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An efficient analytical approach for solving fractional Fokker-Planck equations

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Abstract
The present study focuses on investigating the approximate analytical solutions of linear and non- linear Fokker-Planck equations (FPEs) with space- and time-fractional derivatives using an efficient analytical method, namely the Sumudu transform iterative method (STIM).The fractional derivatives are represented in the terms of Caputo. Analytical outcomes are obtained in the form of a converging series with easily computable components and are shown graphically. The results of the study suggest that the approach is simple to implement and very attractive in terms of computation.

Keywords
Fractional differential equations, Sumudu transform, Fokker-Planck equations, Iterative method, Mittag-Leffler function, Caputo fractional derivative.

AMS Subject Classification
33E12, 26A33, 35A22, 35A24.

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Contents

1. Introduction

Fractional Calculus (FC) is a field of mathematical study that concerns derivatives and integrals of arbitrary orders. Over the past decade, FC has been explored in a variety of fields, such as signal processing, electrochemistry, chemical physics, electromagnetics, engineering, acoustics, fluid mechanics, viscoelasticity, biology and other fields of science [2, 15, 16, 23]. The difficulty of finding accurate and approximate solutions for fractional differential equations (FDEs) in physics and mathematics remains a major challenge requiring new approaches. It is therefore necessary to investigate some effective methods for the solution of FDEs. Integer order differential equations are generalized to FDEs using a fractional calculus. As a result, several differential equation problems have been presented and discussed in the literature on fractional order derivatives.

The Fokker Plank Equation (FPE) was first discovered by Fokker and Plank to describe Brownian motion of the particles. FPE is emerging in various areas of natural science, along with quantum mechanics, circuit theory, chemical physics, solid state physics and theoretical biology. For a little utilization of FPE, reference can be made to works by Xu et al. [27] , Jumarie [10], He and Wu [9], Kamitani and Matsuba [11], and Zak [31]. The motion of the concentration field $w(\xi,t)$ of the single space variable x at the time t for the general FPE shall be given by [22]

$$\frac{\partial w}{\partial t}=\Big[-\frac{\partial}{\partial \xi}A(\xi)+\frac{\partial^2}{\partial \xi^2}B(\xi)\Big]w(\xi,t) \tag{1.1}$$

along with initial condition (IC)

$$w(\xi,0)=f(\xi), \xi\in\mathbb{R}, \tag{1.2}$$

where $A(\xi)$ and $B(\xi)>0$ are referred to as the coefficient of drift and the coefficient of diffusion. Drift coefficients and diffusion coefficients may also depend on time,

$$i.e.\ \ \frac{\partial w}{\partial t}=\Big[-\frac{\partial}{\partial \xi}A(\xi,t)+\frac{\partial^2}{\partial \xi^2}B(\xi,t)\Big]w(\xi,t) \tag{1.3}$$

The generalization of FPEs is known as nonlinear FPEs. Nonlinear FPEs play a vital role in a variety of fields such as surface physics, plasma physics, biophysics, polymer physics, psychology, marketing, population dynamics, engineering, neurosciences, laser physics and pattern formation.
The nonlinear FPE shall be written in the subsequent form

$$\frac{\partial w}{\partial t}=\Big[-\frac{\partial}{\partial \xi}A(\xi,t,w)+\frac{\partial^2}{\partial \xi^2}B(\xi,t,w)\Big]w(\xi,t) \quad (1.4)$$

subject to IC

$$w(\xi,0)=f(\xi),\ \ \xi\in\mathbb{R} \quad (1.5)$$

Nonlinear space-time fractional FPE may be adapted in the subsequent general form

$$\frac{\partial^\alpha}{\partial t^\alpha}w(\xi,t)=\Big[-\frac{\partial^\beta}{\partial \xi^\beta}A(\xi,t,w)+\Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2B(\xi,t,w)\Big]w(\xi,t) \quad (1.6)$$

where $t>0,\xi>0,0<\alpha,\beta\leq 1$. It may be derived from the general FPE by replacing the space and time derivatives with the fractional derivatives of Caputo $\frac{\partial^\alpha}{\partial t^\alpha}$ *and* $\frac{\partial^\beta}{\partial t^\beta}$ defined by eq. (2.2). The function $w(\xi,t)$ is assumed to be a causal function of time and space, i.e., vanishing for $t<0$ and $\xi<0$. Particularly for $\alpha=\beta=1$, the fractional FPE (1.6) reduces to the classical nonlinear FPE given by (1.4) in the case $t>0,\xi>0$.

A number of analytical or numerical methods have been developed for the solution of linear and nonlinear space and/or time fractional Fokker-Planck equations, such as the Optimal Homotopy Asymptotic Method (OHAM) [30], the iterative Laplace Transform Method (ILTM) [28], the Adomian Decomposition Method (ADM) [21], the Homotopy Perturbation Method (HPM) [19], the Homotopy perturbation transform method (HPTM) [13] etc. In recent time, Wang and Liu used Sumudu transform (ST) in conjunction with the iterative method and became a well-known approach known as the Sumudu transform iterative method [STIM] [25] to find approximate analytical solutions for time-fractional Cauchy reaction-diffusion equations. Recently, Kumar and Daftardar-Gejji [12] have successfully implemented the STIM technique in order to obtain analytical solutions for a variety of time and space fractional partial differential equations as well as their systems.
The key advantage of this study is the extension of the work of STIM technique to derive the approximate analytical solutions of space-and time-fractional Fokker-Planck equations.

2. Preliminaries and Basic Definitions

In the present portion, we provide some important definitions and properties relating to the FC and Sumudu transform which are further used in this paper.

Definition 2.1. *A fractional integral of Riemann-Liouville's order $\alpha>0$ of the real value function $w(\xi,t)$ is defined as [20, 24]*

$$I_t^\alpha w(\xi,t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}w(\xi,s)ds, \quad (2.1)$$

where $\Gamma(.)$ is known as the Gamma function.

Definition 2.2. *The Caputo fractional derivative of function $w(\xi,t)$ of order α is defined as [6, 17]*

$$\frac{\partial^\alpha}{\partial t^\alpha}w(\xi,t)=I_t^{m-\alpha}\Big[\frac{\partial^m}{\partial t^m}w(\xi,t)\Big]$$
$$=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_0^t\frac{w^m(\xi,\tau)}{(\xi-\tau)^{\alpha-m+1}}d\rho, \\ \qquad m-1<\alpha\leq m, m\in\mathbb{N}, \\ \frac{\partial^m}{\partial t^m}w(\xi,t) \qquad ,\alpha=m.\end{cases} \quad (2.2)$$

Definition 2.3. *The Sumudu transform over the set of functions $A=\{f(t)|\exists M,\rho_1>0,\rho_2>0$ such that $|f(t)|<Me^{|t|/\rho_j}$ if $t\in(-1)^j\times[0,\infty)\}$ is defined as [3, 26]*

$$S[f(t)]=F(\omega)=\int_0^\infty e^{-t}f(\omega t)dt\ ,\ \omega\in(-\rho_1,\rho_2). \quad (2.3)$$

One of the basic properties of Sumudu transform is

$$S\Big[\frac{t^\alpha}{\Gamma(\alpha+1)}\Big]=\omega^\alpha\ ,\ \alpha>-1. \quad (2.4)$$

inverse Sumudu transforms of ω^α is defined as

$$S^{-1}[\omega^\alpha]=\frac{t^\alpha}{\Gamma(\alpha+1)}\ ,\ \alpha>-1. \quad (2.5)$$

Definition 2.4. *The Sumudu transform of Caputo time fractional derivative of $w(\xi,t)$ of order $\alpha>0$ is defined as [1, 25]*

$$S[\frac{\partial^\alpha w(\xi,t)}{\partial t^\alpha}]=\omega^{-\alpha}S[w(\xi,t)] -\sum_{k=0}^{m-1}\Big[\omega^{-\alpha+k}\frac{\partial^k w(\xi,0)}{\partial t^k}\Big], \quad m-1<\alpha\leq m\ ,\ m\in\mathbb{N}. \quad (2.6)$$

Definition 2.5. *The Mittag-Leffler function is defined as [18]*

$$E_\alpha(z)=\sum_{n=0}^\infty\frac{z^n}{\Gamma(\alpha n+1)}\ ,\alpha\in\mathbb{C},Re(\alpha)>0. \quad (2.7)$$

3. The Basic Concept of STIM

In order to demonstrate the key concept of this method [25], the following general space and time fractional differential equation of the form is considered as

$$\frac{\partial^{\alpha} w}{\partial t^{\alpha}} = F(\xi, w, \frac{\partial^{\beta} w}{\partial \xi^{\beta}}, ..., \frac{\partial^{k\beta} w}{\partial \xi^{k\beta}}),$$
$$l-1 < \alpha \leq l, m-1 < \beta \leq m, k, l, m \in \mathbb{N} \tag{3.1}$$

along with the initial conditions

$$\frac{\partial^k w(\xi,0)}{\partial t^k} = h_k(\xi), k = 0,1,2,...,n-1, \tag{3.2}$$

where $F(\xi, w, \frac{\partial^{\beta} w}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w}{\partial \xi^{l\beta}})$ is a linear/nonlinear operator and $w = w(\xi,t)$ is the unknown function and fractional derivative $\frac{\partial^{l\beta} w(\xi,t)}{\partial \xi^{l\beta}}, l \in N$ is taken as the sequential fractional derivative [17] *i.e.*

$$\frac{\partial^{l\beta} w}{\partial \xi^{l\beta}} = \frac{\partial^{\beta}}{\partial \xi^{\beta}} \frac{\partial^{\beta}}{\partial \xi^{\beta}} \cdots \frac{\partial^{\beta} w}{\partial \xi^{\beta}} \ \ (l \ times). \tag{3.3}$$

Taking the Sumudu transform on the both sides of eq. (3.1), we get

$$S\Big[\frac{\partial^{\alpha} w}{\partial t^{\alpha}}\Big] = S\Big[F(\xi, w, \frac{\partial^{\beta} w}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w}{\partial \xi^{l\beta}})\Big] \tag{3.4}$$

By using eq. (2.6), we have

$$S[w(\xi,t)] = \sum_{k=0}^{m-1} \Big[\omega^k \frac{\partial^k w(\xi,0)}{\partial t^k}\Big] + \omega^{\alpha} S\Big[F\Big(\xi, w, \frac{\partial^{\beta} w}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w}{\partial \xi^{l\beta}}\Big)\Big]. \tag{3.5}$$

The inverse Sumudu transform of eq. (3.5) leads to

$$w(\xi,t) = S^{-1}\Big(\sum_{k=0}^{m-1} \Big[\omega^k \frac{\partial^k w(\xi,0)}{\partial t^k}\Big]\Big) + S^{-1}\Big[\omega^{\alpha} S\Big(F\Big(\xi, w, \frac{\partial^{\beta} w}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w}{\partial \xi^{l\beta}}\Big)\Big)\Big]. \tag{3.6}$$

We can write eq. (3.6) as

$$w(\xi,t) = f(\xi,t) + N\Big(\xi, w, \frac{\partial^{\beta} w}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w}{\partial \xi^{l\beta}}\Big), \tag{3.7}$$

where

$$f(\xi,t) = S^{-1}\Big(\sum_{k=0}^{m-1} \Big[\omega^k \frac{\partial^k w(\xi,0)}{\partial t^k}\Big]\Big) \tag{3.8}$$

$$N\Big(\xi, w, \frac{\partial^{\beta} w}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w}{\partial \xi^{l\beta}}\Big) = S^{-1}\Big[\omega^{\alpha} S\Big(F\Big(\xi, w, \frac{\partial^{\beta} w}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w}{\partial \xi^{l\beta}}\Big)\Big)\Big], \tag{3.9}$$

here N is a linear/nonlinear operator and f is known function .

Further, we apply the iterative technique introduced by Daftardar-Gejji and Jafari [7], which represents a solution $w(\xi,t)$ in components of infinite series

$$w(\xi,t) = \sum_{i=0}^{\infty} w_i(\xi,t), \tag{3.10}$$

Decomposing the operator N as

$$\begin{aligned} N\Big(\xi, \sum_{i=0}^{\infty} w_i, \frac{\partial^{\beta} \sum_{i=0}^{\infty} w_i}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} \sum_{i=0}^{\infty} w_i}{\partial \xi^{l\beta}}\Big) &= N\Big(\xi, w_0, \frac{\partial^{\beta} w_0}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w_0}{\partial \xi^{l\beta}}\Big) \\ &+ \sum_{i=1}^{\infty}\Big(N\Big(\xi, \sum_{j=0}^{i} w_j, \frac{\partial^{\beta} \sum_{j=0}^{i} w_j}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} \sum_{j=0}^{i} w_j}{\partial \xi^{l\beta}}\Big)\Big) \\ &- \sum_{i=1}^{\infty}\Big(N\Big(\xi, \sum_{j=0}^{i-1} w_j, \frac{\partial^{\beta} \sum_{j=0}^{i-1} w_j}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} \sum_{j=0}^{i-1} w_j}{\partial \xi^{l\beta}}\Big)\Big) \end{aligned} \tag{3.11}$$

$$\begin{aligned} S^{-1}\Big[\omega^{\gamma} S\Big(F\Big(\xi, \sum_{i=0}^{\infty} w_i, \frac{\partial^{\beta} \sum_{i=0}^{\infty} w_i}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} \sum_{i=0}^{\infty} w_i}{\partial \xi^{l\beta}}\Big)\Big)\Big] &= S^{-1}\Big[\omega^{\gamma} S\Big(F\Big(\xi, w_0, \frac{\partial^{\beta} w_0}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w_0}{\partial \xi^{l\beta}}\Big)\Big)\Big] \\ &+ \sum_{i=1}^{\infty} S^{-1}\Big[\omega^{\gamma} S\Big(F\Big(\xi, \sum_{j=0}^{i} w_j, \frac{\partial^{\beta} \sum_{j=0}^{i} w_j}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} \sum_{j=0}^{i} w_j}{\partial \xi^{l\beta}}\Big)\Big)\Big] \\ &- \sum_{i=1}^{\infty} S^{-1}\Big[\omega^{\gamma} S\Big(F\Big(\xi, \sum_{j=0}^{i-1} w_j, \frac{\partial^{\beta} \sum_{j=0}^{i-1} w_j}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} \sum_{j=0}^{i-1} w_j}{\partial \xi^{l\beta}}\Big)\Big)\Big]. \end{aligned} \tag{3.12}$$

Using eqs. (3.10) to (3.12) in eq. (3.7), we get

$$\begin{aligned} \sum_{i=0}^{\infty} w_i(\xi,t) &= S^{-1}\Big(\sum_{k=0}^{m-1} \Big[\omega^k \frac{\partial^k w(\xi,0)}{\partial t^k}\Big]\Big) \\ &+ S^{-1}\Big[\omega^{\gamma} S\Big(F\Big(\xi, w_0, \frac{\partial^{\beta} w_0}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} w_0}{\partial \xi^{l\beta}}\Big)\Big)\Big] \\ &+ \sum_{i=1}^{\infty} S^{-1}\Big[\omega^{\gamma} S\Big(F\Big(\xi, \sum_{j=0}^{i} w_j, \frac{\partial^{\beta} \sum_{j=0}^{i} w_j}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} \sum_{j=0}^{i} w_j}{\partial \xi^{l\beta}}\Big)\Big)\Big] \\ &- \sum_{i=1}^{\infty} S^{-1}\Big[\omega^{\gamma} S\Big(F\Big(\xi, \sum_{j=0}^{i-1} w_j, \frac{\partial^{\beta} \sum_{j=0}^{i-1} w_j}{\partial \xi^{\beta}}, ..., \frac{\partial^{l\beta} \sum_{j=0}^{i-1} w_j}{\partial \xi^{l\beta}}\Big)\Big)\Big] \end{aligned} \tag{3.13}$$

The recurrence relations have been defined as follows

$$\left.\begin{aligned}
&w_0(\xi,t) = S^{-1}\Big(\sum_{k=0}^{m-1}\Big[\omega^k \frac{\partial^k w(\xi,0)}{\partial t^k}\Big]\Big)\\
&w_1(\xi,t) = S^{-1}\Big[\omega^\gamma S\Big(F\Big(\xi, w_0, \frac{\partial^\beta w_0}{\partial \xi^\beta}, \ldots, \frac{\partial^{l\beta} w_0}{\partial \xi^{l\beta}}\Big)\Big)\Big]\\
&w_{r+1}(\xi,t)\\
&= S^{-1}\Big[\omega^\gamma S\Big(F\Big(\xi, \sum_{i=0}^{r} w_i, \frac{\partial^\beta \sum_{i=0}^{r} w_i}{\partial \xi^\beta}, \ldots, \frac{\partial^{l\beta} \sum_{i=0}^{r} w_i}{\partial \xi^{l\beta}}\Big)\Big)\Big]\\
&- S^{-1}\Big[\omega^\gamma S\Big(F\Big(\xi, \sum_{i=0}^{r-1} w_i, \frac{\partial^\beta \sum_{i=0}^{r-1} w_i}{\partial \xi^\beta}, \ldots, \frac{\partial^{l\beta} \sum_{i=0}^{r-1} w_i}{\partial \xi^{l\beta}}\Big)\Big)\Big],\\
&\qquad\qquad\qquad\qquad\qquad\qquad r \geq 1.
\end{aligned}\right\} \tag{3.14}$$

Therefore the approximate analytical solution of eqs. (3.1) and (3.2) in truncated series form is given by

$$w(\xi,t) \cong \lim_{N\to\infty} \sum_{r=0}^{N} w_r(\xi,t). \tag{3.15}$$

In general, the above series solutions converge very rapidly. The classic approach to convergence of this type of series has been presented by Daftardar-Gejji and Jafari [7] and Bhalekar and Daftardar-Gejji [4].

4. Solution of Fractional Fokker-planck equations

In this portion, the STIM technique is used to solve linear and nonlinear fractional FPEs.

Example 4.1 Consider the following linear space-time fractional Fokker-plank equation [8]

$$\frac{\partial^\alpha}{\partial t^\alpha} w(\xi,t) = \Big[-\frac{\partial^\beta}{\partial \xi^\beta}\Big(p\xi^\beta\Big) + \Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2\Big(q\xi^{2\beta}\Big)\Big] w(\xi,t),$$
$$t > 0, \xi > 0, 0 < \alpha, \beta \leq 1, p, q \in \mathbb{R} \tag{4.1}$$

with initial condition

$$w(\xi,0) = \xi^{a-1}, a \geq 1. \tag{4.2}$$

Taking the Sumudu transform on the both sides of eq. (4.1), we get

$$\begin{aligned}
&S\Big[\frac{\partial^\alpha}{\partial t^\alpha} w(\xi,t)\Big]\\
&= S\Big[\Big[-\frac{\partial^\beta}{\partial \xi^\beta}\Big(p\xi^\beta\Big) + \Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2\Big(q\xi^{2\beta}\Big)\Big] w(\xi,t)\Big].
\end{aligned} \tag{4.3}$$

By using eq. (2.6), we have

$$\begin{aligned}
S[w(\xi,t)] = w(\xi,0) + \omega^\alpha S\Big[\Big[-\frac{\partial^\beta}{\partial \xi^\beta}\Big(p\xi^\beta\Big)\\
+ \Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2\Big(q\xi^{2\beta}\Big)\Big] w(\xi,t)\Big].
\end{aligned} \tag{4.4}$$

Operating with the inverse Sumudu transform on both sides of eq. (4.4), gives

$$\begin{aligned}
w(\xi,t) = S^{-1}[w(\xi,0)] + S^{-1}\Big(\omega^\alpha S\Big[\Big[-\frac{\partial^\beta}{\partial \xi^\beta}\Big(p\xi^\beta\Big)\\
+ \Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2\Big(q\xi^{2\beta}\Big)\Big] w(\xi,t)\Big]\Big).
\end{aligned} \tag{4.5}$$

Substituting the results from eqs. (3.10) to (3.12) in the eq. (4.5) and applying the eq. (3.14), we find out the components of the STIM solution as follows

$$\begin{aligned}
w_0(\xi,t) &= S^{-1}[w(\xi,0)] = \xi^{a-1},\\
w_1(\xi,t) &= S^{-1}\Big(\omega^\alpha S\Big[\Big[-\frac{\partial^\beta}{\partial \xi^\beta}\Big(p\xi^\beta\Big)\\
&\quad + \Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2\Big(q\xi^{2\beta}\Big)\Big](w_0)\Big]\Big)\\
&= b\xi^{a-1}\frac{t^\alpha}{\Gamma(\alpha+1)}, b = q(a)_{2\beta} - p(a)_\beta,\\
w_2(\xi,t) &= S^{-1}\Big(\omega^\alpha S\Big[\Big[-\frac{\partial^\beta}{\partial \xi^\beta}\Big(p\xi^\beta\Big)\\
&\quad + \Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2\Big(q\xi^{2\beta}\Big)\Big](w_0 + w_1)\Big]\Big)\\
&\quad - S^{-1}\Big(\omega^\alpha S\Big[\Big[-\frac{\partial^\beta}{\partial \xi^\beta}\Big(p\xi^\beta\Big)\\
&\quad + \Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2\Big(q\xi^{2\beta}\Big)\Big](w_0)\Big]\Big)\\
&= \xi^{a-1}\frac{b^2 t^{2\alpha}}{\Gamma(2\alpha+1)},\ b = q(a)_{2\beta} - p(a)_\beta,\\
w_3(\xi,t) &= S^{-1}\Big(\omega^\alpha S\Big[\Big[-\frac{\partial^\beta}{\partial \xi^\beta}\Big(p\xi^\beta\Big)\\
&\quad + \Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2\Big(q\xi^{2\beta}\Big)\Big](w_0 + w_1 + w_2)\Big]\Big)\\
&\quad - S^{-1}\Big(\omega^\alpha S\Big[\Big[-\frac{\partial^\beta}{\partial \xi^\beta}\Big(p\xi^\beta\Big)\\
&\quad + \Big(\frac{\partial^\beta}{\partial \xi^\beta}\Big)^2\Big(q\xi^{2\beta}\Big)\Big](w_0 + w_1)\Big]\Big)\\
&= \xi^{a-1}\frac{b^3 t^{3\alpha}}{\Gamma(3\alpha+1)},\ b = q(a)_{2\beta} - p(a)_\beta
\end{aligned}$$

and so on.

Therefore, the approximate analytical solution can be obtained series form as

$$
\begin{aligned}
w(\xi,t) &= \lim_{N\to\infty}\sum_{r=0}^{N} w_r(\xi,t)\\
&= \xi^{a-1}\Big[1+\frac{bt^{\alpha}}{\Gamma(\alpha+1)}+\frac{(bt^{\alpha})^2}{\Gamma(2\alpha+1)}+,...,\Big]\\
&= \xi^{a-1}E_{\alpha}(bt^{\alpha}), b = q(a)_{2\beta}-p(a)_{\beta}. \qquad (4.6)
\end{aligned}
$$

Special case 4.1.1. Putting $\alpha = 1$, eq. (4.1) along with condition (4.2) reduced to linear space fractional Fokker-planck equation

$$
\frac{\partial w}{\partial t} = \Big[-\frac{\partial^{\beta}}{\partial \xi^{\beta}}\Big(p\xi^{\beta}\Big)+\Big(\frac{\partial^{\beta}}{\partial \xi^{\beta}}\Big)^2\Big(q\xi^{2\beta}\Big)\Big]w(\xi,t),\quad t>0, \xi>0, 0<\beta\le 1, \qquad (4.7)
$$

subject to initial condition

$$
w(\xi,t) = \xi^{a-1}, a \ge 1 \qquad (4.8)
$$

has a solution

$$
w(\xi,t) = \xi^{a-1}e^{bt}, b = q(a)_{2\beta}-p(a)_{\beta}. \qquad (4.9)
$$

Special case 4.1.2. Taking $\beta = 1$, eq. (4.1) along with condition (4.2) reduced to linear time fractional Fokker-planck equation

$$
\frac{\partial^{\alpha} w(\xi,t)}{\partial t^{\alpha}} = \Big[-\frac{\partial}{\partial \xi}(p\xi)+\frac{\partial^2}{\partial \xi^2}(q\xi^2)\Big]w(\xi,t),\quad t>0, \xi>0, 0<\alpha\le 1, \qquad (4.10)
$$

subject to initial condition

$$
w(\xi,0) = \xi^{a-1}, a \ge 1 \qquad (4.11)
$$

and has a solution

$$
w(\xi,t) = \xi^{a-1}E_{\alpha}(bt^{\alpha}), b = qa^2+a(q-p). \qquad (4.12)
$$

Special case 4.1.3. Taking $\alpha = \beta = 1$, eq. (4.1) with condition (4.2) reduced to linear Fokker-planck equation

$$
\frac{\partial w}{\partial t} = \Big[-\frac{\partial}{\partial \xi}(p\xi)+\frac{\partial^2}{\partial \xi^2}(q\xi^2)\Big]w(\xi,t)\ ,\ t>0, \xi>0, \qquad (4.13)
$$

subject to initial condition

$$
w(\xi,0) = \xi^{a-1}, a \ge 1 \qquad (4.14)
$$

and has a solution

$$
w(\xi,t) = \xi^{a-1}e^{bt}, b = qa^2+a(q-p). \qquad (4.15)
$$

Special case 4.1.4. Taking $\alpha = \beta = 1, a = 2, p = 1, q = 1/2$, eq. (4.1) along with condition (4.2) reduced to linear Fokker-planck equation [29]

$$
\frac{\partial w}{\partial t} = \Big[-\frac{\partial}{\partial \xi}(\xi)+\frac{\partial^2}{\partial \xi^2}\Big(\frac{\xi^2}{2}\Big)\Big]w(\xi,t)\ ,\ t>0, \xi>0, \qquad (4.16)
$$

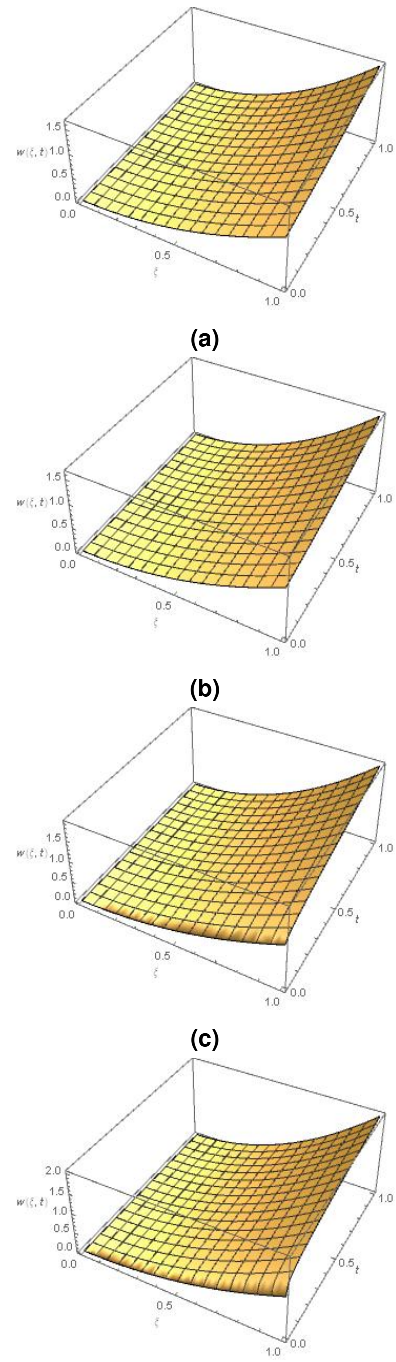

Figure 1. The surface shows the solution $w(\xi,t)$ for example 4.1 : when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.25$.

subject to initial condition

$$
w(\xi,0) = \xi \qquad (4.17)
$$

has a solution

$$
w(\xi,t) = \xi e^{t}. \qquad (4.18)
$$

Special case 4.1.5. Taking $\alpha = \beta = 1, a = 3, p = 1/6, q = 1/12$, eq. (4.1) along with condition (4.2) reduced to linear Fokker-planck equation [29]

$$\frac{\partial w}{\partial t} = \Big[-\frac{\partial}{\partial \xi}\Big(\frac{\xi}{6}\Big) + \frac{\partial^2}{\partial \xi^2}\Big(\frac{\xi^2}{12}\Big)\Big] w(\xi,t)\ ,\ t > 0, \xi > 0, \tag{4.19}$$

subject to initial condition

$$w(\xi,0) = \xi^2 \tag{4.20}$$

has a solution

$$w(\xi,t) = \xi^2 e^{t/2}. \tag{4.21}$$

Example 4.2.Consider the following linear time – fractional Fokker-Planck equation [5]

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \frac{\partial w}{\partial \xi} + \frac{\partial^2 w}{\partial \xi^2}, \quad 0 < \alpha \leq 1, \xi > 0, t > 0 \tag{4.22}$$

with initial condition

$$w(\xi,0) = \xi. \tag{4.23}$$

Taking the Sumudu transform on the both sides of eq. (4.22), we get

$$S\Big[\frac{\partial^\alpha w}{\partial t^\alpha}\Big] = S\Big[\frac{\partial w}{\partial \xi} + \frac{\partial^2 w}{\partial \xi^2}\Big]. \tag{4.24}$$

By using eq. (2.6), we have

$$S[w(\xi,t)] = w(\xi,0) + \omega^\alpha \Big(S\Big[\frac{\partial w}{\partial \xi} + \frac{\partial^2 w}{\partial \xi^2}\Big]\Big). \tag{4.25}$$

Operating with the inverse Sumudu transform on both sides of eq. (4.25), gives

$$w(\xi,t) = S^{-1}[w(\xi,0)] + S^{-1}\Big(\omega^\alpha\Big(S\Big[\frac{\partial w}{\partial \xi} + \frac{\partial^2 w}{\partial \xi^2}\Big]\Big)\Big). \tag{4.26}$$

Substituting the results from eqs. (3.10) to (3.12) in the eq. (4.26) and applying the eq. (3.14), we find out the components of the STIM solution as follows

$$\begin{aligned} w_0(\xi,t) &= S^{-1}[w(\xi,0)] = \xi, \\ w_1(\xi,t) &= S^{-1}\Big(\omega^\alpha\Big(S\Big[\frac{\partial w_0}{\partial \xi} + \frac{\partial^2 w_0}{\partial \xi^2}\Big]\Big)\Big) \\ &= \frac{t^\alpha}{\Gamma(\alpha+1)} \end{aligned}$$

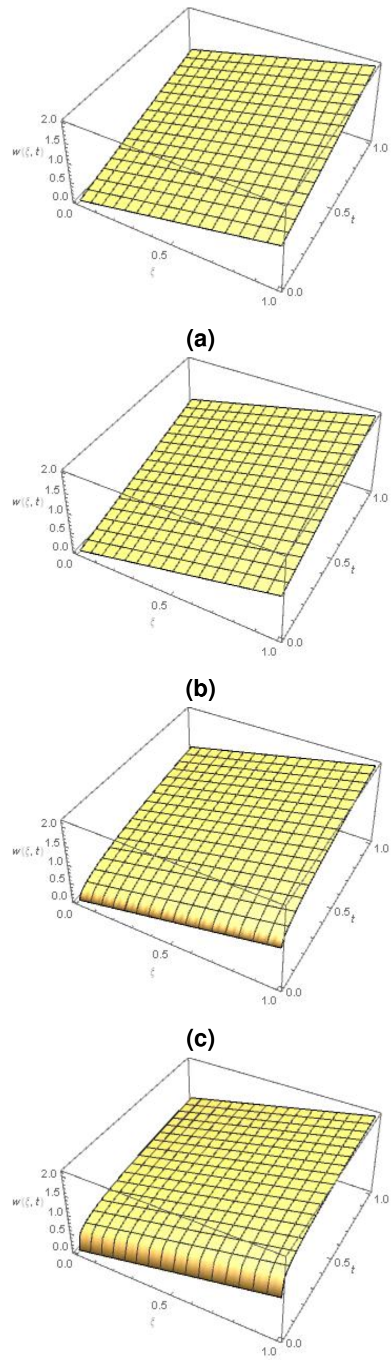

Figure 2. The surface shows the solution $w(\xi,t)$ for example 4.2 : when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.25$.

$$\begin{aligned} w_2(\xi,t) &= S^{-1}\Big(\omega^\alpha\Big(S\Big[\frac{\partial(w_0+w_1)}{\partial \xi} + \frac{\partial^2(w_0+w_1)}{\partial \xi^2}\Big]\Big)\Big) \\ &\quad - S^{-1}\Big(\omega^\alpha\Big(S\Big[\frac{\partial w_0}{\partial \xi} + \frac{\partial^2 w_0}{\partial \xi^2}\Big]\Big)\Big) \\ &= 0 \\ w_n(\xi,t) &= 0\ \forall n \geq 3 \end{aligned}$$

Therefore, the approximate analytical solution can be obtained series form as

$$w(\xi,t) = \lim_{N\to\infty}\sum_{r=0}^{N} w_r(\xi,t) = \xi + \frac{t^{\alpha}}{\Gamma(\alpha+1)}. \tag{4.27}$$

Special case 4.2.1. Taking $\alpha = 1$, eq. (4.22) along with condition (4.23) reduced to linear Fokker-planck equation

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial \xi} + \frac{\partial^2 w}{\partial \xi^2}, \;\; \xi > 0, t > 0, \tag{4.28}$$

with initial condition

$$w(\xi,0) = \xi \tag{4.29}$$

and has a solution

$$w(\xi,t) = \xi + t. \tag{4.30}$$

Example 4.3. Consider the following non-linear time-fractional Fokker-Planck equation [8]

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} w(\xi,t) = \Big[-\frac{\partial}{\partial \xi}\Big(3w - \frac{\xi}{2}\Big) + \frac{\partial^2}{\partial \xi^2}(w\xi)\Big] w(\xi,t), \quad t > 0, \xi > 0, 0 < \alpha \le 1 \tag{4.31}$$

subject to initial condition

$$w(\xi,0) = \xi. \tag{4.32}$$

Taking the Sumudu transform on the both sides of eq. (4.31), we get

$$S\Big[\frac{\partial^{\alpha}}{\partial t^{\alpha}} w(\xi,t)\Big] = S\Big[\Big[-\frac{\partial}{\partial \xi}\Big(3w - \frac{\xi}{2}\Big) + \frac{\partial^2}{\partial \xi^2}(w\xi)\Big] w(\xi,t)\Big]. \tag{4.33}$$

By using eq. (2.6), we get

$$S[w(\xi,t)] = w(\xi,0) + \omega^{\alpha} S\Big[\Big[-\frac{\partial}{\partial \xi}\Big(3w - \frac{\xi}{2}\Big) + \frac{\partial^2}{\partial \xi^2}(w\xi)\Big] w(\xi,t)\Big]. \tag{4.34}$$

Operating with the inverse Sumudu transform on both sides of eq. (4.34), gives

$$w(\xi,t) = S^{-1}[w(\xi,0)] + S^{-1}\Big(\omega^{\alpha} S\Big[\Big[-\frac{\partial}{\partial \xi}\Big(3w - \frac{\xi}{2}\Big) + \frac{\partial^2}{\partial \xi^2}(w\xi)\Big] w(\xi,t)\Big]\Big). \tag{4.35}$$

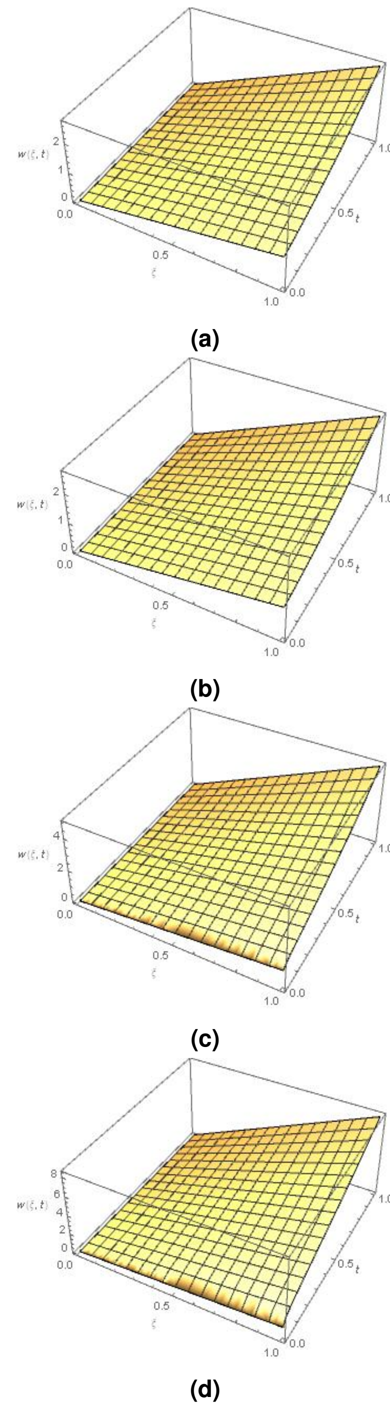

Figure 3. The surface shows the solution $w(\xi,t)$ for example 4.3 : when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.25$.

Substituting the results from eqs. (3.10) to (3.12) in the eq. (4.35) and applying the eq. (3.14), we find out the components of the STIM solution as follows

$$w_0(\xi,t) = S^{-1}[w(\xi,0)] = \xi,$$

$$w_1(\xi,t) = S^{-1}\Big(\omega^\alpha S\Big[\Big[-\frac{\partial}{\partial\xi}\Big(3w_0-\frac{\xi}{2}\Big) + \frac{\partial^2}{\partial\xi^2}(\xi w_0)\Big]w_0\Big]\Big) = \xi\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$w_2(\xi,t) = S^{-1}\Big(\omega^\alpha S\Big[\Big[-\frac{\partial}{\partial x}\Big(3(w_0+w_1)-\frac{\xi}{2}\Big) + \frac{\partial^2}{\partial\xi^2}\Big(\xi(w_0+w_1)\Big)\Big](w_0+w_1)\Big]\Big) - S^{-1}\Big(\omega^\alpha S\Big[\Big[-\frac{\partial}{\partial\xi}\Big(3w_0-\frac{\xi}{2}\Big) + \frac{\partial^2}{\partial\xi^2}(\xi w_0)\Big]w_0\Big]\Big) = \xi\frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$w_3(\xi,t) = \xi\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

and so on.
Therefore, the approximate analytical solution can be obtained series form as

$$w(\xi,t) = \lim_{N\to\infty}\sum_{r=0}^{N} w_r(\xi,t) = \xi\sum_{n=0}^{\infty}\frac{(t^\alpha)^n}{\Gamma(n\alpha+1)} = \xi E_\alpha(t^\alpha). \tag{4.36}$$

Special case 4.3.1. Setting $\alpha = 1$, eq. (4.31) along with condition (4.32) reduced to non linear Fokker-planck equation

$$\frac{\partial w}{\partial t} = \Big[-\frac{\partial}{\partial\xi}\Big(3w-\frac{\xi}{2}\Big) + \frac{\partial^2}{\partial\xi^2}(w\xi)\Big]w(\xi,t)\ ,\ \xi>0, t>0, \tag{4.37}$$

subject to initial condition

$$w(\xi,0) = \xi \tag{4.38}$$

and has a solution

$$w(\xi,t) = \xi e^t. \tag{4.39}$$

Example 4.4. In this example, consider the following nonlinear space-time fractional Fokker-Planck equation [14]

$$\frac{\partial^\alpha w}{\partial t^\alpha} = -\frac{\partial^\beta}{\partial\xi^\beta}\Big(\frac{4w^2}{\xi}-\frac{\xi w}{3}\Big) + \frac{\partial^{2\beta}}{\partial\xi^{2\beta}}(w^2), \quad \xi>0, t>0, 0<\alpha,\beta\le 1 \tag{4.40}$$

with initial condition

$$w(\xi,0) = \xi^2. \tag{4.41}$$

Taking the Sumudu transform on the both sides of eq. (4.40), we get

$$S\Big[\frac{\partial^\alpha w}{\partial t^\alpha}\Big] = S\Big[-\frac{\partial^\beta}{\partial\xi^\beta}\Big(\frac{4w^2}{\xi}-\frac{w\xi}{3}\Big) + \frac{\partial^{2\beta}}{\partial\xi^{2\beta}}(w^2)\Big]. \tag{4.42}$$

By using eq. (2.6), we get

$$S[w(\xi,t)] = w(\xi,0) + \omega^\alpha\Big(S\Big[-\frac{\partial^\beta}{\partial\xi^\beta}\Big(\frac{4w^2}{\xi}-\frac{w\xi}{3}\Big) + \frac{\partial^{2\beta}}{\partial\xi^{2\beta}}(w^2)\Big]\Big). \tag{4.43}$$

Operating with the inverse Sumudu transform on both sides of eq. (4.43), gives

$$w(\xi,t) = S^{-1}[w(\xi,0)] + S^{-1}\Big(\omega^\alpha\Big(S\Big[-\frac{\partial^\beta}{\partial\xi^\beta}\Big(\frac{4w^2}{\xi}-\frac{w\xi}{3}\Big) + \frac{\partial^{2\beta}}{\partial\xi^{2\beta}}(w^2)\Big]\Big)\Big). \tag{4.44}$$

Substituting the results from eqs. (3.10) to (3.12) in the eq. (4.44) and applying the eq. (3.14), we find out the components of the STIM solution as follows

$$w_0(\xi,t) = S^{-1}[w(\xi,0)] = \xi^2,$$

$$w_1(\xi,t) = S^{-1}\Big(\omega^\alpha\Big(S\Big[-\frac{\partial^\beta}{\partial\xi^\beta}\Big(\frac{4w_0^2}{\xi}-\frac{\xi w_0}{3}\Big) + \frac{\partial^{2\beta}}{\partial\xi^{2\beta}}(w_0^2)\Big]\Big)\Big) = \Big[-\frac{22}{\Gamma(4-\beta)}\xi^{3-\beta} + \frac{24}{\Gamma(5-2\beta)}\xi^{4-2\beta}\Big] \times \frac{t^\alpha}{\Gamma(1+\alpha)},$$

$$w_2(\xi,t) = S^{-1}\Big(\omega^\alpha\Big(S\Big[-\frac{\partial^\beta}{\partial\xi^\beta}\Big(\frac{4(w_0+w_1)^2}{\xi} - \frac{\xi(w_0+w_1)}{3}\Big) + \frac{\partial^{2\beta}}{\partial\xi^{2\beta}}(w_0+w_1)^2\Big]\Big)\Big) - S^{-1}\Big(\omega^\alpha\Big(S\Big[-\frac{\partial^\beta}{\partial\xi^\beta}\Big(\frac{4w_0^2}{\xi}-\frac{\xi w_0}{3}\Big) + \frac{\partial^{2\beta}}{\partial\xi^{2\beta}}(w_0^2)\Big]\Big)\Big)$$

$$
\begin{aligned}
&= \frac{506\Gamma(5-\beta)\, t^{2\alpha}\xi^{4-2\beta}}{3\Gamma(1+2\alpha)\Gamma(5-2\beta)\Gamma(4-\beta)} \\
&+ \frac{48\Gamma(7-2\beta)\, t^{2\alpha}\xi^{6-4\beta}}{\Gamma(1+2\alpha)\Gamma(5-2\beta)\Gamma(7-4\beta)} \\
&- \frac{1}{\Gamma(1+2\alpha)\Gamma(6-3\beta)} \\
&\times \Big[\frac{184\Gamma(6-2\beta)}{\Gamma(5-2\beta)} + \frac{44\Gamma(6-\beta)}{\Gamma(4-\beta)}\Big] t^{2\alpha}\xi^{5-3\beta} \\
&- \frac{1936\Gamma(1+2\alpha)\Gamma(6-2\beta)\, t^{3\alpha}\xi^{5-3\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(6-3\beta)\Gamma^2(4-\beta)} \\
&- \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(8-5\beta)\Gamma(5-2\beta)} \\
&\times \Big[\frac{2304\Gamma(8-4\beta)}{\Gamma(5-2\beta)} + \frac{1056\Gamma(8-3\beta)}{\Gamma(4-\beta)}\Big] t^{3\alpha}\xi^{7-5\beta} \\
&+ \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(7-4\beta)\Gamma(4-\beta)} \\
&\times \Big[\frac{484\Gamma(7-2\beta)}{\Gamma(4-\beta)} + \frac{4224\Gamma(7-3\beta)}{\Gamma(5-2\beta)}\Big] t^{3\alpha}\xi^{6-4\beta} \\
&+ \frac{576\Gamma(1+2\alpha)\Gamma(9-4\beta)\, t^{3\alpha}\xi^{8-6\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(9-6\beta)\Gamma^2(5-2\beta)}
\end{aligned}
$$

and so on.

Therefore, the approximate analytical solution can be obtained series form as

$$
\begin{aligned}
w(\xi,t) &= \lim_{N\to\infty}\sum_{r=0}^{N} w_r(\xi,t) \\
&= \xi^2 + \Big[-\frac{22}{\Gamma(4-\beta)}\xi^{3-\beta} \\
&+ \frac{24}{\Gamma(5-2\beta)}\xi^{4-2\beta}\Big]\frac{t^{\alpha}}{\Gamma(1+\alpha)} \\
&+ \frac{506\Gamma(5-\beta)\, t^{2\alpha}\xi^{4-2\beta}}{3\Gamma(1+2\alpha)\Gamma(5-2\beta)\Gamma(4-\beta)} \\
&+ \frac{48\Gamma(7-2\beta)\, t^{2\alpha}\xi^{6-4\beta}}{\Gamma(1+2\alpha)\Gamma(5-2\beta)\Gamma(7-4\beta)} \\
&- \frac{1}{\Gamma(1+2\alpha)\Gamma(6-3\beta)} \\
&\times \Big[\frac{184\Gamma(6-2\beta)}{\Gamma(5-2\beta)} + \frac{44\Gamma(6-\beta)}{\Gamma(4-\beta)}\Big] t^{2\alpha}\xi^{5-3\beta} \\
&- \frac{1936\Gamma(1+2\alpha)\Gamma(6-2\beta)\, t^{3\alpha}\xi^{5-3\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(6-3\beta)\Gamma^2(4-\beta)} \\
&- \frac{\Gamma(1+2\alpha)\, t^{3\alpha}\xi^{7-5\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(8-5\beta)\Gamma(5-2\beta)} \\
&\times \Big[\frac{2304\Gamma(8-4\beta)}{\Gamma(5-2\beta)} + \frac{1056\Gamma(8-3\beta)}{\Gamma(4-\beta)}\Big] \\
&+ \frac{\Gamma(1+2\alpha)}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(7-4\beta)\Gamma(4-\beta)} \\
&\times \Big[\frac{484\Gamma(7-2\beta)}{\Gamma(4-\beta)} + \frac{4224\Gamma(7-3\beta)}{\Gamma(5-2\beta)}\Big] t^{3\alpha}\xi^{6-4\beta} \\
&+ \frac{576\Gamma(1+2\alpha)\Gamma(9-4\beta)\, t^{3\alpha}\xi^{8-6\beta}}{\Gamma^2(1+\alpha)\Gamma(1+3\alpha)\Gamma(9-6\beta)\Gamma^2(5-2\beta)}
\end{aligned}
$$

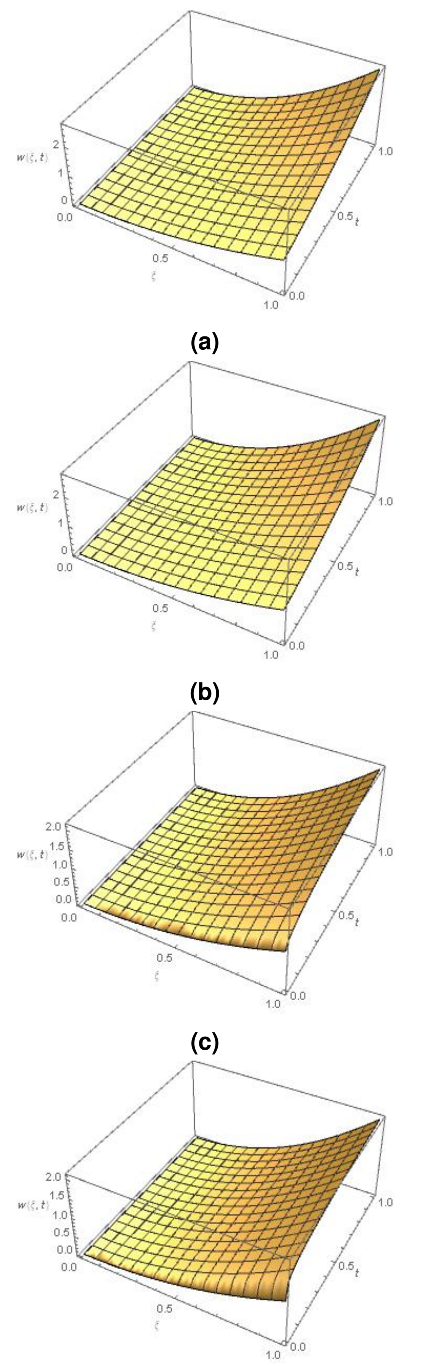

Figure 4. The surface shows the solution $w(\xi,t)$ for example 4.4 : when (a) The exact solution, (b) The approximate solution for $\alpha = 1$, (c) The approximate solution for $\alpha = 0.5$, (d) The approximate solution for $\alpha = 0.25$.

Special case 4.4.1. Setting $\alpha = \beta = 1$, eq. (4.40) along with condition (4.41) reduced to linear Fokker-planck equation

$$\frac{\partial w}{\partial t} = -\frac{\partial}{\partial \xi}\left(\frac{4w^2}{\xi} - \frac{w\xi}{3}\right) + \frac{\partial^2}{\partial \xi^2}(w^2), \ \ \xi > 0, t > 0, \quad (4.45)$$

subject to initial condition

$$w(\xi, 0) = \xi^2 \quad (4.46)$$

and has a solution

$$w(\xi, t) = \xi^2 e^t. \quad (4.47)$$

5. Conclusion

The Sumudu transform iterative method has been used in this study to obtain approximate analytical solutions for linear and nonlinear, space and time fractional Fokker-Planck equations within the fractional Caputo derivatives. The graphical representation reveals a close relationship between approximate solutions and the exact solution. Furthermore, the proposed approach needed fewer calculations and can be applied to solve other problems of fractional order.

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