 & \alpha+[2s+1]J_{1,k} & \alpha+[2s+1+(4s+1)k]J_{1,k}\\ s+2 & \beta+[2s+1+3k]J_{1,k} & \beta+[2s+1+(4s-2)k]J_{1,k}\\ s & \alpha+[2s+1+4k]J_{1,k} & \alpha+[2s+1+(4s-3)k]J_{1,k}\\ \vdots & \vdots & \vdots\\ 2s+1 & \beta+[2s+1+(4s-1)k]J_{1,k} & \beta+[2s+1+2k]J_{1,k}\\ 1 & \alpha+[2s+1+4sk]J_{1,k} & \alpha+[2s+1+k]J_{1,k}\end{array}\right)$$

Here $r_{2i+1}(N)-r_{2i+2}(N)=2i$, $1\le i\le s$. For odd i, we swap the $(2i+1,2k+1-i)$-entry with the $(2i+2,2k+1-i)$-entry.

For even i, we swap the $(2i+1,2k+1-i)$-entry with the $(2i+2,2k+1-i)$-entry of N and swap the $(2i+1,2)$-entry with the $(2i+2,2)$-entry of N. Note that, they work since $1 \le i \le s < k$. The resulting matrix is the required matrix $M \in M_{2s+2,2k+1}$. Note that $c_2 = r_i(M) = k(4sk+4s+2k+3)+s+1$ for $2 \le i \le 2s+2$, $c_3 = c_j(M) = (s+1)(4sk+4s+4k+3)$ for $2 \le j \le 2k+1$ and $c_1 = r_1(M)+c_1(M) = k(8sk+4s+6k+3)+(s+1)(2s+1)$.

Remark 2.9. In the above construction, we use integers in $[2s+2, 4sk+4k+2s+1]$ to form the matrix C. We may use integers in $[1, 4sk+4k]$ to form a new matrix C', namely $C' = C-(2s+1)J_{2s+2,2k}$. The remaining integers of $[4sk+4k+1, 4sk+4k+2s+1]$ form the new matrix PQ', namely $PQ' = PQ+(4sk+4k)J_{2s+1,1}$. By the same procedure as above, we have a new matrix M' with $c_2' = r_i(M') = r_i(M)+2k = k(4sk+4s+2k+5)+s+1$ for $2 \le i \le 2s+2$, $c_3' = c_j(M') = c_j(M)-(2s+2)(2s+1) = (s+1)(4sk+4k+1)$ for $2 \le j \le 2k+1$ and $c_1' = r_1(M')+c_1(M') = r_1(M)+c_1(M)-(2s+1)2k+(2s+1)(4sk+4k) = k(8sk+4s+6k+3)+(2s+1)(4sk+2k)$. So, if $c_2 = c_3$ in the above discussion, then we may change the arrangement M to M' to obtain a Cartesian tri-magic labeling for the $(1, 2s+1, 2k)$-board.

Thus we have

Theorem 2.10. *Suppose $q \ge 3$ is odd and r is even. The $(1,q,r)$-board is Cartesian tri-magic.*

Example 2.11. $(1,5,8)$-board

$$\left(\begin{array}{cccc|cccc}
53 & 52 & 51 & 50 & 49 & 48 & 47 & 46 \\
\hline
6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
21 & 20 & 19 & 18 & 17 & 16 & 15 & 14 \\
22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 \\
37 & 36 & 35 & 34 & 33 & 32 & 31 & 30 \\
38 & 39 & 40 & 41 & 42 & 43 & 44 & 45
\end{array}\right)$$

$$\to C = \left(\begin{array}{cccc|cccc}
53 & 52 & 51 & 50 & 49 & 48 & 47 & 46 \\
\hline
6 & 7 & 8 & 9 & 42 & 43 & 44 & 45 \\
21 & 20 & 19 & 18 & 33 & 32 & 31 & 30 \\
22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 \\
37 & 36 & 35 & 34 & 17 & 16 & 15 & 14 \\
38 & 39 & 40 & 41 & 10 & 11 & 12 & 13
\end{array}\right)$$

The first row is the matrix PR and the last 5 rows form the matrix QR.

Now each column sum is 177, each row sum of QR is 204. But we have to put 1,2,3,4,5 into the matrix PQ (or QP). The average of these numbers is 3. So we have to make each row sum of the augmented matrix $(QP|QR)$ to be 207. Thus we put these numbers into QP as follows:

$$N = \left(\begin{array}{c|cccc|cccc}
* & 53 & 52 & 51 & 50 & 49 & 48 & 47 & 46 \\
\hline
3 & 6 & 7 & 8 & 9 & 42 & 43 & 44 & 45 \\
4 & 21 & 20 & 19 & 18 & 33 & 32 & 31 & 30 \\
2 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 \\
5 & 37 & 36 & 35 & 34 & 17 & 16 & 15 & 14 \\
1 & 38 & 39 & 40 & 41 & 10 & 11 & 12 & 13
\end{array}\right)$$

Now the row sums of QR are 207, 208, 206, 209 and 205. So we must swap some entries of QR. We will pair up rows of QR and QP, namely 2nd and 3rd, 4th and 5th. The 2nd row sum is greater than the 3rd row sum by 2; and the 4th row sum is greater than the 5th row sum by 4. In 2nd and 3rd row of QR, there are two entries at the same column whose difference is 1 (namely 29 and 30); two entries at the same column with difference -1 (namely 37 and 38) and two entries at the same column with difference $+3$ (namely 15 and 12). So, swapping these pairs of integers we get

$$M = \left(\begin{array}{c|cccc|cccc}
* & 53 & 52 & 51 & 50 & 49 & 48 & 47 & 46 \\
\hline
3 & 6 & 7 & 8 & 9 & 42 & 43 & 44 & 45 \\
4 & 21 & 20 & 19 & 18 & 33 & 32 & 31 & 29 \\
2 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 30 \\
5 & 38 & 36 & 35 & 34 & 17 & 16 & 12 & 14 \\
1 & 37 & 39 & 40 & 41 & 10 & 11 & 15 & 13
\end{array}\right)$$

Now $c_1 = 411$, $c_2 = 207$ and $c_3 = 177$.

Or

$$M' = \left(\begin{array}{c|cccc|cccc}
* & 48 & 47 & 46 & 45 & 44 & 43 & 42 & 41 \\
\hline
51 & 1 & 2 & 3 & 4 & 37 & 38 & 39 & 40 \\
52 & 16 & 15 & 14 & 13 & 28 & 27 & 26 & 24 \\
50 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 25 \\
53 & 33 & 31 & 30 & 29 & 12 & 11 & 7 & 9 \\
49 & 32 & 34 & 35 & 36 & 5 & 6 & 10 & 8
\end{array}\right)$$

Now $c_1' = 611$, $c_2' = 215$ and $c_3' = 147$.

Suppose $q = 2s$ and $r = 2k-1$, where $k > s \ge 1$. We want to assign the integers in $[1, 4sk+2k-1]$ to form a matrix M satisfying the properties P.1-P.3.

When $s = 1$, we have the following.

For $r = 3$, consider the Cartesian labeling matrix

$$\left(\begin{array}{c|ccc}
* & 2 & 4 & 6 \\
\hline
8 & 5 & 11 & 3 \\
10 & 9 & 1 & 7
\end{array}\right).$$

Clearly, we get a Cartesian tri-magic design with $c_1 = 30$, $c_2 = 27$, and $c_3 = 16$.

Now for $r \equiv 1 \pmod 4$, $r \ge 5$, let $r = 4s+1, s \ge 1$. Consider

$PR =$

1	3	$\cdots$	$4s-5$	$4s-3$	$4s-1$	$10s+3$	2	4	$\cdots$	$4s-4$	$4s-2$	$4s$

$QP+QR =$

$12s+5$	$12s+3$	$6s+1$	$\cdots$	$4s+5$	$10s+5$	$4s+3$	$4s+1$	$10s+2$	$8s+1$	$\cdots$	$6s+5$	$8s+4$	$6s+3$
$12s+4$	$6s+2$	$12s+2$	$\cdots$	$10s+6$	$4s+4$	$10s+4$	$4s+2$	$8s+2$	$10s+1$	$\cdots$	$8s+5$	$6s+4$	$8s+3$

Clearly, we get a Cartesian tri-magic design with $c_1 = 8s^2+36s+12$, $c_2 = 32s^2+27s+6$, and $c_3 = 18s+6$.

Finally for $r \equiv 3 \pmod 4$, $r \ge 7$, let $r = 4s+3, s \ge 1$. Consider

$PR =$

1	3	5	$\cdots$	$\mathbf{4s-1}$	$4s+1$	$4s+3$	2	4	$\cdots$	$\mathbf{4s-2}$	$4s$	$\mathbf{4s+2}$

$$QP+QR = \begin{array}{|c||c|c|c|c}\hline 12s+10 & 6s+5 & 12s+8 & 6s+3 & \cdots \\ \hline 12s+11 & 12s+9 & 6s+4 & 12s+7 & \cdots \\ \hline \end{array}$$

$$\begin{array}{c|c|c|c||c|c|c|c|c|c|}\hline \cdots & 10s+10 & 4s+5 & 10s+8 & 8s+6 & 10s+6 & \cdots & \mathbf{6s+8} & 8s+8 & 6s+6 \\ \hline \cdots & \mathbf{4s+6} & 10s+9 & 4s+4 & 10s+7 & 8s+5 & \cdots & 8s+9 & 6s+7 & \mathbf{8s+7} \\ \hline \end{array}$$

We now get $c_1 = 8s^2+38s+27$ and $c_3 = 18s+15$. However, we have $y_1 = 32s^2+61s+29$ and $y_2 = 32s^2+63s+31$. To make $y_1 = y_2$, we perform the following exchanges. Note that none of these exchanges would modify the values of $z_k, 1 \le k \le r$. Only the value of c_1 would be changed.

(a) Interchange the labels $4s-2$ and $6s+8$. The value of y_1 is decreased by $2s+10$.

(b) Interchange the labels $4s-1$ and $4s+6$. The value of y_2 is decreased by 7.

(c) Interchange the labels $4s+2$ and $8s+7$. The value of y_2 is decreased by $4s+5$.

In total, the value of y_1 is decreased by $2s+10$, and the value of y_2 is decreased by $4s+12$. Thus, we now have $c_2 = 32s^2+59s+19$ and $c_1 = 8s^2+44s+49$. Clearly, we now have a Cartesian tri-magic design.

We now assume $s \ge 2$. Let A be a $2s \times 2$ magic rectangle using integers in $[0, 4s-1]$. The construction of A can be found in [4]. Hence $r_i(A) = 4s-1$ and $c_j(A) = s(4s-1)$. Exchanging columns and rows if necessary, we may assume the $(1,1)$-entry of A is 0, hence the $(1,2)$-entry of A is $4s-1$.

Let $\Omega = J_{s,2} \otimes \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$ and $\Theta = A \otimes kJ_{1,k}$, where $\otimes$ denotes the Kronecker's multiplication. Thus $r_i(\Omega) = k(k+1)$, $c_j(\Omega) = s(k+1)$, $r_i(\Theta) = (4s-1)k^2$, and $c_j(\Theta) = s(4s-1)k$ for $1 \le i \le 2s$, $1 \le j \le 2k$.

Let $N = \Omega + \Theta$. Then $r_i(N) = 4sk^2+k$ and $c_j(N) = 4s^2k+s$ for $1 \le i \le 2s$, $1 \le j \le 2k$. Now the set of entries of N is $[1, 4sk]$. We set $N = (QP|QR)$. Now, the set of remaining integers is $[4sk+1, 4sk+2k-1]$, which will be arranged to form the matrix PR. Let

$$\gamma = (* \quad 2 \quad 4 \quad \cdots \quad 2k-4 \quad 2k-2 \mid 1 \quad 3 \quad \cdots \quad 2k-3 \quad 2k-1).$$

Insert $\gamma + 4skJ_{1,2k}$ to the first row of N, with $*$ still denoting '$*+4sk$'. The resulting matrix is denoted by N'. Now

$$c_j(N') = \begin{cases} c_j(N) + 4sk + (2j-2), & \text{if } 2 \le j \le k; \\ c_j(N) + 4sk + (2j-1-2k), & \text{if } k+1 \le j \le 2k. \end{cases}$$

Look at the first row of N which is

$$N^{(1)} = (1 \quad 2 \quad \cdots \quad k \mid 4sk-k+1 \quad 4sk-k+2 \quad \cdots \quad 4sk).$$

We swap the j-th entry with the $(k+2-j)$-th entry of $N^{(1)}$, for $2 \le j \le \lceil k/2 \rceil$ and swap the j-th entry with the $(3k+1-j)$-th entry of $N^{(1)}$, for $k+1 \le j \le k + \lfloor k/2 \rfloor$ to get a new row. It is equivalent to reversing the order of the entries from the 2nd to the k-th and reversing the order of the entries from the $(k+1)$-st to the $2k$-th of $N^{(1)}$. Replace $N^{(1)}$ (i.e., the second row of N') by this new row to get a matrix M. Hence $c_j(M) = c_j(N) + 4sk + k = 4s^2k + 4sk + s + k = c_2$, $2 \le j \le 2k$. Note that $r_i(M) = r_{i-1}(N) = 4sk^2 + k = c_3$ for $2 \le i \le 2s+1$; $r_1(M) + c_1(M) = r_1(N') + c_1(N) = 8sk^2 + 4s^2k - 4sk + 2k^2 + s - k = c_1$. Clearly $c_1 > c_2$ and $c_1 > c_3$. Now, $c_3 - c_2 = s[4k(k-s-1)-1] \ne 0$. So M corresponds to a tri-magic $(1, 2s, 2k-1)$-board. So we have

Theorem 2.12. *Suppose $q \ge 2$ is even and r is odd. The $(1,q,r)$-board is Cartesian tri-magic.*

Example 2.13. Consider the graph $(1,6,9)$-board, i.e., $s = 3$ and $k = 5$. Now

$$A = \begin{pmatrix} 0 & 11 \\ 2 & 9 \\ 6 & 5 \\ 7 & 4 \\ 8 & 3 \\ 10 & 1 \end{pmatrix}, \Omega = \left(\begin{array}{ccccc|ccccc} 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 & 5 & 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 & 5 & 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 & 5 & 4 & 3 & 2 & 1 \end{array}\right).$$

Hence

$$\begin{aligned}\Theta &= A \otimes 5J_{1,5} \\ &= \left(\begin{array}{ccccc|ccccc} 0 & 0 & 0 & 0 & 0 & 55 & 55 & 55 & 55 & 55 \\ 10 & 10 & 10 & 10 & 10 & 45 & 45 & 45 & 45 & 45 \\ 30 & 30 & 30 & 30 & 30 & 25 & 25 & 25 & 25 & 25 \\ 35 & 35 & 35 & 35 & 35 & 20 & 20 & 20 & 20 & 20 \\ 40 & 40 & 40 & 40 & 40 & 15 & 15 & 15 & 15 & 15 \\ 50 & 50 & 50 & 50 & 50 & 5 & 5 & 5 & 5 & 5 \end{array}\right)\end{aligned}$$

and

$$N' = \left(\begin{array}{ccccc|ccccc} * & 62 & 64 & 66 & 68 & 61 & 63 & 65 & 67 & 69 \\ \hline 1 & 2 & 3 & 4 & 5 & 56 & 57 & 58 & 59 & 60 \\ 15 & 14 & 13 & 12 & 11 & 50 & 49 & 48 & 47 & 46 \\ 31 & 32 & 33 & 34 & 35 & 26 & 27 & 28 & 29 & 30 \\ 40 & 39 & 38 & 37 & 36 & 25 & 24 & 23 & 22 & 21 \\ 41 & 42 & 43 & 44 & 45 & 16 & 17 & 18 & 19 & 20 \\ 55 & 54 & 53 & 52 & 51 & 10 & 9 & 8 & 7 & 6 \end{array}\right).$$

Now

$$M = \left(\begin{array}{c|ccccccccc} * & 62 & 64 & 66 & 68 & 61 & 63 & 65 & 67 & 69 \\ \hline 1 & 5 & 4 & 3 & 2 & 60 & 59 & 58 & 57 & 56 \\ 15 & 14 & 13 & 12 & 11 & 50 & 49 & 48 & 47 & 46 \\ 31 & 32 & 33 & 34 & 35 & 26 & 27 & 28 & 29 & 30 \\ 40 & 39 & 38 & 37 & 36 & 25 & 24 & 23 & 22 & 21 \\ 41 & 42 & 43 & 44 & 45 & 16 & 17 & 18 & 19 & 20 \\ 55 & 54 & 53 & 52 & 51 & 10 & 9 & 8 & 7 & 6 \end{array}\right).$$

$c_1 = 768$, $c_2 = 248$ and $c_3 = 305$.

By a similar way we have

Example 2.14. The following is a required matrix for $(1,4,7)$-board:

$$\left(\begin{array}{c|ccccccc} * & 34 & 36 & 38 & 33 & 35 & 37 & 39 \\ \hline 1 & 4 & 3 & 2 & 32 & 31 & 30 & 29 \\ 16 & 15 & 14 & 13 & 20 & 19 & 18 & 17 \\ 21 & 22 & 23 & 24 & 9 & 10 & 11 & 12 \\ 28 & 27 & 26 & 25 & 8 & 7 & 6 & 5 \end{array}\right).$$

$c_1 = 318$, $c_2 = 102$ and $c_3 = 132$.

In [1], the authors introduced the concept of local antimagic chromatic number of a graph G, denoted $\chi_{la}(G)$. Observe that for every complete tripartite graph $K(p,q,r)$, $\chi_{la}(K(p,q,r)) = 3$ if and only if the (p,q,r)-board is Cartesian tri-magic. Thus, we have obtained various sufficient conditions such that $\chi_{la}(K(p,q,r)) = 3$. Interested readers may refer to [2, 5–7] for more results on local antimagic chromatic number of graphs. Note that our argument on $(1,2,r)$-board is the proof of Theorem 1 in [5].

3. Cartesian bi-magic

Theorem 3.1. *The $(1,1,r)$-board is Cartesian bi-magic if and only if $r \not\equiv 1 \pmod 4$.*

Proof. [Sufficiency] Suppose $r \not\equiv 1 \pmod 4$. We have three cases.

1. Suppose $r \equiv 0 \pmod 4$. Assign $2r+1$ to PQ. The assignments to PR and QR are given by row 1 and row 2 respectively in the matrix below.

$PR =$	1	$2r-1$	$2r-2$	4	5	$2r-5$	$2r-6$	8	$\cdots$
$QR =$	$2r$	2	3	$2r-3$	$2r-4$	6	7	$2r-7$	$\cdots$

$\cdots$	$r-7$	$r+7$	$r+6$	$r-4$	$r-3$	$r+3$	$r+2$	r
$\cdots$	$r+8$	$r-6$	$r-5$	$r+5$	$r+4$	$r-2$	$r-1$	$r+1$

Clearly, $c_1 = c_2 = r^2 + 5r/2 + 1$ and $c_3 = 2r+1$.

2. Suppose $r \equiv 2 \pmod 4$. For $r = 2$, assign 1 to PQ, assign 2 and 4 to the only row of PR and assign 5 and 3 to the only row of QR. Clearly, $c_1 = c_3 = 7$, $c_2 = 9$. For $r \geq 6$, assign 1 to PQ. The assignments to PR and QR are given by row 1 and row 2 respectively in the matrix below. Note that if $r \geq 10$, we would assign from the 7th column to the last column in a way similar to that for $r \equiv 0 \pmod 4$.

$PR =$	$r-4$	$r+6$	$r-2$	$r+4$	$r+3$	$r+2$	2	$2r$	$2r-1$	5	$\cdots$
$QR =$	$r+7$	$r-3$	$r+5$	$r-1$	r	$r+1$	$2r+1$	3	4	$2r-2$	$\cdots$

$\cdots$	$r-8$	$r+10$	$r+9$	$r-5$
$\cdots$	$r+11$	$r-7$	$r-6$	$r+8$

Clearly, $c_1 = c_2 = r^2 + 3r/2 + 1$ and $c_3 = 2r+3$.

3. Suppose $r \equiv 3 \pmod 4$. Assign $r+1$ to PQ. The assignments to PR and QR are given by row 1 and row 2 respectively in the matrix below. Note that if $r \geq 7$, we would assign from the 4th column to the last column in a way similar to that for $r \equiv 0 \pmod 4$.

$PR =$	$r-2$	$r+3$	$r+2$	1	$2r$	$2r-1$	4	$\cdots$
$QR =$	$r+4$	$r-1$	r	$2r+1$	2	3	$2r-2$	$\cdots$

$\cdots$	$r-6$	$r+7$	$r+6$	$r-3$
$\cdots$	$r+8$	$r-5$	$r-4$	$r+5$

Clearly, $c_1 = c_2 = r^2 + 2r + 1$ and $c_3 = 2r+2$.

[Necessity] Suppose there is a Cartesian bi-magic $(1,1,r)$-board. Clearly, $r > 1$.
Since $rc_3 = (r+1)(2r+1) - s(PQ)$, $2r+1 \leq c_3 \leq 2r+3$. So we have three cases.

(1) Suppose $c_3 = 2r+1$. In this case, $s(PQ) = 2r+1$. It follows that $c_1 = c_2 \neq c_3$. Hence, we must divide $[1,2r]$ into two disjoint sets of r integers with equal total sums. Hence, $r(2r+1)/2$ is even. So that r must be even.

(2) Suppose $c_3 = 2r+2$. In this case, $s(PQ) = r+1$. Hence, we must divide $[1,r] \cup [r+2, 2r+1]$ into two disjoint sets of r integers, say A_1 and A_2, such that (a) $s(A_1) = s(A_2) \neq r+1$, or (b) $s(A_1) = r+1$, $s(A_2) \neq r+1$, where $s(A_i)$ denotes the sum of all integers in A_i, $i = 1,2$. For both case, $s(A_1) + s(A_2) = 2r(r+1)$.

(a) Suppose $s(A_1) = s(A_2) \neq r+1$. This implies that $s(A_1) = r(r+1)$. Since $c_3 = 2r+2$, integers in $[1,r] \cup [r+2, 2r+1]$ must be paired as $(1, 2r+1)$, $(2, 2r)$, $\dots$, $(r, r+2)$ as the corresponding entries in the two $1 \times r$ rectangles PR and QR. Let $PR = \begin{pmatrix} a_1 & \cdots & a_r \end{pmatrix}$ and $QR = \begin{pmatrix} b_1 & \cdots & b_r \end{pmatrix}$. Here $a_i + b_i = c_3 = 2r+2$, $1 \leq i \leq r$. Also we have $\sum\limits_{i=1}^{r} |a_i - b_i| = \sum\limits_{i=1}^{r} 2i = r(r+1)$. Without loss of generality, we may assume $a_i > b_i$ when $1 \leq i \leq k$ and $b_j > a_j$ when $k+1 \leq j \leq r$, for some k. Thus,

$$r(r+1) = \sum_{i=1}^{r} |a_i - b_i| = \sum_{i=1}^{k} (a_i - b_i) + \sum_{j=k+1}^{r} (b_j - a_j). \tag{3.1}$$

Now

$$\begin{aligned} 0 &= s(A_1) - s(A_2) \\ &= \sum_{i=1}^{k} (a_i - b_i) + \sum_{j=k+1}^{r} (a_j - b_j) \\ &= \sum_{i=1}^{k} (a_i - b_i) - \sum_{j=k+1}^{r} (b_j - a_j). \end{aligned}$$

Hence we get $\sum\limits_{i=1}^{k} (a_i - b_i) = \sum\limits_{j=k+1}^{r} (b_j - a_j)$. Combining with (3.1) we have $r(r+1) = 2\sum\limits_{i=1}^{k} (a_i - b_i)$. Since each $a_i - b_i$ is even, $r(r+1) \equiv 0 \pmod 4$. Hence $r \equiv 0, 3 \pmod 4$.

(b) The sum of r distinct positive integers is at least $1 + \cdots + r = r(r+1)/2$. So
$s(A_1) = r+1 \geq r(r+1)/2$. Hence $r = 2$.

(3) Suppose $c_3 = 2r+3$. In this case, $s(PQ) = 1$. Hence, we must divide $[2, 2r+1]$ into two disjoint sets of r integers, say A_1 and A_2, such that (a) $s(A_1) = s(A_2) \neq 2r+2$, or (b) $s(A_1) = 2r+2$ and $s(A_2) \neq 2r+2$. For these case, $s(A_1) + s(A_2) = r(2r+3)$.

In (a), we get $2s(A_1) = r(2r+3)$. Hence r is even.

In (b), similar to the case (2)(b) above, $2r+2 = s(A_1) \geq (r+3)r/2$. Thus $r(r-1) \leq 4$, and hence $r = 2$.

□

Corollary 3.2. *Suppose $1 \le p < r$, where p is odd and r is even. The (p, p, r)-board is Cartesian bi-magic.*

Proof. When $p = 1$, the corollary follows from Theorem 3.1. For $p \ge 3$, we make use of the equations obtained in Theorem 2.1 and let $p = q$. □

Theorem 3.3. *The $(2, 2, r)$-board is Cartesian bi-magic for even r.*

Proof. For $r = 2$, a required design with $c_1 = c_3 = 24$, and $c_2 = 30$ is given below.

$$PQ = \begin{array}{|c|c|}\hline 4 & 10 \\ \hline 5 & 11 \\ \hline\end{array} \qquad PR = \begin{array}{|c|c|}\hline 8 & 2 \\ \hline 1 & 7 \\ \hline\end{array} \qquad QR = \begin{array}{|c|c|}\hline 12 & 9 \\ \hline 3 & 6 \\ \hline\end{array}$$

For $r \ge 4$, let

$$PQ = \begin{array}{|c|c|}\hline 4r+1 & 4r+2 \\ \hline 4r+4 & 4r+3 \\ \hline\end{array}$$

For $r \equiv 0 \pmod 4$, the assignments to PR and QR are then given by rows $1, 2$ and rows $3, 4$ respectively below.

$PR =$	1	8	9	16	$\cdots$	$\mathbf{2r}$	$2r+1$	$\cdots$
	$4r-1$	$4r-6$	$4r-9$	$4r-14$	$\cdots$	$\mathbf{2r+2}$	$2r-1$	$\cdots$
$QR =$	$4r-2$	$4r-5$	$4r-10$	$4r-13$	$\cdots$	$2r+3$	$2r-2$	$\cdots$
	4	5	12	13	$\cdots$	$2r-3$	$2r+4$	$\cdots$

$\cdots$	$4r-15$	$4r-8$	$4r-7$	$4r$
$\cdots$	15	10	7	2
$\cdots$	14	11	6	3
$\cdots$	$4r-12$	$4r-11$	$4r-4$	$4r-3$

Now, interchange entries $2r$ and $2r+2$. We get a Cartesian bi-magic design with $c_1 = c_2 = 2r^2 + 17r/2 + 5$, and $c_3 = 8r + 2$.

For $r \equiv 2 \pmod 4$, the assignments to PR and QR are then given by rows $1, 2$ and rows $3, 4$ respectively below.

$PR =$	4	5	12	13	$\cdots$	$\mathbf{2r}$	$2r+1$	$\cdots$
	$4r-2$	$4r-5$	$4r-10$	$4r-13$	$\cdots$	$\mathbf{2r+2}$	$2r-1$	$\cdots$
$QR =$	$4r-1$	$4r-6$	$4r-9$	$4r-14$	$\cdots$	$2r+3$	$2r-2$	$\cdots$
	1	8	9	16	$\cdots$	$2r-3$	$2r+4$	$\cdots$

$\cdots$	$4r-12$	$4r-11$	$4r-4$	$4r-3$
$\cdots$	14	11	6	3
$\cdots$	15	10	7	2
$\cdots$	$4r-15$	$4r-8$	$4r-7$	$4r$

Now, interchange entries $2r$ and $2r+2$. We also get a Cartesian bi-magic design with $c_1 = c_2 = 2r^2 + 17r/2 + 5$, and $c_3 = 8r + 2$. □

Theorem 3.4. *The (p, p, p)-board is Cartesian bi-magic for all even $p \ge 2$.*

Proof. For $p = 2$, a required design is given in the proof of Theorem 3.3. For $p = 4$, a required design is given as follows with $c_1 = c_3 = 180$, and $c_2 = 228$.

$$PQ = \begin{array}{|c|c|c|c|}\hline 41 & 14 & 15 & 44 \\ \hline 20 & 39 & 38 & 17 \\ \hline 40 & 19 & 18 & 37 \\ \hline 13 & 42 & 43 & 16 \\ \hline\end{array} \qquad PR = \begin{array}{|c|c|c|c|}\hline 29 & 2 & 3 & 32 \\ \hline 8 & 27 & 26 & 5 \\ \hline 28 & 7 & 6 & 25 \\ \hline 1 & 30 & 31 & 4 \\ \hline\end{array}$$

$$QR = \begin{array}{|c|c|c|c|}\hline 45 & 10 & 11 & 48 \\ \hline 24 & 35 & 34 & 21 \\ \hline 36 & 23 & 22 & 33 \\ \hline 9 & 46 & 47 & 12 \\ \hline\end{array}$$

For even $p = 2n \ge 6$, we can get a (p, p, p)-design that is Cartesian magic as follows.

(1) Begin with a PQ, PR and QR each of size 2×2.

(2) Substitute each entry of the above matrices by a magic square of order n using the integers in the given interval accordingly assigned below.

$$PQ = \begin{array}{|c|c|}\hline (4)\ [3n^2+1, 4n^2] & (10)\ [9n^2+1, 10n^2] \\ \hline (5)\ [4n^2+1, 5n^2] & (11)\ [10n^2+1, 11n^2] \\ \hline\end{array}$$

$$PR = \begin{array}{|c|c|}\hline (8)\ [7n^2+1, 8n^2] & (2)\ [n^2+1, 2n^2] \\ \hline (1)\ [1, n^2] & (7)\ [6n^2+1, 7n^2] \\ \hline\end{array}$$

$$QR = \begin{array}{|c|c|}\hline (12)\ [11n^2+1, 12n^2] & (9)\ [8n^2+1, 9n^2] \\ \hline (3)\ [2n^2+1, 3n^2] & (6)\ [5n^2+1, 6n^2] \\ \hline\end{array}$$

Thus, we can get a required Cartesian bi-magic design with $c_1 = c_3 = 22n^3 + 2n$ and $c_2 = 28n^3 + 2n$. □

Theorem 3.5. *The (p, p, p)-board is Cartesian bi-magic for all odd $p \ge 3$.*

Proof. Let $p = 2n + 1$ be odd. Let M be a $p \times p$ magic square, with each row sum is equal to each column sum which is $p(p^2 + 1)/2$.
Define matrices A, B, and C, each a $p \times p$ matrix, as follows. The entries in A are filled row by row. In the first row of A, $A_{1j} = 2$ for $j \in [1, n]$, $A_{1,n+1} = 1$, and $A_{1j} = 0$ for $j \in [n+2, 2n+1]$. Beginning with the second row of A, $A_{i,j} = A_{i-1,j-1}$, and $A_{i1} = A_{i-1,p}$ for $i, j \in [2, p]$. The matrix B is formed column by column such that $B_{i,j} = A_{i,n+1+j}$, for $j \in [1, n]$, and $B_{i,j} = A_{i,j-n}$, for $j \in [n+1, 2n+1]$. Clearly, beginning with the second row of B, each row can be obtained from the previous row using the rearrangements as in the rows of A. The entries in C are filled row by row. In the first row of C, $C_{1j} = 1$ for $j \in [1, n]$, $C_{1,n+1} = 0$, $C_{1j} = 1$ for $j \in [n+2, 2n]$, and $C_{1,2n+1} = 2$. Beginning with the second row of C, each row is obtained from the previous row using the rearrangements as in the rows of A.
Observe that, in each of the matrices A, B, and C, each row sum and each column sum is equal to p. In addition, for $i, j \in [1, p]$, $\{A_{i,j}, B_{i,j}, C_{i,j}\} = \{0, 1, 2\}$.
We now define the matrices PQ, PR, and QR as follows. For $i, j \in [1, p]$,

$$(PQ)_{i,j} = \begin{cases} 3M_{i,j} - A_{i,j} + 1 & \text{if } i = j, \\ 3M_{i,j} - A_{i,j} & \text{if } i \ne j, \end{cases}$$

$$(PR)_{i,j} = \begin{cases} 3M_{i,j} - B_{i,j} - 2 & \text{if } i = j, \\ 3M_{i,j} - B_{i,j} & \text{if } i \ne j, \end{cases}$$

and

$$(QR)_{i,j} = \begin{cases} 3M_{i,j} - C_{i,j} + 1 & \text{if } i = j, \\ 3M_{i,j} - C_{i,j} & \text{if } i \ne j. \end{cases}$$

In addition, note that in PQ and QR, each row sum is equal to each column sum and is equal to $3p(p^2+1)/2 - p + 1$

while in PR, each row sum is equal to each column sum and is equal to $3p(p^2+1)/2-p-2$. Hence, the (p,p,p)-board is Cartesian bi-magic with $c_1=c_3=3p(p^2+1)-2p-1$ and $c_2=3p(p^2+1)-2p+2$. $\square$

We now provide the example of $p=7$.

$$M=\begin{array}{|c|c|c|c|c|c|c|}\hline 30&39&48&1&10&19&28\\\hline 38&47&7&9&18&27&29\\\hline 46&6&8&17&26&35&37\\\hline 5&14&16&25&34&36&35\\\hline 13&15&24&33&42&44&4\\\hline 21&23&32&41&43&3&12\\\hline 22&31&40&49&2&11&20\\\hline\end{array}\quad A=\begin{array}{|c|c|c|c|c|c|c|}\hline 2&2&2&1&0&0&0\\\hline 0&2&2&2&1&0&0\\\hline 0&0&2&2&2&1&0\\\hline 0&0&0&2&2&2&1\\\hline 1&0&0&0&2&2&2\\\hline 2&1&0&0&0&2&2\\\hline 2&2&1&0&0&0&2\\\hline\end{array}$$

$$B=\begin{array}{|c|c|c|c|c|c|c|}\hline 0&0&0&2&2&2&1\\\hline 1&0&0&0&2&2&2\\\hline 2&1&0&0&0&2&2\\\hline 2&2&1&0&0&0&2\\\hline 2&2&2&1&0&0&0\\\hline 0&2&2&2&1&0&0\\\hline 0&0&2&2&2&1&0\\\hline\end{array}\quad C=\begin{array}{|c|c|c|c|c|c|c|}\hline 1&1&1&0&1&1&2\\\hline 2&1&1&1&0&1&1\\\hline 1&2&1&1&1&0&1\\\hline 1&1&2&1&1&1&0\\\hline 0&1&1&2&1&1&1\\\hline 1&0&1&1&2&1&1\\\hline 1&1&0&1&1&2&1\\\hline\end{array}$$

$$PQ=\begin{array}{|c|c|c|c|c|c|c|}\hline 89&115&142&2&30&57&84\\\hline 114&140&19&25&53&81&87\\\hline 138&18&23&49&76&104&111\\\hline 15&42&48&74&100&106&134\\\hline 38&45&72&99&125&130&10\\\hline 61&68&96&123&129&8&34\\\hline 64&91&119&147&6&33&59\\\hline\end{array}$$

$$PR=\begin{array}{|c|c|c|c|c|c|c|}\hline 88&117&144&1&28&55&83\\\hline 113&139&21&27&52&79&85\\\hline 136&17&22&51&78&103&109\\\hline 13&40&47&73&102&108&133\\\hline 37&43&70&98&124&132&12\\\hline 63&67&94&121&128&7&36\\\hline 66&93&118&145&4&32&58\\\hline\end{array}$$

$$QR=\begin{array}{|c|c|c|c|c|c|c|}\hline 90&116&143&3&29&56&82\\\hline 112&141&20&26&54&80&86\\\hline 137&16&24&50&77&105&110\\\hline 14&41&46&75&101&107&135\\\hline 39&44&71&97&126&131&11\\\hline 62&69&95&122&127&9&35\\\hline 65&92&120&146&5&31&60\\\hline\end{array}$$

Theorem 3.6. *For $p\geq 3$, (i) the (p,p,r)-board is Cartesian bi-magic when $r=p$ or r is even, (ii) the (p,r,r)-board is Cartesian bi-magic for even p.*

Proof. (i) View the board as containing a $(2p,r)$-rectangle and another (p,p)-square. Since $p^2<2pr$, we first assign integers in $[1,p^2]$ to the (p,p)-rectangle to get a $p\times p$ magic square with magic constant $p(p^2+1)/2$. Next we assign integers in $[p^2+1,p^2+2pr]$ to get a magic $(2p,r)$-rectangle with row sum constant $r(2pr+1)/2+p^2r$ and column sum constant $p(2pr+1)+2p^3$. Note that the existence of magic rectangle of even order can be found in [4]. Hence, the assignment we have now is Cartesian bi-magic with $c_1=c_2=p(p^2+1)/2+r(2p^2+2pr+1)/2\neq c_3=2p^3+2p^2r+p$.

(ii) We repeat the approach as in (i). Begin with the $(p,2r)$-rectangle and then the (r,r)-rectangle if $2pr<r^2$. Otherwise, reverse the order. $\square$

From the proof of Theorem 2.3, we also have

Corollary 3.7. *Suppose $2\leq p<r$, where p is even and r is odd. The (p,r,r)-board is Cartesian bi-magic.*

From the proof of Theorem 2.8, one can check that $c_1>c_2=c_3$ if $q=r$.

Corollary 3.8. *Suppose $r\geq 2$. The $(1,r,r)$-board is Cartesian bi-magic.*

4. Cartesian magic

Clearly, the (1,1,1)-board is not Cartesian magic. In this section, we always assume $(p,q,r)\neq(1,1,1)$. Let m be the magic constant of a Cartesian magic (p,q,r)-board.

Lemma 4.1. *If a (p,q,r)-board is Cartesian magic, then $p+q+r$ divides $(pq+pr+qr)(pq+pr+qr+1)$.*

Proof. It follows from the fact that $(p+q+r)m=2[1+2+\cdots+(pq+pr+qr)]$. $\square$

Lemma 4.2. *If a (p,q,r)-board is Cartesian magic, then*

(i) $s(PQ)+s(PR)=mp,\ s(PQ)+s(QR)=mq,\ s(PR)+s(QR)=mr$;

(ii) $s(QR)-s(PR)=m(q-p)$;

(iii) $s(QR)-s(PQ)=m(r-p)$;

(iv) $s(PR)-s(PQ)=m(r-q)$.

Proof. By definition, we get (i). Clearly, (ii), (iii) and (iv) follow from (i). $\square$

Theorem 4.3. *If a (p,p,pr)-board is Cartesian magic for $p,r\geq 1$, then $r=1$.*

Proof. Under the hypothesis, by Lemma 4.1, $m=(2p^2r+p^2)(2p^2r+p^2+1)/(pr+2p)$. By Lemma 4.2(i), $s(PR)+s(QR)=prm$. Since $s(PQ)\geq 1$, $s(PR)+s(QR)$ is always less than the sum of all labels. That is, $r(2p^2r+p^2)(2p^2r+p^2+1)/(r+2)<(2p^2r+p^2)(2p^2r+p^2+1)/2$. We have $\frac{r}{r+2}<\frac{1}{2}$. Hence $r=1$. $\square$

Theorem 4.4. *There is no Cartesian magic $(1,q,r)$-board for all $r\geq q\geq 1$.*

Proof. Suppose there is a Cartesian magic $(1,q,r)$-board. By Lemma 4.2(i), $s(PR)+s(QR)=mr=r(q+1)(r+1)(q+r+qr)/(1+q+r)$. Thus we have $r(q+1)(r+1)(q+r+qr)/(1+q+r)<(q+1)(r+1)(q+r+qr)/2$. This implies that $r<1+q$, hence $r=q$. By Lemma 4.2(i) and (iv) we know that m is an even number. So $m/2=\frac{q(q+1)^2(q+2)}{4q+2}=q^2+q+\frac{q^4-q^2}{2(2q+1)}$ is an integer. So $2q+1$ is a factor of $q^2(q+1)(q-1)$. Since $\gcd(2q+1,q)=1$ and $\gcd(2q+1,q+1)=1$, $2q+1$ is a factor of $q-1$. It is impossible except when $q=1$. But when $q=1$, we know that there is no Cartesian magic $(1,1,1)$-board. This completes the proof. $\square$

From [12], we know that $K(p,p,p) \cong C_3 \circ N_p$, the lexicographic product of C_3 and N_p, is supermagic. That is, (p,p,p)-board is Cartesian magic for $p \geq 2$. In the following theorems, we provide another Cartesian magic labeling.

Theorem 4.5. *The (p,p,p)-board is Cartesian magic for all even $p \geq 2$.*

Proof. For $p=2$, a required design with $m=26$ is given by the rectangles below.

$PQ =$

8	5
6	7

$PR =$

4	9
1	12

$QR =$

10	2
11	3

For $p=4$, a required design with $m=196$ is given below.

$PQ =$

32	18	19	29
25	23	22	28
17	31	30	20
24	26	27	21

$PR =$

48	2	3	45
33	15	14	36
1	47	46	4
16	34	35	13

$QR =$

44	6	7	41
37	11	10	40
5	43	42	8
12	38	39	9

For $p=2n \geq 6$, using the approach as in the proof of Theorem 3.4 and the (2,2,2)-design as above. Clearly, the Cartesian magic constant thus obtained is $m = p(3p^2+1)$. □

For example, a Cartesian magic (6,6,6)-design is given below with $m=654$.

$PQ =$

65	72	67	38	45	40
70	68	66	43	41	39
69	64	71	42	37	44
47	54	49	56	63	58
52	50	48	61	59	47
51	46	53	60	55	62

$PR =$

29	36	31	74	81	76
34	32	30	79	77	75
33	28	35	78	73	80
2	9	4	101	108	103
7	5	3	106	104	102
6	1	8	105	100	107

$QR =$

83	90	85	11	18	13
88	86	84	16	14	12
87	82	89	15	10	17
92	99	94	20	27	22
97	95	93	25	23	21
96	91	98	24	19	26

Theorem 4.6. *The (p,p,p)-board is Cartesian magic for all odd $p \geq 3$.*

Proof. Let A, B, C and M be as defined in the proof of Theorem 3.5. We now define the matrices PQ, PR, and QR as follows. For $i,j \in [1,p]$, $(PQ)_{i,j} = 3M_{i,j} - A_{i,j}$, $(PR)_{i,j} = 3M_{i,j} - B_{i,j}$, and $(QR)_{i,j} = 3M_{i,j} - C_{i,j}$. Thus,

$$\{(PQ)_{i,j}, (PR)_{i,j}, (QR)_{i,j} : i,j \in [1,p]\}$$
$$= \{3M_{i,j} - k : i,j \in [1,p], k \in [0,2]\} = [1, 3p^2].$$

In addition, note that each row sum and each column sum is equal to $3p(p^2+1)/2 - p$. Hence, the (p,p,p)-board is Cartesian magic with $m = 3p(p^2+1) - 2p$. □

We now provide the example of $p=7$. Only matrices PQ, PR and QR are shown.

$PQ =$

88	115	142	2	30	57	84
114	139	19	25	53	81	87
138	18	22	49	76	104	111
15	42	48	73	100	106	134
38	45	72	99	124	130	10
61	68	96	123	129	7	34
64	91	119	147	6	33	58

$PR =$

90	117	144	1	28	55	83
113	141	21	27	52	79	85
136	17	24	51	78	103	109
13	40	47	75	102	108	133
37	43	70	98	126	132	12
63	67	94	121	128	9	36
66	93	118	145	4	32	60

$QR =$

89	116	143	3	29	56	82
112	140	20	26	54	80	86
137	16	23	50	77	105	110
14	41	46	74	101	107	135
39	44	71	97	125	131	11
62	69	95	122	127	8	35
65	92	120	146	5	31	59

5. Miscellany and unsolved problems

Here are some ad hoc examples:

Example 5.1. Modifying a 3×3 magic square we get a Cartesian bi-magic labeling for the (1,2,3)-board whose corresponding labeling matrix is

$$\left(\begin{array}{c|cc|ccc} * & 1 & 2 & 5 & 11 & 4 \\ \hline 1 & * & * & 6 & 7 & 9 \\ 2 & * & * & 10 & 3 & 8 \\ \hline 5 & 6 & 10 & * & * & * \\ 11 & 7 & 3 & * & * & * \\ 4 & 9 & 8 & * & * & * \end{array}\right) \left(\begin{array}{c} 23 \\ \hline 23 \\ 23 \\ \hline 21 \\ 21 \\ 21 \end{array}\right)$$

Example 5.2. A Cartesian tri-magic labeling for the (1,3,3)-board whose corresponding labeling matrix is

$$\left(\begin{array}{c|ccc|ccc} * & 10 & 13 & 7 & 12 & 4 & 8 \\ \hline 10 & * & * & * & 1 & 6 & 15 \\ 13 & * & * & * & 3 & 11 & 5 \\ 7 & * & * & * & 14 & 9 & 2 \\ \hline 12 & 1 & 3 & 14 & * & * & * \\ 4 & 6 & 11 & 9 & * & * & * \\ 8 & 15 & 5 & 2 & * & * & * \end{array}\right) \left(\begin{array}{c} 54 \\ \hline 32 \\ 32 \\ 32 \\ \hline 30 \\ 30 \\ 30 \end{array}\right)$$

Example 5.3. A Cartesian tri-magic labeling for the (2,3,3)-board whose corresponding labeling matrix is

$$\left(\begin{array}{cc|ccc|ccc} * & * & 14 & 11 & 12 & 21 & 17 & 18 \\ * & * & 13 & 15 & 10 & 16 & 19 & 20 \\ \hline 14 & 13 & * & * & * & 4 & 2 & 7 \\ 11 & 15 & * & * & * & 3 & 5 & 6 \\ 12 & 10 & * & * & * & 8 & 9 & 1 \\ \hline 21 & 16 & 4 & 3 & 8 & * & * & * \\ 17 & 19 & 2 & 5 & 9 & * & * & * \\ 18 & 20 & 7 & 6 & 1 & * & * & * \end{array}\right) \left(\begin{array}{c} 93 \\ 93 \\ \hline 40 \\ 40 \\ 40 \\ \hline 52 \\ 52 \\ 52 \end{array}\right)$$

It is well known that magic rectangles and magic squares have wide applications in experimental designs [8–10]. Thus, results on Cartesian magicness can be potential tools for use in situations yet unexplored. We end this paper with the following open problems and conjectures.

Problem 5.4. *Characterize Cartesian tri-magic $(1,r,r)$-boards for $r \geq 4$.*

Problem 5.5. *Characterize Cartesian tri-magic (p,p,r)-boards for $r > p$ where p is odd, $p \geq 3$ and r is even.*

Problem 5.6. *Characterize Cartesian tri-magic (p,r,r)-boards for $r > p$ where p is even, $p \geq 2$ and r is odd.*

Problem 5.7. *Characterize Cartesian tri-magic (p,q,r)-boards for $r \geq q \geq p \geq 2$ where exactly two of the parameters are even.*

Conjecture 5.8. *Almost all (p,q,r)-boards are Cartesian bi-magic.*

Conjecture 5.9. *Almost all (p,q,r)-boards are not Cartesian magic.*

Finally, we may say a (p,q,r)-board is *Pythagorean magic* if $\{c_1,c_2,c_3\}$ is a set of Pythagorean triple. Thus, both $(1,1,1)$- and $(1,1,2)$-boards are Pythagorean magic. The study of Pythagorean magic is another interesting and difficult research problem.

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