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Method of lower - upper solutions of fractional differential equations with initial time difference and applications

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Abstract

This paper deals with the method of lower - upper solutions of Riemann-Liouville (R-L)fractional differential equations with initial time difference. Monotone technique coupled with lower - upper solutions of the problem is developed and is successfully applied to obtain existence and uniqueness results with initial time difference.

Keywords

Lower - Upper Solutions, Monotone technique, Existence and uniqueness, Initial time difference, Fractional differential equations.

2010 Mathematics Subject Classification

34A12, 34C60, 34A45.

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Contents

1. Introduction

The study of theory of fractional differential equations [7, 9, 14] parallel to the well-known theory of ordinary differential equations [5, 6] has been attracted researchers since last three decades due to wide applications in many branches of sciences, engineering, nature and social sciences. V. Lakshmikantham et.al [8] proved local and global existence of solutions of Riemann-Liouville fractional differential equations and uniqueness of solutions. Monotone iterative method for Riemann-Liouville fractional differential equations with initial conditions is studied by McRae [10]. Vasundhara Devi established [1] the general monotone method for periodic boundary value problem of Caputo fractional differential equation. The Caputo fractional differential equation with periodic boundary conditions have been studied in [2, 3] and developed monotone method for the problem. Existence and uniqueness of solution of Riemann-Liouville fractional differential equation with integral boundary conditions is obtained in [11, 12]. Recently, initial value problems involving Riemann-Liouville fractional derivative was studied by authors [4, 13]

In 2011, C. Yaker et.al. [15] developed monotone iterative method for the following nonlinear fractional differential equations with initial time difference with locally H$\ddot{o}$lder continuous function and obtained existence and uniqueness of solution of the problem.

$$D^q u(t) = f(t,u), \quad u^0 = u(t)(t-t_0)^{1-q}\}_{t=t_0} \tag{1.1}$$

where $0<q<1$ and $f \in C[\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}], D^q$ is the Riemann-Liouville fractional derivative of order q.

In this paper, we improve the results obtained by Yaker et.al. for the class of all continuous functions

$u(t) \in C_p(J,R) = \{u(t) \in C(J,R) \ and \ u(t)(t-t_0)^p \in C(J,R)\}$, $J=[t_0,T]$.

In section 2, we consider some basic definitions and results required in main results. In section 3, monotone iterative method is developed for the problem and as an application of the method existence and uniqueness results for Riemann-Liouville fractional differential equation with initial time difference are established.

2. Preliminaries

Here, we consider some basic definitions and results for the development of our main results.

Definition 2.1. *A pair of functions $v(t)$ and $w(t)$ in $C_p(J,\mathbb{R})$ is said to be lower and upper solutions of the problem (1.1) if*

$$D^q v(t) \le f(t,v(t)), \quad v^0 \le u^0, \quad D^q w(t) \ge f(t,w(t)), \quad w^0 \ge u^0.$$

where $u^0 = u(t)(t-t_0)^{1-q}\}_{t=t_0}$

Lemma 2.2. *[1] Let $m \in C_p([t_0,T],\mathbb{R})$ and for any $t_1 \in (t_0,T]$ we have $m(t_1)=0$ and $m(t)<0$ for $t_0 \le t \le t_1$. Then it follows that $D^q m(t_1) \ge 0$.*

Theorem 2.3. *[11] Assume that*

(i) *$v(t)$ and $w(t)$ in $C_p(J,\mathbb{R})$ are lower and upper solutions of (1.1)*

(ii) *$f(t,u(t))$ satisfy one-sided Lipschitz condition*

$$f(t,u)-f(t,v) \le L(u-v), \quad ,u \ge v, \quad L \in (0, \frac{1}{\Gamma(1-q)T^q})$$

Then $v^0 \le w^0$ implies that $v(t) \le w(t), \quad t_0 \le t \le T$.

Corollary 2.4. *The function $f(t,u)=\sigma(t)u$, where $\sigma(t) \le L$, is admissible in Theorem 2.3 to yield $u(t) \le 0$ on $t_0 \le t \le T$.*

The nonhomogeneous linear fractional differential equation with initial conditions

$$D^q u(t) = \lambda u + f(t), \quad u^0 = u(t)(t-t_0)^{1-q}\}_{t=t_0}$$

has following integral representation of solution

$$u(t) = u^0(t) + \frac{\lambda}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}u(s)ds + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}E_{q,q}(\lambda(t-s)^q f(s)ds \tag{2.1}$$

$t \in [t_0,T]$ where $E_{q,q}$ denotes the two parameter Mittag-Leffler function. Now, we prove the existence of solutions in the closed set $\Omega = \{(t,u) | v(t)| \le u(t) \le w(t), t \in [t_0,T]\}$, if we know the existence of lower and upper solutions v,w satisfying $v(t) \le w(t), t \in [t_0,T]$, of the initial value problem (IVP) (1.1).

Theorem 2.5. *Let $v,w \in C_p([t_0,T],\mathbb{R})$ be lower and upper solutions of the IVP (1.1) such that $v(t) \le w(t), t \in [t_0,T]$ and $f \in C(\Omega,\mathbb{R})$. Then there exists a solution $u(t)$ of the IVP (1.1) satisfying $v(t) \le u(t) \le w(t)$ on $[t_0,T]$.*

Following existence and comparison results are analogous to the existence and comparison results proved by Yaker et.al. [15] when the lower and upper solutions start at different initial times. The proof of these results are similar to that of the proof of the existence and comparison results proved by Yaker et.al.[15].

Theorem 2.6. *Assume that*

(i) *$v \in C_p[t_0,t_0+T],\mathbb{R}], t_0, T > 0, w \in C_p^*[\tau_0,\tau_0+T],\mathbb{R}]$ is continuous and $p = 1-q$ where*
$C_p(J,R) = \{u(t) \in C(J,R)$ and $u(t)(t-t_0)^p \in C(J,R)\}$,
$J = [t_0,t_0+T], C_p^*(J^*,R) = \{u(t) \in C(J^*,R)$ and $u(t)(t-t_0)^p \in C(J^*,R)\}$,
$J^* = [\tau_0,\tau_0+T]$,
$f \in C[[t_0,\tau_0+T] \times \mathbb{R},\mathbb{R}]$ and

$$D^q v(t) = f(t,u(t)), \quad t_0 \le t \le t_0+T, \quad D^q w(t) = f(t,w(t)), \quad \tau_0 \le t \le \tau_0+T,$$

$v^0 \le u^0 \le w^0$, where $v^0 = v(t)(t-t_0)^{1-q}\}_{t=t_0}, w^0 = w(t)(t-\tau_0)^{1-q}\}_{t=\tau_0}$

(ii) *$f(t,u)$ satisfies Lipschitz condition,*

$$f(t,u)-f(t,v) \le L[u-v], \text{for } u \ge v, \text{and}, M \ge 0.$$

(iii) *$\tau_0 > t_0$ and $f(t,u)$ is nondecreasing in t for each u*

Then we have

(a) *$v(t) \le w(t+\eta), t_0 \le t \le t_0+T$,*

(b) *$v(t-\eta) \le w(t), \tau_0 \le t \le \tau_0+T$, where $\eta = \tau_0 - t_0$.*

Existence result with initial time difference is given in the following theorem:

Theorem 2.7. *Assume that*

(i) *Assumption (i) of Theorem 2.6 holds.*

(ii) *$f(t,u)$ is nondecreasing in t for each u and $v(t) \le w(t+\eta), t_0 \le t \le t_0+T$, where $\eta = \tau_0 - t_0$*

Then there exists a solution $u(t)$ of (1.1) with $u^0 = u(t)(t-t_0)^{1-q}\}_{t=t_0}$ satisfying $v(t) \le u(t) \le w(t+\eta)$ on $t_0, t_0+T]$.

3. Main Results

In this section, we apply monotone iterative method by choosing lower and upper solutions with initial time difference for IVP (1.1) to obtain the existence and uniqueness of solution of the IVP (1.1).

Theorem 3.1. *Assume that*

(E_1) *$v \in C_p[t_0,t_0+T],\mathbb{R}], t_0, T > 0, w \in C_p^*[\tau_0,\tau_0+T],\mathbb{R}]$ is continuous and $p = 1-q$ where $C_p(J,R) = \{u(t) \in C(J,R)$ and $u(t)(t-t_0)^p \in C(J,R)\}, J = [t_0,t_0+T]$, $C_p^*(J^*,R) = \{u(t) \in C(J^*,R)$ and $u(t)(t-t_0)^p \in C(J^*,R)\}$,*
$J^* = [\tau_0,\tau_0+T], f \in C[[t_0,\tau_0+T] \times \mathbb{R},\mathbb{R}]$ and

$$D^q v(t) = f(t,u(t)), \quad t_0 \le t \le t_0+T, \quad D^q w(t) = f(t,w(t)), \quad \tau_0 \le t \le \tau_0+T,$$

$v^0 \le u^0 \le w^0$, where $v^0 = v(t)(t-t_0)^{1-q}\}_{t=t_0}$, $w^0 = w(t)(t-\tau_0)^{1-q}\}_{t=\tau_0}$

(E_2) *$f(t,u)$ satisfies one-sided Lipschitz condition,*

$$f(t,u)-f(t,v)\leq L[u-v], \textit{for } u\geq v, \textit{and}, M\geq 0.$$

(E_3) *$f(t,u)$ is nondecreasing in t for each u and $v(t)\leq w(t+\eta), t_0\leq t\leq t_0+T$, where $\eta=\tau_0-t_0$*

Then there exists monotone sequences $\{v_n(t)\}$ and $\{w_n(t)\}$ in $C_p(J,\mathbb{R})$ which converges uniformly and monotonically on $[t_0,t_0+T]$ such that

$$\{v_n(t)\}\to v(t) \quad and \quad \{w_n(t)\}\to w(t) \quad as \quad n\to\infty$$

where $v(t)$ and $w(t)$ are minimal and maximal solutions of IVP (1.1) respectively.

Proof. Let $w_0(t)=w(t+\eta)$ and $v_0(t)=v(t)$ for $t_0\leq t\leq t_0+T$, where $\eta=\tau_0-t_0$. Since $f(t,u)$ is nondecreasing in t for each u we have

$$D^q w_0(t)=D^q w(t+\eta)\geq f(t+\eta,w(t+\eta))\geq f(t,w_0(t))$$

and $w_0^0=w_0(t)(t-t_0)^{1-q}\}_{t=t_0}=w(t+\eta)(t-t_0)^{1-q}\}_{t=t_0}=w(t)(t-t_0)^{1-q}\}_{t=t_0}=w^0$
Also,

$$D^q v_0(t)=D^q v(t)\leq f(t,v_0(t))$$

and $v_0^0=v_0(t)(t-t_0)^{1-q}\}_{t=t_0}=v(t)(t-t_0)^{1-q}\}_{t=t_0}=v^0$, $v^0\leq u^0\leq w^0$
which proves that v_0 and w_0 are lower and upper solutions of IVP (1.1) respectively.
For any μ in $C_p(J,\mathbb{R})$ such that for $v_0(t)\leq\mu\leq w_0(t)$ on J, consider the following linear fractional differential equation

$$D^q u(t)=f(t,\mu)-M(u-\mu), \quad u^0=u(t)(t-t_0)^{1-q}\}_{t=t_0}. \tag{3.1}$$

Since the right hand side of IVP (3.1) satisfies Lipschitz condition, unique solution of IVP (3.1) exists. Let A be a mapping such that $A\mu=u$ and construct sequences $\{v_n\}$ and $\{w_n\}$. We prove that $v_0\leq Av_0, w_0\geq Aw_0$ and A is a monotone operator on the segment $[v_0,w_0]=\{u\in C_p(J,\mathbb{R}): v_0\leq u\leq w_0\}$ To prove $v_0\leq Av_0, w_0\geq Aw_0$, set $Av_0=v_1$ where v_1 is the unique solution of IVP (3.1) with $\mu=v_0$. Setting $p(t)=v_0(t)-v_1(t)$ we have

$$D^q p(t)=D^q v_0(t)-D^q v_1(t)\leq -Mp(t)$$
$$\text{and} \quad p(t)=0.$$

By Corollary 2.4, it follows that $v_0(t)\leq v_1(t)$. Similarly, we prove $w_0(t)\geq w_1(t)$. Nextly, we prove A is monotone operator. Let $\mu_1,\mu_2\in[v_0,w_0]$ be such that $\mu_1\leq\mu_2$. Also, suppose that $u_1=A\mu_1$ and $u_2=A\mu_2$. Set $p(t)=u_1(t)-u_2(t)$, then

$$\begin{aligned}D^q p(t)&=D^q u_1(t)-D^q u_2(t)=f(t,\mu_1)-\\&M(u_1-\mu_1)-f(t,\mu_2)+M(u_2-\mu_2)\leq\\&-M(\mu_1-\mu_2)-M(u_1-\mu_1)+M(u_2-\mu_2)\\&=-M(u_1-u_2)=-Mp(t)\end{aligned}$$
$$\text{and} \quad p(t)=0.$$

Applying Corollary 2.4, it follows that, $A\mu_1\leq A\mu_2$. Now we define the sequences $v_n=Av_{n-1}$ and $w_n=Aw_{n-1}$ and conclude that

$$v_0(t)\leq v_1(t)\leq\ldots\leq v_n(t)\leq w_n(t)\leq\ldots\leq w_1(t)\leq w_0(t).$$

Clearly, the sequences $\{v_n\}$ and $\{w_n\}$ are uniformly bounded on J and it follows that $\{D^q v_n\}$ and $\{D^q w_n\}$ are also uniformly bounded. Hence the sequences $\{v_n\}$ and $\{w_n\}$ are equicontinuous on J and consequently by Ascoli-Arzela's theorem there exist subsequences $\{v_{nk}\}$ and $\{w_{nk}\}$ that converge uniformly on J. Thus it follows that the sequences $\{v_n\}$ and $\{w_n\}$ converge uniformly and monotonically to v and w respectively, as $n\to\infty$. Now, we prove that v and w are solutions of IVP (1.1). Using corresponding Volterra integral equations for (1.1), we have

$$\begin{aligned}v_{n+1}(t)&=\frac{u^0(t-t_0)^{q-1}}{\Gamma(q)}+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\{f(s,v_n(s))-\\&M(v_{n+1}-v_n)\}ds\\w_{n+1}(t)&=\frac{u^0(t-t_0)^{q-1}}{\Gamma(q)}+\frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}\{f(s,w_n(s))-\\&M(w_{n+1}-w_n)\}ds\end{aligned}$$

as $n\to\infty$, we get

$$\begin{aligned}v(t)&=\frac{u^0(t-t_0)^{q-1}}{\Gamma(q)}+\frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}f(s,v_n(s))ds\\w(t)&=\frac{u^0(t-t_0)^{q-1}}{\Gamma(q)}+\frac{1}{\Gamma(q)}\int_{t_0}^{T}(t-s)^{q-1}f(s,w_n(s))ds\end{aligned}$$

where $u^0=u(t)(t-t_0)^{1-q}|_{t=t_0}$. Lastly, we prove that $v(t)$ and $w(t)$ are the minimal and maximal solutions of IVP (1.1), respectively. Let $u(t))$ be any solution of IVP (1.1) such that $v_0\leq u\leq w_0$ on J. For this, suppose that for some k, $v_k(t)\leq u(t)\leq w_k(t)$ on J and set $p(t)=v_{k+1}(t)-u(t)$ so that

$$\begin{aligned}D^q p(t)&=D^q v_{k+1}(t)-D^q u(t)\leq -M(v_{k+1}-u)-\\&M(v_{k+1}-v_k)=-Mp(t)\end{aligned}$$
$$\text{and} \quad p(t)(t-t_0)^{1-q}|_{t=t_0}=0.$$

By Corollary 2.4, it follows that $v_{k+1}\leq u$ on J. Similarly, we can prove $u(t)\leq w_{k+1}(t)$ on J. By induction it follows that $v_n(t)\leq u(t)\leq w_n(t)$ for all n on J . Taking limit as $n\to\infty$, it follows that $v(t)\leq u(t)\leq w(t)$ on J. □

Theorem 3.2. *Assume that*

(U_1) *Assumptions E_1 and E_3 of Theorem 3.1 holds.*

(U_2) *$f(t,u)$ satisfies Lipschitz condition,*

$$|f(t,u)-f(t,v)|\leq L|u-v|, \textit{for } u\geq v, \textit{and}, M\geq 0.$$

then there exists unique solution of IVP (1.1).

Proof. It is sufficient to prove $v(t) \geq w(t)$. If $p(t) = w(t) - v(t)$ then

$$D^q p(t) \leq M(w(t) - v(t)) \leq -Mp(t)$$
$$\text{and} \quad p(0) = 0.$$

Thus, by Corollary 2.4, we get $p(t) \leq 0$ implies $w(t) \leq v(t)$. Hence $v(t) = u(t) = w(t)$ is the unique solution of (1.1) on $[t_0, t_0 + T]$. □

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