Malaya Journal of Matematik, Vol. 8, No. 3, 1240-1242, 2020

https://doi.org/10.26637/MJM0803/0090

A study on fuzzy semi-stratifiable and fuzzy stratifiable spaces

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Abstract

The notion of stratifiable space and semi-stratifiable space were studied by Gary Gruenhage [4]. Alexander P. Sostak extended the concept of stratifiable space to the fuzzy case [1]. In this paper we investigate some properties of fuzzy closure-preserving collection, fuzzy stratifiable space and fuzzy semi-stratifiable space.

Keywords

Fuzzy closure-preserving collection, fuzzy stratifiable space, fuzzy semi-stratifiable space and fuzzy M_1- space.

AMS Subject Classification

54A40, 54E35, 03E72.

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Article History: Received **03** April **2020**; Accepted **12** July **2020**

Contents

1. Introduction

The concepts like stratifiable space and semi-stratifiable space were extensively studied by Gary Gruenhage.For a detailed discussion reference may be made of [4]. The class of stratifiable spaces does not differ too much from the class of metrizable spaces. Alexander P. Sostak extended the concept of stratifiable space to the fuzzy case [1].In this paper we establish some properties of fuzzy closure-preserving collection, fuzzy stratifiable space and fuzzy semi-stratifiable space.

2. Preliminaries

We use the term fuzzy topological space in Chang's sense [2]. A fuzzy point x_α is a fuzzy set which takes every element in X to 0 except one element $x \in X$ and its value at x is α where $(0 < \alpha \leq 1)$.$\underline{0}$ is the fuzzy set which maps all element in X to 0 and $\underline{1}$ is the fuzzy set which maps every element in X to 1. The closure of a fuzzy set U in a fuzzy topological space is denoted by $\overline{U}$. In this paper each fuzzy topological space is assumed to be T_1 and regular.$\mathbb{N}$ denotes set of all natural numbers.We list some of the definitions which we are using in this paper.

Definition 2.1. *[11] A collection $\mathscr{A}$ of fuzzy subsets of a fuzzy topological space (X,F) is called discrete if for each fuzzy point x_α, there is a $G \in F$ with $x_\alpha \leq G$ and $G \wedge A \neq \underline{0}$ holds for at most one element A of $\mathscr{A}$.*

Definition 2.2. *[9] Let $\mathscr{A}$ be a cover of a fuzzy topological space (X,F).For $\alpha \in (0,1]$ and a fuzzy point x_α, $st(x_\alpha, \mathscr{A}) = \vee \{B : B \in \mathscr{A}, B(x) \geq \alpha\}$ and for a fuzzy set G, $st(G, \mathscr{A}) = \vee \{B : B \in \mathscr{A}$ and $B \wedge G \neq \underline{0}\}$*

Definition 2.3. *[9] Let (X,F) be a fuzzy topological space.A fuzzy point $x_\alpha, \alpha \in (0,1]$ is called a cluster point of the set $\{(x_n)_\alpha : n \in \mathbb{N}\}$, where $(x_n)_\alpha$ is a fuzzy set with support x_n and value α, if for each fuzzy set $G \in F$ such that $x_\alpha \leq G$,there exists $n_0 \in \mathbb{N}$ with $x_{n_0} \neq x$ and $(x_{n_0})_\alpha \leq G$.*

Definition 2.4. *[7] Let (X,F) be a fuzzy topological space and let $F_0 \subset F$. Then F_0 is called a fuzzy base of F if $F = \{\vee \mathscr{A} : \mathscr{A} \subset F_0\}$.*

3. Fuzzy closure-preserving collection

In this section we define fuzzy closure-preserving collection and fuzzy σ-closure-preserving collection.

Definition 3.1. *A collection $\mathscr{H}$ of fuzzy subsets of a fuzzy topological space (X,F) is called fuzzy closure-preserving if $\overline{\vee\mathscr{H}'} = \vee\{\overline{H} : H \in \mathscr{H}'\}$ for each $\mathscr{H}' \subset \mathscr{H}$. If $\mathscr{H} = \cup_{n=1}^{\infty}\mathscr{H}_n$ where each $\mathscr{H}_n$ is fuzzy closure-preserving, then $\mathscr{H}$ is called a fuzzy σ-closure-preserving.*

Theorem 3.2. *Fuzzy locally finite collections are fuzzy closure-preserving.*

Proof. Let $\mathscr{A}$ be a fuzzy locally finite collection of fuzzy subsets of X. Let $\mathscr{A}' \subset \mathscr{A}$. Let $A' \in \mathscr{A}'$. Now $A' \leq \vee\{A' : A' \in \mathscr{A}'\}$ which implies $\overline{A'} \leq \overline{\vee\{A' : A' \in \mathscr{A}'\}}$.That is $\overline{A'} \leq \overline{\vee\mathscr{A}'}$ for each $A' \in \mathscr{A}'$. So $\vee\{\overline{A'} : A' \in \mathscr{A}'\} \leq \overline{\vee\mathscr{A}'}$. Thus we need to prove the other side. Clearly, $\vee\{A' : A' \in \mathscr{A}'\} \leq \vee\{\overline{A'} : A' \in \mathscr{A}'\}$. If $\vee\{\overline{A'} : A' \in \mathscr{A}'\}$ is closed, then $\overline{\vee\{A' : A' \in \mathscr{A}'\}} \leq \vee\{\overline{A'} : A' \in \mathscr{A}'\}$, that is $\overline{\vee\mathscr{A}'} \leq \vee\{\overline{A'} : A' \in \mathscr{A}'\}$. So it suffices to prove that, $\vee\{\overline{A'} : A' \in \mathscr{A}'\}$ is closed.

Let $x_\alpha \leq (\vee\{\overline{A'} : A' \in \mathscr{A}'\})^c$. Since $\mathscr{A}$ is locally finite collection , $\mathscr{A}'$ is also locally finite collection and hence so is $\{\overline{A'} : A' \in \mathscr{A}'\}$. Let G be a fuzzy open set with $x_\alpha \leq G$ and $G \wedge \overline{A'_i} \neq \underline{0}$ holds for finitelymany $A'_i \in \mathscr{A}'$, say $1 \leq i \leq n$. Then $x_\alpha \leq G \wedge \wedge_{i=1}^{n}\overline{A'_i}^c$ and $(G \wedge \wedge_{i=1}^{n}\overline{A'_i}^c) \wedge (\vee\{\overline{A'} : A' \in \mathscr{A}'\}) = \underline{0}$. Therefore for every fuzzy point $x_\alpha \leq (\vee\{\overline{A'} : A' \in \mathscr{A}'\})^c$, there exists a fuzzy open set $G \wedge \wedge_{i=1}^{n}\overline{A'_i}^c$ with $x_\alpha \leq G \wedge \wedge_{i=1}^{n}\overline{A'_i}^c$ and $G \wedge \wedge_{i=1}^{n}\overline{A'_i}^c \leq (\vee\{\overline{A'} : A' \in \mathscr{A}'\})^c$. Therefore $(\vee\{\overline{A'} : A' \in \mathscr{A}'\})^c$ is open and hence $\vee\{\overline{A'} : A' \in \mathscr{A}'\}$ is closed. □

4. Fuzzy semi-stratifiable and fuzzy stratifiable spaces

Definition 4.1. *[1] A fuzzy topological space (X,F) is called stratifiable if to every $U \in F$, one can assign a sequence of fuzzy sets $U_n \in F$ such that*
(1)$\overline{U_n} \leq U$ for all $n \in \mathbb{N}$
(2) $\vee_n U_n = U$
(3)$U \leq V (V \in F)$ implies $U_n \leq V_n$ for all $n \in \mathbb{N}$

When studying the properties of fuzzy stratifiable spaces it is convinient to use the following dual characterization.

Theorem 4.2. *A fuzzy topological space (X,F) is fuzzy stratifiable if there is a function G which assigns to each $n \in \mathbb{N}$ and fuzzy-closed subset H of X, a fuzzy-open set $G(n,H)$ with $H \leq G(n,H)$ such that*
(1)$H = \wedge_n G(n,H)$
(2)$H \leq K \implies G(n,H) \leq G(n,K)$
(3)$H = \wedge_n \overline{G(n,H)}$

Remark 4.3. *we can assume the following condition satisfies:*
(4)$G(n+1,H) \leq G(n,H)$ for each $n \in \mathbb{N}$.

Definition 4.4. *A fuzzy topological space (X,F) is called fuzzy semi-stratifiable if there is a function G which assigns to each $n \in \mathbb{N}$ and fuzzy closed subset H of X, a fuzzy open set $G(n,H)$ with $H \leq G(n,H)$ such that*
(1)$H = \wedge_n G(n,H)$
(2)$H \leq K \implies G(n,H) \leq G(n,K)$

Definition 4.5. *[10] A sequence $(\mathscr{A}_n)$ of fuzzy open covers of a fuzzy topological space (X,F) is called a fuzzy development for X, if for $\alpha \in (0,1]$,a fuzzy point x_α, the set $\{st(x_\alpha, \mathscr{A}_n) : n \in \mathbb{N}\}$ is a base at x_α.A fuzzy topological space (X,F) is fuzzy developable if it has a fuzzy development.*

Theorem 4.6. *Every fuzzy developable space is fuzzy semi-stratifiable.*

Proof. Suppose (X,F) is a fuzzy developable space. Let $(\mathscr{A}_n)$ be a fuzzy development for X. Let $n \in \mathbb{N}$ and H be a fuzzy closed subset of X. Let $G(n,H) = st(H, \mathscr{A}_n)$.Clearly $H \leq st(H, \mathscr{A}_n) = G(n,H)$ and $H = \wedge_n st(H, \mathscr{A}_n) = \wedge_n G(n,H)$.If $H \leq K$ then $st(H, \mathscr{A}_n) \leq st(K, \mathscr{A}_n)$. Therefore (X,F) is fuzzy semi-stratifiable.

□

Definition 4.7. *A fuzzy topological space (X,F) is called a fuzzy M_1-space if X has a fuzzy σ-closure preserving base.*

Theorem 4.8. *Every fuzzy M_1-space is fuzzy stratifiable.*

Proof. Suppose (X,F) is a fuzzyM_1-space.Let $\mathscr{B} = \cup_n \mathscr{B}_n$ be a fuzzy σ-closure preserving base for X. Let H be a fuzzy closed subset of X and let $n \in \mathbb{N}$.Let $G(n,H) = (\vee_n\{\overline{B} : B \in \mathscr{B}_n, \overline{B} \wedge H = \underline{0}\})^c$. Then $H \leq G(n,H)$. Clearly $H = \wedge_n G(n,H) = \wedge_n \overline{G(n,H)}$. Also $H \leq K \implies G(n,H) \leq G(n,K)$. Therefore (X,F) is fuzzy stratifiable. □

Theorem 4.9. *A fuzzy topological space (X,F) is fuzzy semi-stratifiable if and only if for each fixed $\alpha \in (0,1]$ there is a function $g : \mathbb{N} \times \{x_\alpha : x \in X\} \to F$ such that*
(i) $x_\alpha = \wedge_n g(n, x_\alpha)$
(ii) $y_\alpha \leq g(n, (x_n)_\alpha) \implies (x_n)_\alpha \to y_\alpha$
(X,F) is fuzzy stratifiable is we can also obtain
(iii)if $y_\alpha \not\leq H$,where H is fuzzy-closed subset, then $y_\alpha \not\leq \overline{\vee\{g(n,x_\alpha) : x_\alpha \leq H\}}$ for some $n \in \mathbb{N}$.

Proof. Suppose (X,F) is fuzzy semi-stratifiable. Fix $\alpha \in (0,1]$. Let $g(n,x_\alpha) = G(n,x_\alpha)$, where G satisfies definition 4.4, with $G(n+1,H) \leq G(n,H)$ for each n. Since $H = \wedge_n G(n,H)$, $x_\alpha = \wedge_n G(n,x_\alpha) = \wedge_n g(n,x_\alpha)$. Therefore (i) holds true. To see (ii), assume that $y_\alpha \leq g(n,(x_n)_\alpha)$ and $(x_n)_\alpha \nrightarrow y_\alpha$. Then there is an infinite subset $A \subset \mathbb{N}$ with $y_\alpha \not\leq \overline{\vee\{(x_n)_\alpha : n \in A\}}$. Then $y_\alpha \not\leq G(m, \overline{\vee\{(x_n)_\alpha : n \in A\}})$ for some $m \in \mathbb{N}$. Choose $n \in A$ with $n \geq m$.Then $y_\alpha \leq g(n,(x_n)_\alpha) = G(n,(x_n)_\alpha) \leq G(n, \overline{\vee\{(x_n)_\alpha : n \in A\}}) \leq G(m, \overline{\vee\{(x_n)_\alpha : n \in A\}})$, since $n \geq m$. This is a contradiction. Therefore $(x_n)_\alpha \to y_\alpha$.

Suppose (X,F) is fuzzy stratifiable. To check (iii), assume that $y_\alpha \not\leq H$ where H is a closed fuzzy subset. Since $H = \wedge_n \overline{G(n,H)}$, there exists $n \in \mathbb{N}$ with $y_\alpha \not\leq \overline{G(n,H)}$. But $\vee_{x_\alpha \leq H} G(n,x_\alpha) \leq G(n,H)$. So $y_\alpha \not\leq \vee_{x_\alpha \leq H} G(n,x_\alpha)$ and hence

$y_\alpha \not\leq \overline{\vee\{g(n,x_\alpha) : x_\alpha \leq H\}}$.

Conversely, assume that there is a function g satisfying the conditions of theorem. Let $n \in \mathbb{N}$ and H be a closed fuzzy subset. Let $G(n,H) = \vee_{\alpha=H(x)} g(n,x_\alpha)$. Clearly $H \leq G(n,H)$. Then $\wedge_n G(n,H) = \wedge_n \vee_{\alpha=H(x)} g(n,x_\alpha) = \vee_{\alpha=H(x)}(\wedge_n g(n,x_\alpha)) = \vee_{\alpha=H(x)} x_\alpha = H$.
Suppose $H \leq K$ where K is a closed fuzzy subset. Then $G(n,H) = \vee_{\alpha=H(x)} g(n,x_\alpha) \leq \vee_{\alpha=K(x)} g(n,x_\alpha) = G(n,K)$.Therefore (X,F) is fuzzy semi-stratifiable. Clearly $\wedge_n \overline{G(n,H)} = \wedge_n \overline{\vee_{\alpha=H(x) g(n,x_\alpha)}} = H$ by (iii). Hence (X,F) is fuzzy stratifiable. □

Acknowledgment

The author is thankful to the reviewers for their valuable suggestions and comments to improve the quality of the paper.

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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