Lie group investigation of fractional partial differential equation using symmetry

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Abstract

In this paper, we make use of Lie group investigation of the following non-linear of fractional partial differential equations using symmetry.

$$D^{\alpha_1} u_1 = \frac{1}{2} u_1$$

$$D^{\alpha_2} u_2 = u_2 + u_1{}^2, \tag{0.1}$$

where α_1 and α_2 are real constant, $0 < \alpha_1 \alpha_2 \leq 1$ and u_1 and u_2 are functions of independent variable x and $D^{\alpha} u$ is a fractional derivative of u w.r.t. x which can be the following R-L type.

$$\frac{\partial^{\alpha} u}{\partial x^{\alpha}} = \begin{cases} \frac{\partial^m u}{\partial x^m} & \alpha = m \in N \\ \\ \frac{1}{(m-\alpha)!} \frac{\partial^m}{\partial x^m} \int_0^t (t-\tau)^{m-\alpha-1} u(\tau, x) d\tau & m-1 < \alpha < m, m \in N \end{cases}$$

Keywords

Lie group, Nonlinear fractional differential equation, Reimann Liouville integral and derivative, Symmetry analysis, Invariant solution.

AMS Subject Classification:

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1. Introduction

Fractional differential equations (FDEs) have a large significance as they have vast applications in relating familiar phenomena to nearly every field of Mathematics, Science, Engineering and Technology etc.[3–7]. The most important advantage of using FDEs in these and other applications is their nonlocal property. Constructing fractional mathematical models for precise phenomena and developing numerical or analytical solutions for these fractional mathematical models are most important issues in mathematics, physics, and engineering.

A well-known technique for studying FDEs is the Lie symmetry method. It was in the beginning proposed for integer-order differential equations [1, 2, 8] and after that recognized for fractional-order single differential equations [9–11, 18]. Lie group analysis and its applications to differential equations[12–14, 19–21] is one of the efficient methods to handle fractional partial differential equation (FPDE), for such problems a large amount labours have been spent in recent years to extend techniques to deal with FDEs because there exists no precise method to resolve it thoroughly.

There are some efforts in group analysis of FDEs and group properties of FDEs such as Scaling transformations of the time-fractional linear wave-diffusion equation and its group invariant solutions[15], Using group of scaling transformations, self-similarity solutions to a nonlinear fractional diffusion, Burgers and KdV equations[16, 17], Lie symmetries of the fractional nonlinear anomalous diffusion equations[22], Entire group classification and symmetry reductions of the fractional fifth-order KdV type of equations[23] e.t.c.

2. Preliminaries

In this section, we use some definitions and notations which are given in [9, 14, 15, 24, 27] with details and present technical preparation needed for further discussion.

Definition 2.1. [24] *The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\mu, \mu \geq -1$ is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_\alpha^x (x-\mu)^{\alpha-1} f(\mu) d\mu, \ \alpha > 0, x > 0.$$

$j^0 f(x) = f(x)$
when we formulate the model of real world problem with fractional calculus, the Riemann-Liouville have certain disadvantages. Caputo proposed in his work on the theory of viscoelasticity a modified fractonal differential operator D^α.

Definition 2.2. [24]*The fractional derivative of f(x) in Caputo sense is defined as*

$$D^\alpha f(x) = J^{m-\alpha} D^m f(x)$$
$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\eta)^{m-\alpha-1} f^m(\eta) d\eta$$

for $m-1 < \alpha \leq m, m \in N, x > 0, f \in C_{-1}^m$

Definition 2.3. [9, 27] *The Caputo space-fractional derivative of order $\alpha > 0$ is defined as*

$$D_y^\alpha u(x,y) = \frac{\partial^\alpha u(x,y)}{\partial y^\alpha}$$
$$= \frac{1}{\Gamma(m-\alpha)} \int_0^y (y-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau$$

if $m-1 < \alpha \leq m, m \in N$
and the Caputo fractional derivative of order $\beta > 0$ is defined as

$$D_y^\beta u(x,y) = \frac{\partial^\beta u(x,y)}{\partial x^\beta}$$
$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\eta)^{m-\alpha-1} \frac{\partial^m u(\eta,y)}{\partial \eta^m} d\eta$$

if $m-1 < \alpha \leq m, m \in N$

Definition 2.4. [9, 27] *For m to be a smallest integer that exceeds α, the Caputo time fractional derivative operator of order $\alpha > 0$ is defined as*

$$D_t^\alpha u(x,t) = \frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$$
$$= \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m}, m-1 < \alpha < m \\ \\ \frac{\partial^m u(x,t)}{\partial t^m}, \quad \alpha = m \in N \end{cases}$$

Let us consider the equation

$$F(u) = 0, \ u = u(x,t) \tag{2.1}$$

Definition 2.5. [9, 14] *A one parameter family of scaling transformations, denoted by T_λ, is a transformation of (x,t,u)-space of the form*

$$\bar{x} = \lambda^a x, \ \bar{t} = \lambda^b t, \ \bar{u} = \lambda^c u, \tag{2.2}$$

where a, b and c are constants and λ is real parameter restricted to an open interval I containing $\lambda = 1$.

Definition 2.6. [9, 14]*The equation (2.1) is invariant under the one-parameter family T_λ of scaling transformation (2.2) iff T_λ takes any solution u of (2.1) to a solution $\bar{u}$ of the same equation:*

$$\bar{u} = T_\lambda u \text{ and } F(\bar{u}) = 0$$

Definition 2.7. [15]*The real- valued function $\eta(x,t,u)$ is called an invariant of the one-parameter family of T_λ, if it is unaffected by the transformations, in other words:*

$$\eta(T_\lambda(x,t,u)) = \eta(x,t,u) \text{ for all } \lambda \in I.$$

On the half-space $\{(x,t,u), x > 0, t > 0\}$, the invariants of the family of scaling transformation (2.2) are provided by the functions

$$\eta_1(x,t,u) = xt^{-a/b}, \ \eta_2(x,t,u) = t^{-c/b}u.$$

if the equation (2.1) is a second order partial differential equation of the form

$$G(x,t,u,u_x,u_t,u_{xx},u_{tt},u_{xt}) = 0 \tag{2.3}$$

and this equation is invariant under T_λ, given by (2.2), then the transformation

$$u(x,t) = t^{c/b}v(z), \ z = xt^{-a/b}$$

reduces the equation (2.3) to a second order ordinary differential equation of the form

$$g(z,v,v',v'') = 0.$$

Theorem 2.8. [15] *The invariants of scaling transformations are given by the expressions*

$$\eta_1(x,t,u) = xt^{-\alpha/\beta}, \ \ \eta_2(x,t,u) = t^{-\gamma}u,$$

γ being a arbitrary constant.

3. Main Results

Lie investigation for FPDE's:

Take a scalar time FPDE's of the following forms

$$\frac{\partial^{\alpha_1} u_1(x)}{\partial x^{\alpha_1}} = F_1(x, u_1, u_2, u_{1x}, u_{2x}, u_{1xx}, u_{2xx}...)$$

$$\frac{\partial^{\alpha_2} u_1(x)}{\partial x^{\alpha_2}} = F_2(x, u_1, u_2, u_{1x}, u_{2x}, u_{1xx}, u_{2xx}...)$$

where the subscripts denote the partial derivatives.
We imagine that the above FPDE is invariant under a one parameter (ε). In view of lie theory, the creation of the symmetry group is equivalent to the determinant of its infinitesimal transformation.

$$\overline{x} = x + \varepsilon\xi(x, u_1, u_2) + O(\varepsilon^2)$$

$$\overline{u_1} = u_1 + \varepsilon\eta_1(x, u_1, u_2) + O(\varepsilon^2)$$

$$\overline{u_2} = u_2 + \varepsilon\eta_2(x, u_1, u_2) + O(\varepsilon^2)$$

$$\frac{\partial^{\alpha_1}\overline{u_1}(x)}{\partial\overline{x}^{\alpha_1}} = \frac{\partial^{\alpha_1}u_1(x)}{\partial x^{\alpha_1}} + \varepsilon\zeta^0_{\alpha_1} + O(\varepsilon^2)$$

$$\frac{\partial^{\alpha_2}\overline{u_2}(x)}{\partial\overline{x}^{\alpha_2}} = \frac{\partial^{\alpha_2}u_2(x)}{\partial x^{\alpha_2}} + \varepsilon\zeta^0_{\alpha_2} + O(\varepsilon^2)$$

$$\frac{\partial\overline{u_1}}{\partial\overline{x}} = \frac{\partial u_1}{\partial x} + \varepsilon\zeta^1_1 + O(\varepsilon^2)$$

$$\vdots \quad (3.1)$$

where ξ, η_1 and η_2 are infinitesimal and ζ^1_1, $\zeta^0_{\alpha_1}$, and $\zeta^0_{\alpha_2}$ are extended infinitesimal of order 1, α_1, α_2 respectively. Infinitesimal generator takes the form

$$X = \xi\frac{\partial}{\partial\alpha} + \eta_1\frac{\partial}{\partial u_1} + \eta_2\frac{\partial}{\partial u_2}$$

The following equation is the α_1^{th} extended infinitesimal related to R-L fractional time derivative.

$$\zeta^0_{\alpha_1} = D_x^{\alpha 1}(\eta_1) - \alpha_1 D_x(\xi)D_x^{\alpha 1}u_1 - \sum_{n=1}^{\infty}\binom{\alpha_1}{n+1}D_x^{n+1}(\xi)I_x^{\alpha_1-n}(u_1).$$

Here the operator $D_x^{\alpha 1}$ stands for the total fractional derivative operator. The first term in the right hand side of equation can be written as

$$D_x^{\alpha_1}(\eta_1) = \frac{\partial^{\alpha_1}\eta_1}{\partial x^{\alpha_1}} + \eta_{1u_1}\frac{\partial^{\alpha_1}u_1}{\partial x^{\alpha 1}} - u_1\frac{\partial^{\alpha_1}\eta_{1u_1}}{\partial x^{\alpha_1}} + \eta_{1u_2}\frac{\partial^{\alpha_1}u_2}{\partial x^{\alpha_1}} - u_2\frac{\partial^{\alpha_1}\eta_{1u_2}}{\partial x^{\alpha_1}}$$

$$+ \sum_{n=1}^{\infty}\binom{\alpha_1}{n}\frac{\partial^n(\eta_{1u_1})}{\partial x^n}I_x^{\alpha_1-n}u_1 + \sum_{n=1}^{\infty}\binom{\alpha_1}{n}\frac{\partial^n(\eta_{1u_2})}{\partial x^n}I_x^{\alpha_1-n}u_2 + \mu^*,$$

where

$$\mu^* = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha_1}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{x^{n-\alpha_1}}{\Gamma(n+1-\alpha_1)}\times$$

$$\left\{(-u_1)^r\frac{\partial^m(u_1^{k-r})}{\partial x^m}\frac{\partial^{n-m+k}\eta_1}{\partial x^{n-m}\partial u_1^k} + (-u_2)^r\frac{\partial^m(u_2^{k-r})}{\partial x^m}\frac{\partial^{n-m+k}\eta_1}{\partial x^{n-m}\partial u_2^k}\right\}.$$

Thus, the α_1^{th} extended infinitesimal becomes

$$\zeta^0_{\alpha_1} = \frac{\partial^{\alpha_1}\eta_1}{\partial x^{\alpha_1}} + (\eta_{1u_1} - \alpha_1 D_x(\xi))\frac{\partial^{\alpha_1}u_1}{\partial x^{\alpha_1}} - u_1\frac{\partial^{\alpha_1}\eta_{1u_1}}{\partial x^{\alpha_1}}$$

$$+ \eta_{1u_2}\frac{\partial^{\alpha_1}u_2}{\partial x^{\alpha_1}} - u_2\frac{\partial^{\alpha 1}\eta_{1u_2}}{\partial x^{\alpha_1}} + \mu^*$$

$$+ \sum_{n=1}^{\infty}\left[\binom{\alpha_1}{n}\frac{\partial^n(\eta_{1u_1})}{\partial x^n} - \binom{\alpha_1}{n+1}D_x^{n+1}(\xi)\right]I_x^{\alpha_1-n}(u_1) \quad (3.2)$$

$$+ \sum_{n=1}^{\infty}\binom{\alpha}{n}\frac{\partial^n(\eta_{1u_2})}{\partial x^n}I_x^{\alpha_1-n}(u_2)$$

and similarly,

$$\zeta^0_{\alpha_2} = D_x^{\alpha_2}(\eta_2) - \alpha_2 D_x\xi D_x^{\alpha_2}u_2 - \sum_{n=1}^{\infty}\binom{\alpha_2}{n+1}D_x^{n+1}(\xi)I_x^{\alpha_2-n}u_2$$

and

$$D_x^{\alpha_2}(\eta_2) = \frac{\partial^{\alpha_2}\eta_2}{\partial x^{\alpha_2}} + \eta_{2u_1}\frac{\partial^{\alpha_2}u_1}{\partial x^{\alpha_2}} - u_1\frac{\partial^{\alpha_2}\eta_{2u_1}}{\partial x^{\alpha_2}} + \eta_{2u_2}\frac{\partial^{\alpha_2}u_2}{\partial x^{\alpha_2}} - u_2\frac{\partial^{\alpha_2}\eta_{2u_2}}{\partial x^{\alpha_2}}$$

$$+ \sum_{n=1}^{\infty}\binom{\alpha_2}{n}\frac{\partial^n(\eta_{2u_1})}{\partial x^n}I_x^{\alpha_2-n}u_1 + \sum_{n=1}^{\infty}\binom{\alpha_2}{n}\frac{\partial^n(\eta_{2u_2})}{\partial x^n}I_x^{\alpha_2-n}u_2 + \mu^{**},$$

where

$$\mu^{**} = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{k=2}^{m}\sum_{r=0}^{k-1}\binom{\alpha_2}{n}\binom{n}{m}\binom{k}{r}\frac{1}{k!}\frac{x^{n-\alpha_2}}{\Gamma(n+1-\alpha_2)}\times$$

$$\left\{(-u_1)^r\frac{\partial^m u_1^{k-r}}{\partial x^m}\frac{\partial^{n-m+k}\eta_2}{\partial x^{n-m}\partial u_1^k} + (-u_2)^r\frac{\partial^m u_2^{k-r}}{\partial x^m}\frac{\partial^{n-m+k}\eta_2}{\partial x^{n-m}\partial u_2^k}\right\}.$$

Thus, the α_2^{th} extended infinitesimal is given

$$\zeta^0_{\alpha_2} = \frac{\partial^{\alpha_2}\eta_2}{\partial x^{\alpha_2}} + (\eta_{2u_1} - \alpha_2 D_x(\xi))\frac{\partial^{\alpha_2}u_1}{\partial x^{\alpha_2}} - u_1\frac{\partial^{\alpha_2}\eta_{2u_1}}{\partial x^{\alpha_2}}$$

$$+ \eta_{2u_2}\frac{\partial^{\alpha_2}u_2}{\partial x^{\alpha_2}} - u_2\frac{\partial^{\alpha_2}\eta_{2u_2}}{\partial x^{\alpha_2}} + \mu^{**}$$

$$+ \sum_{n=1}^{\infty}\left[\binom{\alpha_2}{n}\frac{\partial^n\eta_{2u_1}}{\partial x^n}I_x^{\alpha_2-n}(u_1)\right] \quad (3.3)$$

$$+ \sum_{n=1}^{\infty}\binom{\alpha_2}{n}\left[\frac{\partial^{\alpha_2}\eta_{2u_2}}{\partial x^n} - \binom{\alpha}{n+1}D_x^{n+1}(\xi)\right]I_x^{\alpha_2-n}(u_2)$$

Lie investigation using Symmetry of Non-linear system of Fractional Differential Equation:

With the help of Lie theory, we accept that the non linear system of FDE equation (0.1) is invariant under the one parameter transformation equation (3.1) so the equation (0.1)

$$D^{\alpha_1} u_1 = \frac{u_1}{2}$$

$$D^{\alpha_2} u_2 = u_2 + u_1^2, 0 < \alpha_1, \alpha_2 \leq 1$$

take the form

$$\zeta_{\alpha_1}^0 - \frac{\eta_1}{2} = 0, 0 < \alpha_1 \leq 1 \qquad (3.4)$$
$$\zeta_{\alpha_2}^0 - \eta_2 - 2u_1 \eta_1 = 0, 0 < \alpha_2 \leq 1$$

which depends on the variables $I_x^{\alpha_1-n} u_1, I_x^{\alpha_1-n} u_2, I_x^{\alpha_2-n} u_1$ and $I_x^{\alpha_2-n} u_2$ for $n = 1, 2, 3...$ which are considered to be independent.

The formation of equation (3.4) permit us to reduce it to a system of infinitely many linear FDE's by substituting the expression for $\zeta_{\alpha_1}^0$ and $\zeta_{\alpha_2}^0$ given in (3.2,3.3) and (3.4) and equating various powers of derivatives of u_1 and u_2 to zero, we get the following an over determined system of the linear equations

$$\xi_{u_1} = \xi_{u_2} = 0$$

$$\eta_{1u_1} = \eta_{1u_2u_2} = \eta_{1xu_2} = 0$$

$$\eta_{2u_1} = \eta_{2u_2u_2} = \eta_{2xu_2} = 0$$

$$\frac{\partial^{\alpha_1} \eta_1}{\partial x^{\alpha_1}} - (\eta_{1u_1} - \alpha_1 \bar{\xi})(\frac{\mu_1}{2}) - u_1 \frac{\partial^{\alpha_1} \eta_{1u_1}}{\partial x^{\alpha_1}} - \frac{\eta_1}{2} = 0$$

$$\frac{\partial^{\alpha_2} \eta_2}{\partial x^{\alpha_2}} - (\eta_{2u_2} - \alpha_2 \bar{\xi})(u_2 + u_1^2) - u_1 \frac{\partial^{\alpha_1} \eta_{2u_1}}{\partial x^{\alpha_1}} - \eta_2 - 2u_1 \eta_1 = 0$$

$$\binom{\alpha_1}{n} \frac{\partial^n \eta_{1u_1}}{\partial x^n} - \binom{\alpha_1}{n+1} D_x^{n+1}(\xi) = 0, \ n \in N$$

$$\binom{\alpha_2}{n} \frac{\partial^n \eta_{2u_2}}{\partial x^n} - \binom{\alpha_2}{n+1} D_x^{n+1}(\xi) = 0, \ n \in N$$

by solving this system, we conclude the following explicit form of infinitesimal

$$\xi = c_1, \eta_1 = \frac{c_2}{2} \mu_1, \eta_2 = (c_2 + c_2)\mu_2$$

i.e.

$$\xi = c_1, \eta_1 = \frac{c_2}{2} \mu_1, \eta_2 = (2c_2)\mu_2$$

where c_1 and c_2 are arbitrary constants, hence the infinitesimal operator becomes

$$X = c_1 \frac{\partial}{\partial x} + \frac{c_2}{2} \mu_1 \frac{\partial}{\partial u_1} + 2c_2 \mu_2 \frac{\partial}{\partial u_2}$$

All the similarity variables associated with Lie symmetries can be divided by solving the following characteristic equation

$$\frac{dx}{c_1} = \frac{du_1}{c_2 \frac{\mu_1}{2}} = \frac{du_2}{2c_2 \mu_2}$$

The similarity functions associated with infinitesimal generator X take the following forms

$$u_1 = e^{k_1 x} \varphi_1, u_2 = xe^{4k_1 x} \varphi_2$$

where $k_1 = \frac{c_2}{2c_1}$ is an arbitrary constant and

$$\varphi_2 = 4^{1-\alpha_2} k_1^{-\alpha_2} \varphi_1^2$$

therefore, the solution for the non-linear system of fractional differential equation (0.1) will take the following form

$$\mu_1 = \varphi_1 e^{k_1 x}, \mu_2 = 4^{1-\alpha_2} k_1^{-\alpha_2} \varphi_1^2 xe^{2k_1 x}$$

where $k_1 = \frac{c_2}{2c_1}$ is an arbitrary constant.

4. Conclusion

The usual method to investigate symmetry properties of ordinary or partial differential equations is the group analysis method. The Lie group investigation effectively drawn-out to the investigation of symmetry properties of fractional differential equations and successfully use for making exact solutions of these Fractional Differential Equation's.

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