Malaya Journal of Matematik, Vol. 8, No. 3, 1228-1233, 2020
https://doi.org/10.26637/MJM0803/0087

Ball convergence of a novel bi-parametric iterative scheme for solving equations

Ioannis K. Argyros [1] and Santhosh George[2]*

Abstract
The aim of this article is to establish a ball convergence result for a bi-parametric iterative scheme for solving equations involving Banach space valued operators. In contrast to earlier approaches in the less general setting of the k-dimensional Euclidean space where hypotheses on the seventh derivative are used, we only use hypotheses on the first derivative. Hence, we extend the applicability of the method. Moreover, the radius of convergence as well as error bounds on the distances are given based on Lipschitz-type functions. Numerical examples are given to test our conditions. These examples show that earlier convergence conditions are not satisfied but ours are satisfied.

Keywords
Bi-parametric iterative scheme, Banach space, Ball convergence, Lipschitz-type conditions, Fréchet-derivative.

AMS Subject Classification
65G99, 65H10, 65J15, 49M15.

[1] *Department of Mathematical Sciences, Cameron University, Lawton, OK 73505, USA.*
[2] *Department of Mathematical and Computational Sciences, National Institute of Technology Karnataka, India-575025.*
***Corresponding author**: [1] iargyros@cameron.edu; [2]sgeorge@nitk.edu.in
Article History: Received **17** November **2018**; Accepted **09** April **2020**

Contents

1. Introduction

Let X,Y stand for Banach spaces, $D \subseteq X$ be a nonempty convex and open set and let $\mathscr{L}(X,Y)$ stand for the space of bounded linear operators from X into Y.

Numerous problems in mathematics, sciences and engineering are written in a form like

$$G(x) = 0, \tag{1.1}$$

where $G : D \longrightarrow Y$ stands for a differentiable operator in the sense of Fréchet. Iterative schemes are used to generate a sequence approximating a solution x_* of equation (1.1), which is unique in a neighborhood of x_*. This is done, since closed type or analytic type solutions are usually hard to find.

A great effort is made recently to develop high convergence order iterative schemes [1–22]. In particular, we define for each $n = 0,1,2,\ldots$ the three-step iterative scheme [5]

$$\begin{aligned} y_n &= x_n - \frac{2}{3}G'(x_n)^{-1}G(x_n) \\ z_n &= x_n - [I + \frac{3}{4}M_n(I + 6(4I - 3\alpha M_n)^{-1}M_n] \\ &\quad \times G'(x_n)^{-1}G(x_n) \\ x_{n+1} &= z_n - [(\beta G'(x_n) + \gamma G'(y_n))^{-1}(G'(x_n) \\ &\quad + \delta G'(y_n))]G'(x_n)^{-1}G(z_n), \end{aligned} \tag{1.2}$$

where $x_0 \in D$ is a given initial point, $M_n = I - G'(x_n)^{-1}G'(y_n)$, $\alpha \in \mathbb{R}$ $\beta = \frac{2-3\gamma}{5}$, $\delta = \frac{2\gamma-3}{5}$ and $\gamma \in \mathbb{R} - \{-1\}$. The ball convergence of iterative scheme (1.2) was given in [5] when $X = Y = \mathbb{R}^j$ (j a natural number). The motivation, development and advantages of this iterative scheme over other competing scheme be also found in [5]. Using hypotheses on the Fréchet-derivatives reaching the seventh order as well as Taylor series expansions, the sixth convergence order of the method was established. However, the high order derivatives restrict the applicability of method (1.2) to cases when G is at least seven times Fréchet-differentiable. As an academic and motivational example, consider function G on $D = [-\frac{1}{2}, \frac{5}{2}]$ by

$$G(x) = \begin{cases} x^3 lnx^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

We have that $x^* = 1$,

$$G'(x) = 3x^2 lnx^2 + 5x^4 - 4x^3 + 2x^2,$$

$$G''(x) = 6xlnx^2 + 20x^3 - 12x^2 + 10x$$

and

$$G'''(x) = 6lnx^2 + 60x^2 - 24x + 22.$$

Function $G'''(x)$ is unbounded on D. Moreover, the radius of convergence or computable error bounds on the distances $\|x_n - x_*\|$ or uniqueness of the solution result of how close x_0 should be to x_* for convergence are not given in [5]. In our article, we address these concerns. First of all, we only use hypotheses on the first Fréchet-derivative. The radius of convergence, upper error bounds on $\|x_n - x_*\|$ and uniqueness results are given using Lipschitz-type constants. Moreover, the convergence order is computed using computational order of convergence (COC) or approximate computational order of convergence (ACOC) [1–3], which do not utilize higher than one Fréchet derivatives. It is worth mentioning that ball convergence results provide the degree of difficulty for choosing initial points, so they are very useful.

The rest of the article is structured: Section 2 deals with the Ball convergence of iterative scheme (1.2). The numerical examples can be found in Section 3 that completes this article.

2. Ball convergence

We base the ball convergence of method (1.2) on some parameters and scalar functions. Suppose that equation

$$\lambda_0(t) = 1 \tag{2.1}$$

has at least one positive root. Denote by ρ_0 the smallest such root, where $\lambda_0 : [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous and increasing function with $\lambda_0(0) = 0$. Let $\lambda : [0,\rho_0) \longrightarrow [0,+\infty)$, $\mu : [0,\rho_0) \longrightarrow [0,+\infty)$ be also continuous and increasing functions with $\lambda(0) = 0$. Define functions φ_1 and ψ_1 on the interval $[0,\rho_0)$ by

$$\varphi_1(t) = \frac{\int_0^1 \lambda((1-\theta)t)d\theta + \frac{1}{3}\int_0^1 \mu(\theta t)d\theta}{1-\lambda_0(t)}$$

and

$$\psi_1(t) = \varphi_1(t) - 1.$$

Suppose that

$$\mu(0) < 3. \tag{2.2}$$

We have $\psi_1(0) < 0$ and $\psi_1(t) \longrightarrow +\infty$ as $t \longrightarrow \rho_0^-$. By the intermediate value theorem (IVT) equation $\psi_1(t) = 0$ has at least one positive root. Denote by r_1 the smallest such root. By IVT equation

$$\frac{3}{4}|\alpha|\frac{\lambda_0(t)+\lambda_0(\varphi_1(t)t)}{1-\lambda_0(t)} = 1 \tag{2.3}$$

has at least one positive root in $(0,\rho_0)$. Denote by ρ_1 the smallest such root. Define functions $p, q\varphi_2, \psi_2$ on $[0,\min\{\rho_0,\rho_1\})$, by

$$p(t) = 6[|4-3\alpha| + 3|\alpha|\int_0^1 \mu(\theta\varphi_1(t)t)td\theta]$$

$$q(t) = p(t) - 1,$$

$$\varphi_2(t) = \varphi_1(t) + \frac{3}{4}\frac{\left(\frac{\lambda_0(t)+\lambda_0(\varphi_1(t)t)}{1-\lambda_0(t)}\right)^2\left(\frac{\int_0^1\mu(\theta t)d\theta}{1-\lambda_0(t)}\right)}{1-p(t)}$$

and

$$\psi_2(t) = \varphi_2(t) - 1.$$

Suppose that equation

$$p(t) = 1 \tag{2.4}$$

has at least one positive root. Denote by ρ_2 the smallest such root. We get again $\psi_2(0) < 0$ and $\psi_2(t) \longrightarrow +\infty$ as $t \longrightarrow \min\{\rho_0,\rho_1\}^-$. Denote by r_2 the smallest positive root of equation $\psi_2(t) = 0$. Notice that $\beta+\gamma \neq 0$. By IVT equation

$$\frac{|\beta|\lambda_0(t)+|\gamma|\lambda_0(\varphi_1(t)t)}{|\beta+\gamma|} = 1$$

has at least one positive root in $(0,\rho_0)$. Denote by ρ_2, the smallest such root. Define functions $p_1, q_1, \varphi_3, \psi_3$ on $[0,\min\{\rho_0,\rho_1,\rho_2,\rho_3\})$ by

$$p_1(t) = \frac{|\beta|\lambda_0(t)+|\gamma|\lambda_0(\varphi_1(t)t)}{|\beta+\gamma|},$$

$$q_1(t) = p_1(t) - 1,$$

$$\varphi_3(t) = \left[1 + \frac{(\mu(t)+|\delta|\mu(\varphi_1(t)t))\int_0^1\mu(\theta\varphi_2(t)t)d\theta}{(1-p_1(t))|\beta+\gamma|(1-\lambda_0(t))}\right]\varphi_2(t)$$

and

$$\psi_3(t) = \varphi_3(t) - 1.$$

Suppose that

$$\left(1 + \frac{(1+|\delta|)\mu(0)^2}{|\beta+\gamma|}\right)\frac{\mu(0)}{3} < 1 \tag{2.5}$$

We obtain $\psi_3(0) < 0$ and $\psi_3(t) \longrightarrow +\infty$ as $t \longrightarrow \min\{\rho_0,\rho_1,\rho_2,\rho_3\}$. Denote by r_3 the smallest positive root of equation $\psi_3(t) = 0$. Define the radius of convergence r by

$$r = \min\{r_i\},\ i = 1,2,3. \tag{2.6}$$

Then, it follows that for each $t \in [0,r)$

$$0 \leq \lambda_0(t) < 1 \tag{2.7}$$

$$0 \leq p(t) < 1 \tag{2.8}$$

$$0 \leq p_1(t) < 1 \tag{2.9}$$

and

$$0 \leq \varphi_i < 1. \tag{2.10}$$

Denote by $B(u,a), \bar{B}(u,a)$, respectively the open and closed balls in X with center $u \in X$ and radius $a > 0$.

We consider the conditions (H):

(h1) $G : D \subseteq X \longrightarrow Y$ is continuously Fréchet differentiable and there exists $x_* \in D$ such that $G(x_*) = 0$ and $G'(x_*)^{-1} \in \mathscr{L}(Y,X)$.

(h2) There exists function $\lambda_0 : [0,+\infty) \longrightarrow [0,+\infty)$ continuous and increasing with $\lambda_0(0) = 0$ such that for each $x \in D$

$$\|G'(x_*)^{-1}(G'(x) - G'(x_*))\| \leq \lambda_0(\|x - x_*\|).$$

Set $D_0 = D \cap B(x_*, \rho_0)$, where ρ_0 is defined in (2.1).

(h3) There exist functions $\lambda : [0,\rho_0) \longrightarrow [0,+\infty)$, $\mu : [0,\rho_0) \longrightarrow [0,+\infty)$ such that for each $x,y \in D_0$

$$\|G'(x_*)^{-1}(G'(y) - G'(x))\| \leq \lambda(\|y - x\|)$$

and

$$\|G'(x_*)^{-1}G'(x)\| \leq \mu(\|x - x_*\|).$$

(h4) $\bar{B}(x_*, r) \subseteq D$, (2.2) and (2.5) hold where r is defined in (2.6).

(h5) There exist $r_* \geq r$ such that

$$\int_0^1 \lambda_0(\theta r_*)d\theta < 1.$$

Set $D_1 = D \cap \bar{B}(x_*, r_*)$.

THEOREM 2.1. *Under the conditions (H), further suppose that $x_0 \in B(x_*, r_*) - \{x_*\}$. Then, the following hold $\lim_{n\longrightarrow+\infty} x_n = x_*$, $\{x_n\} \subseteq B(x_*, r_*)$,*

$$\|y_n - x_*\| \leq \varphi_1(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| < r, \quad (2.11)$$

$$\|z_n - x_*\| \leq \varphi_2(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| \quad (2.12)$$

and

$$\|x_{n+1} - x_*\| \leq \varphi_3(\|x_n - x_*\|)\|x_n - x_*\| \leq \|x_n - x_*\| \quad (2.13)$$

for each $n = 0,1,2,\ldots$, where functions $\varphi_i, i = 1,2,3$ are given previously and r is defined in (2.6). Moreover, x_* is the only solution of equation $G(x) = 0$ in the set D_1.

Proof. Let $x \in B(x_*, r) - \{x_*\}$. By (h2), we have in turn that

$$\|G'(x_*)^{-1}(G'(x) - G'(x_*))\| \leq \lambda_0(\|x - x_*\|) \leq \lambda_0(r) < 1, \quad (2.14)$$

which together with the Banach perturbation Lemma [14] gives $G'(x)^{-1} \in \mathscr{L}(Y,X)$,

$$\|G'(x)^{-1}G'(x_*)\| \leq \frac{1}{1 - \lambda_0(\|x - x_*\|)} \quad (2.15)$$

and y_0 is well defined by the first substep of method (1.2). Let us write using (h1) that

$$G(x) = G(x) - G(x_*) = \int_0^1 G'(x_* + \theta(x - x_*))d\theta(x - x_*). \quad (2.16)$$

Then, by the second hypothesis in (h3)

$$\begin{aligned} \|G'(x_*)^{-1}G(x) &= \|\int_0^1 G'(x_* + \theta(x - x_*))d\theta(x - x_*)\| \\ &\leq \mu(\|x - x_*\|)\|x - x_*\|. \end{aligned} \quad (2.17)$$

We also have by the second hypothesis in (h3), (h1) and (2.15)

$$\begin{aligned} & \|x_0 - x_* - G'(x_0)^{-1}G(x_0)\| \\ \leq\ & \|G'(x_0)^{-1}G'(x_*)\| \\ & \|G'(x_*)^{-1}(G'(x_0)(x_0 - x_*) - G(x_0))\| \\ \leq\ & \frac{\int_0^1 \lambda((1-\theta)\|x_0 - x_*\|)d\theta\|x_0 - x_*\|}{1 - \lambda_0(\|x_0 - x_*\|)}. \end{aligned} \quad (2.18)$$

Using (2.6), (2.10) (for $i = 1$), (2.15), (2.17) (for $x = x_0$) and (2.18), we get

$$\begin{aligned} & \|y_0 - x_*\| \\ =\ & \|(x_0 - x_* - G'(x_0)^{-1}G(x_0)) \\ & + \frac{1}{3}G'(x_0)^{-1}G(x_0)\| \\ \leq\ & \|x_0 - x_* - G'(x_0)^{-1}G(x_0)\| \\ & + \frac{1}{3}\|G'(x_0)^{-1}G'(x_*)\|\|G'(x_*)^{-1}G(x_0)\| \\ \leq\ & \frac{\int_0^1 \lambda((1-\theta)\|x_0 - x_*\|)d\theta + \int_0^1 \mu(\theta\|x_0 - x_*\|)d\theta}{1 - \lambda_0(\|x_0 - x_*\|)} \\ & \times \|x_0 - x_*\| \\ =\ & \varphi_1(\|x_0 - x_*\|)\|x_0 - x_*\| \\ \leq\ & \|x_0 - x_*\| < r, \end{aligned} \quad (2.19)$$

so (2.11) holds for $n = 0$ and $y_0 \in B(x_0, x_*)$. We need an estimate on M_0 which is well defined by (2.15), so

$$\begin{aligned} \|M_0\| &= \|I - G'(x_0)^{-1}G'(y_0)\| \\ &\leq \|G'(x_0)^{-1}G'(x_*)\|\|G'(x_*)^{-1}(G'(x_0) - G'(y_0))\| \\ &\leq \|G'(x_0)^{-1}G'(x_*)\|[\|G'(x_*)^{-1}(G'(x_0) - G'(x_*))\| \\ &\quad + \|G'(x_*)^{-1}(G'(y_0) - G'(x_*))\|] \\ &\leq \frac{\lambda_0(\|x_0 - x_*\|) + \lambda_0(\|y_0 - x_*\|)}{1 - \lambda_0(\|x_0 - x_*\|)} \\ &\leq \frac{\lambda_0(\|x_0 - x_*\|) + \lambda_0(\varphi_1(\|x_0 - x_*\|)\|x_0 - x_*\|)}{1 - \lambda_0(\|x_0 - x_*\|)}. \end{aligned} \quad (2.20)$$

We shall show that $(I + 6(4I - 3\alpha M_0))^{-1} \in \mathscr{L}(Y,X)$. Using

(2.6), (2.8), (2.15) and (2.17), we get in turn that

$$\begin{aligned}
& \|6(4I-3\alpha M_0)\| \\
= \;& 6\|(4-3\alpha)I+3\alpha F'(x_0)^{-1}F'(y_0)\| \\
\leq \;& 6[|4-3\alpha| \\
& +\frac{3|\alpha|\int_0^1 \mu(\theta\varphi_1(\|x_0-x_*\|)\|x_0-x_*\|d\theta\|x_0-x_*\|}{1-\lambda_0(\|x_0-x_*\|)}] \\
\leq \;& p(\|x_0-x_*\|)\leq p(r)<1, \qquad (2.21)
\end{aligned}$$

so

$$\|(I+6(4I-3\alpha M_0))^{-1}\|\leq\frac{1}{1-p(\|x_0-x_*\|)} \qquad (2.22)$$

and z_0 is well defined by second substep of method (1.2). Using (2.6), (2.10) (for $i=2$), (2.15), (2.17), (2.18), (2.19), (2.20) and (2.22), we get in turn by the second substep of method (1.2).

$$\begin{aligned}
& \|z_0-x_*\| \\
\leq \;& \|x_0-x_*-G'(x_0)^{-1}G(x_0)\| \\
& +\frac{3}{4}\|M_0\|^2\|(I+6(4I-3\alpha M_0))^{-1}\| \\
& \times\|G'(x_0)^{-1}G'(x_*)\|\|G'(x_*)^{-1}G(x_0)\| \\
\leq \;& [\varphi_1(\|x_0-x_*\| \\
& +\frac{3}{4}\frac{\left(\frac{\lambda_0(\|x_0-x_*\|)+\lambda_0(\varphi_1(\|x_0-x_*\|)\|x_0-x_*\|)}{1-\lambda_0(\|x_0-x_*\|)}\right)^2}{1-p(\|x_0-x_*\|)} \\
& \times\frac{\int_0^1\mu(\theta\|x_0-x_*\|)d\theta}{1-\lambda_0(\|x-x_*\|)}]\|x_0-x_*\| \\
= \;& \varphi_2(\|x_0-x_*\|)\|x_0-x_*\|\leq\|x_0-x_*\|<r, \quad (2.23)
\end{aligned}$$

so (2.12) holds for $n=0$ and $z_0\in B(x_*,r)$. Next, we show that $(\beta G'(x_0)+\gamma G'(y_0))^{-1}\in\mathscr{L}(Y,X)$. Using (2.6), (2.9), (h2) and (2.19), we have in turn that

$$\begin{aligned}
& \|(\beta G'(x_*)+\gamma G'(x_*))^{-1} \\
& \times(\beta G'(x_0)+\gamma G'(y_0)-(\beta+\gamma)G'(x_*)]\| \\
\leq \;& \frac{1}{|\beta+\gamma|}[|\beta|\|G'(x_*)^{-1}(G'(x_0)-G'(x_*))\| \\
& +|\gamma|\|G'(x_*)^{-1}(G'(y_0)-G'(x_*0)\|] \\
\leq \;& \frac{1}{|\beta+\gamma|}[|\beta|\lambda_0(\|x_0-x_*\|) \\
& +|\gamma|\lambda_0(\varphi_1(\|x_0-x_*\|)\|x_0-x_*\|)] \\
= \;& p_1(\|x_0-x_*\|)\leq p_1(r)<1, \qquad (2.24)
\end{aligned}$$

so

$$\|(\beta G'(x_0)+\gamma G'(y_0))^{-1}G'(x_*)\|\leq\frac{1}{1-p_1(\|x_0-x_*\|)} \quad (2.25)$$

and x_1 is defined by the third substep of method (1.2). Then, using (2.6), (2.10) (for $i=3$), (2.15) (for $x=x_0$), (2.17) (for $x=z_0$), (2.19), (2.23), (2.25) and the third substep of method (1.2):

$$\begin{aligned}
& \|x_1-x_*\| \\
\leq \;& \|z_0-x_*\| \\
& +\|(\beta G'(x_0)+\gamma G'(y_0))^{-1}G'(x_*)\| \\
& \times\|G'(x_*)^{-1}(G'(x_0)+\delta G'(y_0))\| \\
& \times\|G'(x_0)^{-1}G'(x_*)\|\|G'(x_*)^{-1}G(z_0)\| \\
\leq \;& \left[1+\frac{\Pi}{(1-p_1(\|x_0-x_*\|))|\beta+\gamma|(1-\lambda_0(\|x_0-x_*\|))}\right] \\
& \times\varphi_2(\|x_0-x_*\|)\|x_0-x_*\| \\
= \;& \varphi_3(\|x_0-x_*\|)\|x_0-x_*\|\leq\|x_0-x_*\|<r, \qquad (2.26)
\end{aligned}$$

where

$$\begin{aligned}
\Pi \;:=\; & [\mu(\|x_0-x_*\|)+|\delta|\mu(\varphi_1(\|x_0-x_*\|)\|x_0-x_*\|)] \\
& \times\int_0^1\mu(\varphi_2(\|x_0-x_*\|)\|x_0-x_*\|)d\theta,
\end{aligned}$$

so (2.3) holds and $x_1\in B(x_*,r)$. Substitute x_0,y_0,z_0,x_1 by x_m,y_m,z_m,x_{m+1} in the preceding estimates to terminate the mathematical induction for estimates (2.11)–(2.13). Then, using the estimation

$$\|x_{m+1}-x_*\|\leq s\|x_m-x_*\|<r,\, s=\varphi_3(\|x_0-x_*\|)\in[0,1) \qquad (2.27)$$

we conclude that $\lim_{m\longrightarrow+\infty}x_m=x_*$ and $x_{m+1}\in B(x_*,r)$. The uniqueness part, is shown by considering $y_*\in D_1$ with $F(y_*)=0$ and setting $T=\int_0^1G'(x_*+\theta(y_*-x_*))d\theta$ to obtain by (h2) and (h5) that

$$\begin{aligned}
\|G'(x_*)^{-1}(T-G'(x_*))\| \;\leq\; & \int_0^1\lambda_0(\theta\|x_*-y_*\|)d\theta \\
\leq\; & \int_0^1\lambda_0(\theta r_*)d\theta<1 \qquad (2.28)
\end{aligned}$$

so $T^{-1}\in\mathscr{L}(Y,X)$. Finally, from the identity

$$0=G(y_*)-G(x_*)=T(y_*-x_*), \qquad (2.29)$$

we deduce that $x_*=y_*$.

□

REMARK 2.2. *(a) Let $\lambda_0(t)=L_0t$, $\lambda(t)=Lt$. The radius $r_A=\frac{2}{2L_0+L}$ was obtained by Argyros as the convergence radius for Newton's method under condition (h1)-(h3). Notice that the convergence radius for Newton's method given independently by Rheinboldt [20] and Traub [21] is given by*

$$\tilde{\rho}=\frac{2}{3L}<r_A.$$

Let $f(x)=e^x-1$. Then $x^*=0$. Set $\Omega=B(0,1)$. Then, we have that $L_0=e-1<L=e^{\frac{1}{L_0}}$, so $\tilde{\rho}=0.24252961<\tilde{\rho}_1=0.3827$.

Moreover, the new error bounds [2–7] are:

$$\|x_{n+1}-x^*\| \leq \frac{L}{1-L_0\|x_n-x^*\|}\|x_n-x^*\|^2,$$

whereas the old ones [20, 21]

$$\|x_{n+1}-x^*\| \leq \frac{L}{1-L\|x_n-x^*\|}\|x_n-x^*\|^2.$$

Clearly, the new error bounds are more precise, if $L_0 < L$. Clearly, the radius of convergence of method (1.2) given by ρ^* is smaller than $\tilde{\rho}_1$.

(b) *Method (1.2) stays the same if we use the new instead of the old conditions [5]. We can use the computational order of convergence (COC)[1–3]*

$$\xi = \frac{ln\frac{\|x_{n+2}-x_{n+1}\|}{\|x_{n+1}-x_n\|}}{ln\frac{\|x_{n+1}-x_n\|}{\|x_n-x_{n-1}\|}}, \quad \text{for each } n = 1,2,\ldots$$

or the approximate computational order of convergence (ACOC)[1–3]

$$\xi^* = \frac{ln\frac{\|x_{n+2}-x^*\|}{\|x_{n+1}-x^*\|}}{ln\frac{\|x_{n+1}-x^*\|}{\|x_n-x^*\|}}, \quad \text{for each } n = 0,1,2,\ldots.$$

(c) *Using (h2) and*

$$\begin{aligned} & \|G'(x^*)^{-1}G'(x)\| \\ = \; & \|G'(x^*)^{-1}(G'(x)-G'(x^*))+I\| \\ \leq \; & 1+\|G'(x^*)^{-1}(G'(x)-G'(x^*))\| \\ \leq \; & 1+q_0(\|x-x^*\|) \end{aligned}$$

the second condition in (h3) can be replaced by

$$\mu(t) = 1+\lambda_0(t)$$

or

$$\mu(t) = 1+\lambda_0(\rho_0).$$

Notice that if in particular, $\lambda_0(t) = L_0 t$, then, we can choose $\mu(t) = 2$, since $t \in [0, \frac{1}{L_0})$.

3. Numerical examples

The numerical examples are presented in this section for $\alpha = \frac{4}{3}$ and $\beta = \gamma = \delta = 1$.

EXAMPLE 3.1. *Let $\mathscr{B}_1 = \mathscr{B}_2 = \mathbb{R}^3, \Omega = \bar{U}(0,1), x^* = (0,0,0)^T$. Define function F on Ω for $u = (x,y,z)^T$ by*

$$F(u) = (e^x-1, \frac{e-1}{2}y^2+y, z)^T.$$

Then, the Fréchet-derivative is given by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the (2.8)-(2.12), conditions, we get $\lambda_0(t) = (e-1)t, \lambda(t) = e^{\frac{1}{e-1}}t, \mu(t) = e^{\frac{1}{e-1}}$.

Then using the definition of r, we have that
$r_1 = 0.15440695135715407082521721804369$
$r_2 = 0.022891367094531062803541843209132$
$r_3 = 0.021455242177788026725071546252366 = r.$

EXAMPLE 3.2. *Let $\mathscr{B}_1 = \mathscr{B}_2 = C[0,1]$, the space of continuous functions defined on $[0,1]$ and be equipped with the max norm. Let $\Omega = \overline{U}(0,1)$. Define function F on Ω by*

$$F(\varphi)(x) = \varphi(x) - 5\int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{3.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15\int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in \Omega.$$

Then, we get that $x^* = 0$, $\lambda_0(t) = 7.5t, \lambda(t) = 15t, \mu(t) = 2$. This way, we have that
$r_1 = 0.022222222222222222222222222222222$
$r_2 = 0.013998246548730894703305160930995 = r$
$r_3 = 0.019957165953580074946316358364129.$

EXAMPLE 3.3. *Let us return back to the motivational example. Then, we get that $\lambda_0(t) = \lambda(t) = 147t, \mu(t) = 2$. So, we obtain*
$r_1 = 0.0015117157974300831443688586545729$
$r_2 = 0.00097050808874623369516126958345126 = r$
$r_3 = 0.0049435619795227554035266237519863.$

EXAMPLE 3.4. *Let $\mathscr{B}_1 = \mathscr{B}_2 = C[0,1], \Omega = \bar{U}(x^*,1)$ and consider the nonlinear integral equation of the mixed Hammerstein-type [1, 2, 6–9, 12] defined by*

$$x(s) = \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt,$$

where the kernel G is the Green's function defined on the interval $[0,1]\times[0,1]$ by

$$G(s,t) = \begin{cases} (1-s)t, & t \leq s \\ s(1-t), & s \leq t. \end{cases}$$

The solution $x^*(s) = 0$ is the same as the solution of equation (1.1), where $F : C[0,1] \longrightarrow C[0,1])$ is defined by

$$F(x)(s) = x(s) - \int_0^1 G(s,t)(x(t)^{3/2} + \frac{x(t)^2}{2})dt.$$

Notice that

$$\|\int_0^1 G(s,t)dt\| \leq \frac{1}{8}.$$

Then, we have that

$$F'(x)y(s) = y(s) - \int_0^1 G(s,t)(\frac{3}{2}x(t)^{1/2} + x(t))dt,$$

so since $F'(x^*(s)) = I$,

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq \frac{1}{8}(\frac{3}{2}\|x-y\|^{1/2} + \|x-y\|).$$

Then, we get that $\lambda_0(t) = \lambda(t) = \frac{1}{8}(\frac{3}{2}t^{1/2} + t), \mu(t) = 1 + \lambda_0(t)$. *So, we obtain*
$r_1 = 1.4154$,
$r_2 = 0.0010070574734927378608012604743749$,
$r_3 = 0.00048321696805947146830320648724921 = r$.

References

[1] Argyros, I. K., George, S., Magreñán, A. A., Local convergence for multi-point-parametric Chebyshev-Halley-type methods of high convergence order, *Journal of Computational and Applied Mathematics*, 282(2015), 215–224.

[2] Argyros, I. K., George, S., Thapa, N., *Mathematical Modeling For The Solution Of Equations And Systems Of Equations With Applications*, Volume-I, Nova Publishes, NY, 2018.

[3] Argyros, I. K., George, S., Thapa, N.,, *Mathematical Modeling For The Solution Of Equations And Systems Of Equations With Applications*, Volume-II, Nova Publishes, NY, 2018.

[4] Argyros, I. K, Kansal, M., Kanwar, K., Local convergence for multi-point methods using only the first derivative, SeMA, DOI 10.1007/s40324-016-0075-z.

[5] Bahl. A, Cordero. A, Sharma. R, Torregrosa.J, Analysing complex dynamics of a novel bi-parametric sixth order iterative scheme for solving nonlinear systems, *Appl. Math. Comput.,* (To appear).

[6] Candela, V., Marquina, A. , Recurrence relations for rational cubic methods I: The Halley method, *Computing*, 44(2)(1990), 169–184.

[7] Candela, V., Marquina, A. , Recurrence relations for rational cubic methods II: The Halley method, *Computing*, 45(4)(1990), 355-367.

[8] Chen, J., Argyros, I. K., Agarwal, R. P., Majorizing functions and two-point Newton-type methods, *J. Comput. Appl. Math.*, 224(5)(2010), 1473–1484.

[9] Ezquerro, J.A., González,D., Hernández, M.A., On the local convergence of Newton's method under generalized conditions of Kantorovich, *Applied Mathematics Letters*, 26(2013), 566–570.

[10] Gutierrez, J. M., Hernández, M. A., An acceleration of Newtons method: super-Halley method, *Appl. Math. Comput.*, 117(2001), 223–239.

[11] Hernandez, M. A., Romero, N., On a characterization of some Newton-like method of $r-$order at least three, *J. Comput. Appl. Math.*, 183(2005), 53–66.

[12] Hernandez, M. A. H., Martinez, E., On the semilocal convergence of a three steps Newton-type iterative process under mild convergence conditions, *Numer. Algor.,* 70(2)(2015), 377–392.

[13] Jarratt, P., Some fourth order multipoint iterative methods for solving equations, *Math. Comput.*, 20(1966), 434–437.

[14] Kantorovich, L.V., Akilov, G.P., *Functional Analysis*, Pergamon Press, Oxford, 1982.

[15] Kumar. A, Gupta. D. K., Local convergence of Super Halley's method under weaker conditions on Fréchet derivative in Banach spaces, *The Journal of Analysis*, (2017), Doi.org/10.1007/s41478-017-0034-9.

[16] Madhu, K., Sixth order Newton-type method for solving system of nonlinear equations and its applications, *Applied Mathematics E- Notes*, 7(2017), 221–230.

[17] Magreñán, Á. A., Different anomalies in a Jarratt family of iterative root-finding methods, *Appl. Math. Comput.,* 233(2014), 29–38.

[18] Magreñán, Á. A., A new tool to study real dynamics: The convergence plane, *Applied Mathematics and Computation*, 248(2014), 215–224.

[19] Parida,P. K. , Gupta, D. K., Recurrence relations for a Newton-like method in Banach space, *J. Comput. Appl. Math.*, 206(2007), 873–887.

[20] Rheinboldt, W.C., An adaptive continuation process for solving systems of nonlinear equations, *Polish Academy of Science, Banach Ctr. Publ.,* 3(1)(1978), 129–142.

[21] Traub, J. F., *Iterative Methods for Solution of Equations*, Printice-Hall, Inc., Englewood Cliffs, N. J., 1964.

[22] Wang, K., Kou, J., Gu, C., Semilocal convergence of a sixth-order Jarratt method in Banach spaces, *Numer. Algor.*, 57(2011), 441–456.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆