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On intuitionistic fuzzy quasi ideals of rings with respect to a t-norm

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Abstract

In this paper, the notion of fuzzy quasi ideal is extended to intuitionistic fuzzy quasi ideal with respect to a triangular norm and various results of such intuitionistic fuzzy ideals are studied. It has also been established that if an intuitionistic fuzzy subring with respect to a triangular norm is an intuitionistic fuzzy quasi ideal with respect to that norm, then it is also intuitionistic fuzzy quasi ideal with respect to the annihilation of that triangular norm.

Keywords

Intuitionistic fuzzy set, Intuitionistic fuzzy subring, Intuitionistic fuzzy quasi ideal, t-norm.

AMS Subject Classification

08A72, 16D25.

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Contents

1. Introduction

Fuzzy set was initiated by Zadeh [18] in 1965 and since that time various algebraic structures like groups, rings, modules have been characterized by many researchers in fuzzy setting. Rosenfeld [16] defined fuzzy subgroup of a group. Around 1982, W.Liu [10] introduced the concept of fuzzy ideals of ring. As these algebraic structures play very important role in mathematics and in many other branches of science, so these concepts have been generalized by many researchers. The intuitionistic fuzzy set, which is a generalization of a fuzzy set was introduced by K.T.Atanassov [2, 3] is one of them. In 2003, Banerjee and Basnet [4] studied intuitionistic fuzzy subring and intuitionistic fuzzy ideals. Rahman and Saikia [14] studied intuitionistic fuzzy sub-modules with respect to triangular norm. Recently, in 2019 intuitionistic fuzzy subrings in near rings with respect to a triangular norm and t- co-norm is studied by Murugadas and Vetrivel [12]. The notion of quasi ideals for rings was first introduced by Steinfeld [17]. Mohanraaj and Dheena [5] studied fuzzy quasi ideals of rings in terms of a triangular norm. In this present study, intuitionistic fuzzy quasi ideal is defined with respect to a triangular norm and a t co-norm and few results on such intuitionistic fuzzy ideals are investigated.

2. Preliminaries

Through the whole paper R indicates a non-commutative ring with unity, T indicates a 't-norm' and S_T indicates a 't co-norm' if not otherwise specified.

Definition 2.1. *An arbitrary mapping $\mu : X \to [0,1]$ is said to be a fuzzy subset of X.*

Definition 2.2. [2] *For a non-empty set X, by an intuitionistic fuzzy set(abbreviated as IFS) we mean a structure of the form*

$$A = \{(r, \mu_A(r), \nu_A(r) | r \in X)\}$$

where μ_A and ν_A are mappings from X to $[0,1]$ i.e. fuzzy sets in X and denote the membership degree (viz. $\mu_A(r)$) and non-membership degree (viz. $\nu_A(r)$), $\forall r \in X$ to the set A respectively together with the condition $0 \leq \mu_A(r) + \nu_A(r) \leq 1, \forall r \in X$. The set of all IFSs of X is denoted by IFS(X). We denote $A = \{(r, \mu_A(r), \nu_A(r) | r \in X)\}$ simply by $A = (\mu_A, \nu_A)$.

Definition 2.3. [2] *Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two IFSs of X. Then* *(i)* $A \subseteq B$ *iff* $\mu_A(x^*) \leq \mu_B(x^*)$ *and* $\nu_A(x^*) \geq \nu_B(x^*)$ *for all* $x^* \in X$.

(ii) $A = B$ *iff* $\mu_A(x^*) = \mu_B(x^*)$ *and* $\nu_A(x^*) = \nu_B(x^*)$ *for all* $x^* \in X$

(iii) $A \cap B = \{(x^*, \mu_A(x^*) \wedge \mu_B(x^*), \nu_A(x^*) \vee \nu_B(x^*)) | x^* \in X\}$

(iv) $A \wedge B = \{(x^*, T(\mu_A(x^*), \mu_B(x^*)), S(\nu_A(x^*), \nu_B(x^*))) | x^* \in X\}$

(v) $A \cup B = \{(x^*, \mu_A(x^*) \vee \mu_B(x^*), \nu_A(x^*) \wedge \nu_B(x^*)) | x^* \in X\}$

(vi) $\Box A = \{(x^*, \mu_A(x^*), \mu_A^c(x^*)) | x^* \in X\}$

(vii) $\diamond A = \{(x^*, \nu_A^c(x^*), \nu_A(x^*)) | x^* \in X\}$

Definition 2.4. [7] *An arbitrary mapping* T *from* $[0,1] \times [0,1]$ *to* $[0,1]$ *is said to be a t-norm (or triangular norm) if* $\forall x^*, y^*, z^* \in [0,1]$ *the following postulates are satisfied:*
$T_1)T(x^*,1) = T(1,x^*) = x^*$
$T_2)$*If* $y^* \leq z^*$ *then* $T(x^*,y^*) \leq T(x^*,z^*)$
$T_3)T(x^*,y^*) = T(y^*,x^*)$
$T_4)T(x^*,T(y^*,z^*)) = T(T(x^*,y^*),z^*)$.

Definition 2.5. [7] *An arbitrary mapping* S *from* $[0,1] \times [0,1]$ *to* $[0,1]$ *is said to be a fuzzy union (or t co-norm) if* $\forall x^*, y^*, z^* \in [0,1]$ *the following postulates are satisfied:*
$S_1)S(x^*,1) = S(1,x^*) = x^*$
$S_2)$*If* $y^* \leq z^*$ *then* $S(x^*,y^*) \leq S(x^*,z^*)$
$S_3)S(x^*,y^*) = S(y^*,x^*)$
$S_4)S(x^*,S(y^*,z^*)) = S(S(x^*,y^*),z^*)$.

The pair (T, S_T) axe called dual in respect of fuzzy complement if
(i) $T(x^*,y^*) = 1 - S_T(1-x^*, 1-y^*)$
(ii) $S_T(x^*,y^*) = 1 - T(1-x^*, 1-y^*)$, fox all $x^*, y^* \in [0,1]$.

Definition 2.6. T_c*, the C-annihihation of* T *is defined as:*
$T_{(c)} : [0,1] \times [0,1] \to [0,1]$

$$T_{(c)}(x^*,y^*) = \begin{cases} 0; & if\ x^* \leq 1 - y^* \\ T(x^*,y^*); & otherwise \end{cases}$$

fox all $x^*, y^* \in [0,1]$

Definition 2.7. [4] *For* $A, B \in IFS(X)$*, their sum* $A + B$ *is defined as*

$$\mu_{A+B}(p) = \begin{cases} \sup_{p=q+r} \{\mu_A(q) \wedge \mu_B(r)\} \\ 0; \quad else \end{cases} \quad and$$

$$\nu_{A+B}(p) = \begin{cases} \inf_{p=q+r} \{\nu_A(q) \vee \nu_B(r)\} \\ 1; \quad else \end{cases}$$

Definition 2.8. [4] *For* $A, B \in IFS(X)$*, their product* AB *is defined as*

$$\mu_{AB}(p) = \begin{cases} \sup_{p=qr} \{\mu_A(q) \wedge \mu_B(r)\} \\ 0; \quad else \end{cases} \quad and$$

$$\nu_{AB}(p) = \begin{cases} \inf_{p=qr} \{\nu_A(q) \vee \nu_B(r)\} \\ 1; \quad else \end{cases}$$

Definition 2.9. [4] *Let* A *be a subset of a non-empty set* X*. Then an IFS* $\chi_A = (\mu_{\chi_A}, \nu_{\chi_A})$ *is called an intuitionistic fuzzy characteristic function and is defined as*

$$\mu_{\chi_A}(x^*) = \begin{cases} 1; \ if\ x^* \in A \\ 0; \ if\ x^* \notin A \end{cases} \quad and\ \nu_{\chi_A}(x^*) = \begin{cases} 0; \ if\ x^* \in A \\ 1; \ if\ x^* \notin A \end{cases}$$

Definition 2.10. [4] *Let* R *be be a ring. Then* χ_0 *and* χ_R *are IFSs on* R *defined by*

$$\chi_0(r) = (\mu_{\chi_0}(r), \nu_{\chi_0}(r)) \ and \quad \chi_R(r) = (\mu_{\chi_R}(r), \nu_{\chi_R}(r)),$$

where

$$\mu_{\chi_0}(r) = \begin{cases} 1; \ when\ r = 0 \\ 0; \ when\ r \neq 0 \end{cases} ; \nu_{\chi_0}(r) = \begin{cases} 0; \ when\ r = 0 \\ 1; \ when\ r \neq 0 \end{cases}$$

and $\mu_{\chi_R}(r) = 1$ *;* $\nu_{\chi_R}(r) = 0 \ \forall r \in R$

Definition 2.11. [13] *An operator* T_m *in terms of* T *is defined as*
$T_m(x_1^*, x_2^*, \ldots, x_m^*) = T(x_j^*, T_{m-1}(x_1^*, x_2^*, \ldots, x_{j-1}^*, x_{j+1}^*, \ldots, x_m^*))$
for all $1 \leq j \leq m, x_j^* \in [0,1], m \geq 3, T_2 = T$.
Again T_∞ *is defined as,* $T_\infty(x_1^*, x_2^*, \ldots) = \lim_{m\to\infty} T_m(x_1^*, x_2^*, \ldots, x_m^*)$

Definition 2.12. [14] *An operator* S_m *in terms of* S *is defined as*
$S_m(x_1^*, x_2^*, \ldots, x_m^*) = S(x_j^*, S_{m-1}(x_1^*, x_2^*, \ldots, x_{j-1}^*, x_{j+1}^*, \ldots, x_m^*))$
for all $1 \leq j \leq m, x_j^* \in [0,1], m \geq 3, S_2 = S$.
Again S_∞ *is defined as,* $S_\infty(x_1^*, x_2^*, \ldots) = \lim_{m\to\infty} T_m(x_1^*, x_2^*, \ldots, x_m^*)$

Definition 2.13. [13] *The intersection* $\mu_1 \cap \mu_2 \cap \ldots \cap \mu_k$ *with respect to (in brief wrt)* T *of the collection* $\{\mu_1, \mu_2, \ldots, \mu_k\}$ *of fuzzy subsets in* X *is defined as,* $(\mu_1 \cap \mu_2 \cap \ldots \cap \mu_k)(y) = T_k(\mu_1(y), \mu_2(y), \ldots, \mu_k(y))$ *for all* $y \in X$*. Again the fuzzy set* $\bigcap_T \mu_j(y) = T_\infty(\mu_1(y), \mu_2(y), \ldots)$ *means the intersection of the collection* $\{\mu_1, \mu_2, \ldots\}$ *of fuzzy sets in* X *wrt* T.

Definition 2.14. [14] *The union* $\mu_1 \cup \mu_2 \cup \ldots \cup \mu_k$ *wrt* S *of the collection* $\{\mu_1, \mu_2, \ldots, \mu_k\}$ *of fuzzy subsets in* X *is defined as,* $(\mu_1 \cup \mu_2 \cup \ldots \cup \mu_k)(y) = S_k(\mu_1(y), \mu_2(y), \ldots, \mu_k(y))$ *fox all* $y \in X$*. Again the fuzzy set* $\bigcup_S \mu_j(y) = S_\infty(\mu_1(y), \mu_2(y), \ldots)$ *means the union of the collection* $\{\mu_1, \mu_2, \ldots\}$ *of fuzzy sets in* X *wrt* S.

Definition 2.15. [14] *For the collection* $\{A_1, A_2, \ldots, A_m\}$ *here* $A_k = (\mu_k, \nu_k), k = 1, 2, \ldots, m$ *of IFSs of* X *their intersection* $A_1 \cap A_2 \cap \ldots \cap A_m$ *wrt* T *is an IFS of* X*, defined by*

$$A_1 \cap A_2 \cap \ldots \cap A_m = \{(y, T_m(\mu_1(y), \mu_2(y), \ldots, \mu_m(y)), S_{T_m}(\nu_1(y), \nu_2(y), \ldots, \nu_m(y))) | y \in X\}$$

Again the IFS,

$$\bigcap_T A_k(y) = \{(y, T_\infty(A_1(y), A_2(y), \ldots), S_{T_\infty}(A_1(y), A_2(y), \ldots)) | y \in X\}$$

means the intersection of the collection $\{A_1, A_2, \ldots\}$ *of IFSs of* X *wrt* T.

Definition 2.16. [14] *For the collection* $\{A_1, A_2, \ldots, A_m\}$ *here* $A_k = (\mu_k, \nu_k), k = 1, 2, \ldots, m$ *of IFSs of* X *their union* $A_1 \cup A_2 \cup \ldots \cup A_m$ *wrt* T *is an IFS of* X*, defined by*

$$A_1 \cup A_2 \cup \ldots \cup A_m = \{(y, S_{T_m}(\mu_1(y), \mu_2(y), \ldots, \mu_m(y)), T_m(\nu_1(y), \nu_2(y), \ldots, \nu_m(y)))|y \in X\}$$

Again the IFS,

$$\bigcup_T A_k(y) = \{(y, S_{T_\infty}(A_1(y), A_2(y), \ldots), T_\infty(A_1(y), A_2(y), \ldots))|y \in X\}$$

means the union of the collection $\{A_1, A_2, \ldots\}$ *of IFSs of* X *wrt* T.

Definition 2.17. [15] $A \in IFS(R)$ *is called an intuitionistic fuzzy subring (in short IFSR) of* R *wrt* T *if* $\forall r, s \in R$ *it satisfies*

(i) $\mu_A(r-s) \geq T(\mu_A(r), \mu_A(s))$ *and* $\nu_A(r-s) \leq S_T(\nu_A(r), \nu_A(s))$.

(ii) $\mu_A(rs) \geq T(\mu_A(r), \mu_A(s))$ *and* $\nu_A(rs) \leq S_T(\nu_A(r), \nu_A(s))$.

Lemma 2.18. $A \in IFS(R)$ *is an IFSR of* R *wrt* T *iff* $\Box A$ *and* $\diamond A$ *are IFSRs of* R *wrt* T.

Proof. Suppose A is an IFSR of R wrt T. Let $r, s \in R$. Then
$\mu_A^c(r-s) = 1 - \mu_A(r-s)$
$\leq 1 - T(\mu_A(r), \mu_A(s)) = S_T(1-\mu_A(r), 1-\mu_A(s))$.
Thus $\mu_A^c(r-s) \leq S_T(\mu_A^c(r), \mu_A^c(s))$.
Also, $\mu_A^c(rs) = 1 - \mu_A(rs)$
$\leq 1 - T(\mu_A(r), \mu_A(s)) = S_T(1-\mu_A(r), 1-\mu_A(s)) = S_T(\mu_A^c(r), \mu_A^c(s))$. The other two axioms can be found from the hypothesis that A is an IFSR of R wrt T. Hence $\Box A$ is an IFSR of R wrt T. Also,
$\nu_A^c(r-s) = 1 - \nu_A(r-s)$
$\geq 1 - S_T(\nu_A(r), \nu_A(s)) = T(1-\nu_A(r), 1-\nu_A(s))$.
Therefore $\nu_A^c(x-s) \geq T(\nu_A^c(x), \nu_A^c(s))$. Again, $\nu_A^c(rs) = 1 - \nu_A(rs)$
$\geq 1 - T(\nu_A(r), \nu_A(s)) = S_T(1-\nu_A(r), 1-\nu_A(s)) = S_T(\nu_A^c(r), \nu_A^c(s))$. The other two axioms can be found from the hypothesis that A is an IFSR of R wrt T. Therefore $\diamond A$ is an IFSR of R wrt T. The reverse part of the lemma follows directly. □

Lemma 2.19. *Suppose C-annihilation* $T_{(c)}$ *of* T *provides a t-norm. If* A *is an IFSR of* R *wrt* T*, then* $\Box A$ *and* $\diamond A$ *are IFSRs of* R *wrt* $T_{(c)}$.

Proof. Let the C-annihilation of T be $T_{(c)}$. Let us define $S_{T_{(c)}}$ as

$$S_{T_{(c)}}(x^*, y^*) = \begin{cases} 1; & if \ 1-x^* \leq y^* \\ 1 - T(1-x^*, 1-y^*); & otherwise \end{cases}$$

fox all $x^*, y^* \in [0,1]$. Then $S_{T_{(c)}}$ is the dual of $T(c)$.
Since A is an IFSR of R wrt T, so we get
$\mu_A(r-s) \geq T(\mu_A(r), \mu_A(s)) \geq T_{(c)}(\mu_A(r), \mu_A(s))$ and
$\mu_A(rs) \geq T(\mu_A(r), \mu_A(s)) \geq T_{(c)}(\mu_A(r), \mu_A(s))$.

We have

$$S_{T_{(c)}}(\mu_A^c(r), \mu_A^c(s)) = \begin{cases} 1; & if \ \mu_A(r) \leq 1 - \mu_A(s) \\ 1 - T(1-\mu_A^c(r), 1-\mu_A^c(s)); & else \end{cases}$$

Now, $S_{T_{(c)}}(\mu_A^c(r), \mu_A^c(s))$
$\geq 1 - T(1-\mu_A^c(r), 1-\mu_A^c(s))$
$= 1 - T(\mu_A(r), \mu_A(s)) = 1 - \mu_A(r-s)$
[since $\mu_A(r-s) \geq T(\mu_A(r), \mu_A(s))$].
Thus we have $\mu_A^c(r-s) \leq S_{T_{(c)}}(\mu_A^c(r), \mu_A^c(s))$. Similarly it can be shown that $\mu_A^c(rs) \leq S_{T_{(c)}}(\mu_A^c(r), \mu_A^c(s))$. Hence $\Box A$ is an IFSR of R wrt $T_{(c)}$.
For the next part we have,

$$T_{(c)}(\nu_A^c(r), \nu_A^c(s)) = \begin{cases} 0; & if \ \nu_A^c(r) \leq 1 - \nu_A^c(s) \\ T(\nu_A^c(r), \nu_A^c(s)); & otherwise \end{cases}$$

This implies, $T_{(c)}(\nu_A^c(r), \nu_A^c(s))$
$\leq T(\nu_A^c(r), \nu_A^c(s))$
$= 1 - S_T(\nu_A(r), \nu_A(s)) \leq 1 - \nu_A(r-s)$.
Thus we have $\nu_A^c(r-s) \geq T_{(c)}(\nu_A^c(r), \nu_A^c(s))$. Similarly it can be shown that $\nu_A^c(rs) \geq T_{(c)}(\nu_A^c(r), \nu_A^c(s))$.
Since A is an IFSR of R wrt T, therefore,
$\nu_A(r-s) \leq S_T(\nu_A(r), \nu_A(s)) \leq S_T(c)(\nu_A(r), \nu_A(s))$ and
$\nu_A(rs) \leq S_T(\nu_A(r), \nu_A(s)) \leq S_T(c)(\nu_A(r), \nu_A(s))$. Hence $\diamond A$ is an IFSR of R wrt $T_{(c)}$. □

3. Intuitionistic Fuzzy Quasi Ideal wrt a t-norm

Definition 3.1. $A \in IFS(R)$ *is called a intuitionistic fuzzy quasi ideal (in short IFQI) if for all* $r, s \in R$ *it satisfies*

(i) $\mu_A(r-s) \geq \min(\mu_A(r), \mu_A(s))$ *and* $\nu_A(r-s) \leq \max(\nu_A(r)\nu_A(s))$.

(ii) $(A.\chi_R) \cap (\chi_R.A) \subseteq A$

Definition 3.2. $A \in IFS(R)$ *is called a intuitionistic fuzzy quasi ideal (in short IFQI) wrt* T *if for all* $r, s \in R$ *it satisfies*

(i) $\mu_A(r-s) \geq T(\mu_A(r), \mu_A(s))$ *and* $\nu_A(r-s) \leq S_T(\nu_A(r), \nu_A(s))$.

(ii) $(A.\chi_R) \wedge (\chi_R.A) \subseteq A$

Example 3.3. *Let us consider* $R = \mathbf{Z_4} = \{0,1,2,3\}$ *under addition and multiplication modulo 4. Now we define fuzzy subsets* μ_A *and* ν_A *on* R *as follows:*
$\mu_A(0) = 0.3, \mu_A(1) = 0.25, \mu_A(2) = 0.2, \mu_A(3) = 0.1$ *and*
$\nu_A(0) = 0.6, \nu_A(1) = 0.65, \nu_A(2) = 0.7, \nu_A(3) = 0.8$
Then $A = (\mu_A, \nu_A)$ *is an IFQI of* R *wrt the pair of triangular norms and conorms given below:*
$(i) x^*y^*, x^*+y^*-x^*y^*, (ii) \max(0, x^*+y^*-1), \min(1, x^*+y^*)$.
But $A = (\mu_A, \nu_A)$ *is not an IFQI of* R*, as*
$\mu_A(0-1) = \mu_A(3) = 0.1 \ngeq 0.25 = \min(\mu_A(0), \mu_A(1))$

Theorem 3.4. *An IFSR A of R wrt T is an IFQI wrt T iff* $\mu_A(x) \geq T\big[\sup_{x=yz} \mu_A(y), \sup_{x=yz} \mu_A(z)\big]$ *and* $\nu_A(x) \leq S_T\big[\inf_{x=yz} \nu_A(y), \inf_{x=yz} \nu_A(z)\big]$, *for all* $x \in R$.

Proof. Let A be an IFSR of R wrt T. First let us assume that A be IFQI of R wrt T and $x \in R$. Then $(A.\chi_R) \wedge (\chi_R.A) \subseteq A$, implies,

$$\mu_A(x) \geq T(\mu_{A.\chi_R}(x), \mu_{\chi_R.A}(x)) = T\big[\sup_{x=yz} T(\mu_A(y), \mu_{\chi_R}(z)), \sup_{x=yz} T(\mu_{\chi_R}(y), \mu_A(z))\big] = T\big[\sup_{x=yz} \mu_A(y), \sup_{x=yz} \mu_A(z)\big]$$

and

$$\nu_A(x) \leq S_T(\nu_{A.\chi_R}(x), \nu_{\chi_R.A}(x)) = S_T\big[\inf_{x=yz} S(\nu_A(y), \nu_{\chi_R}(z)), \inf_{x=yz} S_T(\nu_{\chi_R}(y), \nu_A(z))\big] = S_T\big[\inf_{x=yz} \nu_A(y), \inf_{x=yz} \nu_A(z)\big].$$

Conversely, let A satisfies the conditions, $\mu_A(x) \geq T\big[\sup_{x=yz} \mu_A(y), \sup_{x=yz} \mu_A(z)\big]$ and $\nu_A(x) \leq S_T\big[\inf_{x=yz} \nu_A(y), \inf_{x=yz} \nu_A(z)\big]$, for all $x \in R$. By our assumption we have,

$$\mu_A(x) \geq T\big[\sup_{x=yz} \mu_A(y), \sup_{x=yz} \mu_A(z)\big] = T\big[\sup_{x=yz} T(\mu_A(y), \mu_{\chi_R}(z)), \sup_{x=yz} T(\mu_{\chi_R}(y), \mu_A(z))\big] = T(\mu_{A.\chi_R}(x), \mu_{\chi_R.A}(x)) = \mu_{(A.\chi_R)\wedge(\chi_R.A)}(x).$$

Also we have,

$$\nu_A(x) \leq S_T\big[\inf_{x=yz} \nu_A(y), \inf_{x=yz} \nu_A(z)\big] = S_T\big[\inf_{x=yz} S(\nu_A(y), \nu_{\chi_R}(z)), \inf_{x=yz} S_T(\nu_{\chi_R}(y), \nu_A(z))\big] = S_T(\nu_{A.\chi_R}(x), \nu_{\chi_R.A}(x)) = \nu_{(A.\chi_R)\wedge(\chi_R.A)}(x).$$

Therefore $(A.\chi_R) \wedge (\chi_R.A) \subseteq A$. The other two axioms can be found from the hypothesis that A is an IFSR of R wrt T. Hence A is an IFQI of R wrt T. ☐

Theorem 3.5. *An IFSR A of R wrt T is an IFQI wrt T iff* $\Box A$ *and* $\diamond A$ *are IFQIs of R wrt T.*

Proof. Suppose A is an IFSR of R wrt T. Then by lemma (2.18) $\Box A$ and $\diamond A$ are IFSRs of R wrt T. First let us assume that A be IFQI of R wrt T. Then by theorem (3.4), we have $\mu_A(x) \geq T\big[\sup_{x=yz} \mu_A(y), \sup_{x=yz} \mu_A(z)\big] \ldots\ldots (1)$

Now, $S_T\big[\inf_{x=yz} \mu_A^c(y), \inf_{x=yz} \mu_A^c(z)\big]$
$= 1 - T\big[1 - \inf_{x=yz} \mu_A^c(y), 1 - \inf_{x=yz} \mu_A^c(z)\big]$
$= 1 - T\big[\sup_{x=yz} \mu_A(y), \sup_{x=yz} \mu_A(z)\big]$
$\geq 1 - \mu_A(x)$ [Using (1)].

Therefore, $\mu_A^c(x) \leq S_T\big[\inf_{x=yz} \mu_A^c(y), \inf_{x=yz} \mu_A^c(z)\big] \ldots\ldots (2)$. Thus from (1) and (2) by theorem (3.4), we have $\Box A = (\mu_A, \mu_A^c)$ is an IFQI of R wrt T.

Also, since A is an IFQI of R wrt T, so by theorem (3.4), we have $\nu_A(x) \leq S_T\big[\inf_{x=yz} \nu_A(y), \inf_{x=yz} \nu_A(z)\big] \ldots\ldots (3)$

Again, $T\big[\sup_{x=yz} \nu_A^c(y), \sup_{x=yz} \nu_A^c(z)\big]$
$= 1 - S_T\big[1 - \sup_{x=yz} \nu_A^c(y), 1 - \sup_{x=yz} \nu_A^c(z)\big]$
$= 1 - S_T\big[\sup_{x=yz} \nu_A(y), \sup_{x=yz} \nu_A(z)\big]$
$\leq 1 - \nu_A(x)$ [Using (3)].

Therefore, $\nu_A^c(x) \geq T\big[\sup_{x=yz} \nu_A^c(y), \sup_{x=yz} \nu_A^c(z)\big] \ldots\ldots (4)$. Thus from (3) and (4) by theorem (3.4), we have $\diamond A = (\nu_A^c, \nu_A)$ is an IFQI of R wrt T.

The reverse part of the theorem obviously follows from theorem (3.4). ☐

Theorem 3.6. *Suppose* $T_{(c)}$, *C-annihilation T provides a t-norm. If an IFSR A of R wrt T is an IFQI, then* $\Box A$ *and* $\diamond A$ *are IFQIs of R wrt* $T_{(c)}$.

Proof. Let A be an IFSR of R wrt T such that it is an IFQI. Then by lemma (2.19) $\Box A$ and $\diamond A$ are IFSRs of R wrt $T_{(c)}$. Since A is an IFQI of R wrt T, so for all $x \in R$ by theorem (3.4), we get

$$\mu_A(x) \geq T\big[\sup_{x=yz} \mu_A(y), \sup_{x=yz} \mu_A(z)\big] \geq T_{(c)}\big[\sup_{x=yz} \mu_A(y), \sup_{x=yz} \mu_A(z)\big] \quad (3.1)$$

and

$$\nu_A(x) \leq S_T\big[\inf_{x=yz} \nu_A(y), \inf_{x=yz} \nu_A(z)\big] \leq S_{T_{(c)}}\big[\inf_{x=yz} \nu_A(y), \inf_{x=yz} \nu_A(z)\big] \quad (3.2)$$

Now, $S_{T_{(c)}}\big[\inf_{x=yz} \mu_A^c(y), \inf_{x=yz} \mu_A^c(z)\big]$
$= 1 - T_{(c)}\big[1 - \inf_{x=yz} \mu_A^c(y), 1 - \inf_{x=yz} \mu_A^c(z)\big]$
$= 1 - T_{(c)}\big[\sup_{x=yz} \mu_A(y), \sup_{x=yz} \mu_A(z)\big]$
$\geq 1 - \mu_A(x)$ [Using (1)].

Therefore, $\mu_A^c(x) \leq S_{T_{(c)}}\big[\inf_{x=yz} \mu_A^c(y), \inf_{x=yz} \mu_A^c(z)\big] \ldots\ldots (3)$.

Thus from (1) and (3) by theorem (3.4), we have $\Box A = (\mu_A, \mu_A^c)$ is an IFQI of R wrt $T_{(c)}$.

Again, $T_{(c)}\big[\sup_{x=yz} \nu_A^c(y), \sup_{x=yz} \nu_A^c(z)\big]$
$= 1 - S_{T_{(c)}}\big[1 - \sup_{x=yz} \nu_A^c(y), 1 - \sup_{x=yz} \nu_A^c(z)\big]$
$= 1 - S_{T_{(c)}}\big[\sup_{x=yz} \nu_A(y), \sup_{x=yz} \nu_A(z)\big]$
$\leq 1 - \nu_A(x)$ [Using (2)].

Therefore, $\nu_A^c(x) \geq T_{(c)}\big[\sup_{x=yz} \nu_A^c(y), \sup_{x=yz} \nu_A^c(z)\big] \ldots\ldots (4)$. Thus from (2) and (4) by theorem (3.4), we have $\diamond A = (\nu_A^c, \nu_A)$ is an IFQI of R wrt $T_{(c)}$.

☐

Theorem 3.7. *Let* $\{A_1, A_2, \ldots, A_m\}$ *where* $A_k = (\mu_k, \nu_k), k = 1, 2, \ldots, m$ *be m IFQIs of R wrt T. Then* $A_1 \cap A_2 \cap \ldots \cap A_m$ *is also an IFQI of R wrt T.*

Proof. Let $A = A_1 \cap A_2 \cap \ldots \cap A_m$. We will use the induction method to prove that A is an IFQI of R wrt T.

If $m = 1$, then $A = A_1$ and therefore A is an IFQI of R wrt T.

We suppose, the intersection of $(m-1)$ IFQIs of R wrt T is again an IFQI of R wrt T. By our assumption $A_2 \cap A_3 \cap \ldots \cap A_m$ is an IFQI of R wrt T. Let $x, y \in R$. Then

$(\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(x-y)$
$= T_m(\mu_1(x-y), \mu_2(x-y), \ldots, \mu_m(x-y))$
$= T(\mu_1(x-y), T_{m-1}(\mu_2(x-y), \ldots, \mu_m(x-y)))$
$\geq T(T(\mu_1(x)\mu_1(y)), T(T_{m-1}(\mu_2(x), \ldots, \mu_m(x)), T_{m-1}(\mu_2(y), \ldots, \mu_m(y)))$
(since $A_2 \cap A_3 \cap \ldots \cap A_m$ is an IFQI of R)
$= T(T(\mu_1(y)\mu_1(x)), T(T_{m-1}(\mu_2(x), \ldots, \mu_m(x)), T_{m-1}(\mu_2(y), \ldots, \mu_m(y)))$
$= T(\mu_1(y), T(T(\mu_1(x), T_{m-1}(\mu_2(x), \ldots, \mu_m(x))), T_{m-1}(\mu_2(y), \ldots, \mu_m(y))))$
$= T(\mu_1(y), T(T_m(\mu_1(x), \mu_2(x), \ldots, \mu_m(x)), T_{m-1}(\mu_2(y), \ldots, \mu_m(y))))$
$= T(\mu_1(y), T(T_{m-1}(\mu_2(y), \ldots, \mu_m(y))), T_m(\mu_1(x), \mu_2(x), \ldots, \mu_m(x)))$
$= T(T_m(\mu_1(y), \mu_2(y), \ldots, \mu_m(y)), T_m(\mu_1(x), \mu_2(x), \ldots, \mu_m(x)))$
$= T((\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(x), (\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(y))$

Thus $(\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(x-y) \geq T((\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(x), (\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(y))$

Let (T, S_T) be the dual pair wrt fuzzy complement. Then

$(\nu_1 \cup \nu_2 \cup \ldots \cup \nu_m)(x-y)$
$= S_{T_m}(\nu_1(x-y), \nu_2(x-y), \ldots, \nu_m(x-y))$
$= S_T(\nu_1(x-y), S_{T_{m-1}}(\nu_2(x-y), \ldots, \nu_m(x-y)))$
$\leq S_T(S_T(\nu_1(x)\nu_1(y)), S_T(S_{T_{m-1}}(\nu_2(x), \ldots, \nu_m(x)), S_{T_{m-1}}(\nu_2(y), \ldots, \nu_m(y)))$
(since $A_2 \cap A_3 \cap \ldots \cap A_m$ is an IFQI of R)
$= S_T(S_T(\nu_1(y)\nu_1(x)), S_T(S_{T_{m-1}}(\nu_2(x), \ldots, \nu_m(x)), S_{T_{m-1}}(\nu_2(y), \ldots, \nu_m(y)))$
$= S_T(\nu_1(y), S_T(S_T(\nu_1(x), S_{T_{m-1}}(\nu_2(x), \ldots, \nu_m(x))), S_{T_{m-1}}(\nu_2(y), \ldots, \nu_m(y))))$
$= S_T(\nu_1(y), S_T(S_{T_m}(\nu_1(x), \nu_2(x), \ldots, \nu_m(x)), S_{T_{m-1}}(\nu_2(y), \ldots, \nu_m(y))))$
$= S_T(\nu_1(y), S_T(S_{T_{m-1}}(\nu_2(y), \ldots, \nu_m(y))), S_{T_m}(\nu_1(x), \nu_2(x), \ldots, \nu_m(x)))$
$= S_T(S_{T_m}(\nu_1(y), \nu_2(y), \ldots, \nu_m(y)), S_{T_m}(\nu_1(x), \nu_2(x), \ldots, \nu_m(x)))$
$= S_T((\nu_1 \cup \nu_2 \cup \ldots \cup \nu_m)(x), (\nu_1 \cup \nu_2 \cup \ldots \cup \nu_m)(y))$

Thus $(\nu_1 \cup \nu_2 \cup \ldots \cup \nu_m)(x-y) \leq S_T((\nu_1 \cup \nu_2 \cup \ldots \cup \nu_m)(x), (\nu_1 \cup \nu_2 \cup \ldots \cup \nu_m)(y))$

Again, $(\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(x)$
$= T_m(\mu_1(x), \mu_2(x), \ldots, \mu_m(x))$
$= T(\mu_1(x), T_{m-1}(\mu_2(x), \mu_3(x), \ldots, \mu_m(x)))$
$\geq T\left[T(\sup_{x=yz} \mu_1(y), \sup_{x=yz} \mu_1(z)), T(\sup_{x=yz} T_{m-1}(\mu_2(y), \ldots, \mu_m(y)), \sup_{x=yz} T_{m-1}(\mu_2(z), \ldots, \mu_m(z)))\right]$
(since $A_2 \cap A_3 \cap \ldots \cap A_m$ is an IFQI of R)
$= T\left[T(\sup_{x=yz} \mu_1(z), \sup_{x=yz} \mu_1(y)), T(\sup_{x=yz} T_{m-1}(\mu_2(y), \ldots, \mu_m(y)), \sup_{x=yz} T_{m-1}(\mu_2(z), \ldots, \mu_m(z)))\right]$
$= T\left[T(\sup_{x=yz} \mu_1(z), T(\sup_{x=yz} \mu_1(y), \sup_{x=yz} T_{m-1}(\mu_2(y), \ldots, \mu_m(y)))), \sup_{x=yz} T_{m-1}(\mu_2(z), \ldots, \mu_m(z)))\right]$
$\geq T\left[T(\sup_{x=yz} \mu_1(z), \sup_{x=yz} T(\mu_1(y), T_{m-1}(\mu_2(y), \ldots, \mu_m(y)))), \sup_{x=yz} T_{m-1}(\mu_2(z), \ldots, \mu_m(z)))\right]$
$= T\left[T(\sup_{x=yz} \mu_1(z), \sup_{x=yz} T_m(\mu_1(y), \mu_2(y), \ldots, \mu_m(y)), \sup_{x=yz} T_{m-1}(\mu_2(z), \ldots, \mu_m(z)))\right]$
$= T\left[T(\sup_{x=yz} \mu_1(z), \sup_{x=yz} T_{m-1}(\mu_2(z), \ldots, \mu_m(z))), \sup_{x=yz} T_m(\mu_1(y), \mu_2(y), \ldots, \mu_m(y))\right]$
$\geq T\left[\sup_{x=yz} T(\mu_1(z), T_{m-1}(\mu_2(z), \ldots, \mu_m(z))), \sup_{x=yz} T_m(\mu_1(y), \mu_2(y), \ldots, \mu_m(y))\right]$
$= T\left[\sup_{x=yz} T_m(\mu_1(z), \mu_2(z), \ldots, \mu_m(z))), \sup_{x=yz} T_m(\mu_1(y), \mu_2(y), \ldots, \mu_m(y))\right]$
$= T\left[\sup_{x=yz} (\mu_1 \cap \mu_2 \cap \ldots, \cap \mu_m)(y), \sup_{x=yz} (\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(z)\right]$
$= T\left[\sup_{x=yz} T((\mu_1 \cap \mu_2 \cap \ldots, \cap \mu_m)(y), \mu_{\chi_R}(z)), \sup_{x=yz} T(\mu_{\chi_R}(y), (\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(z))\right]$

Thus $(\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m)(x) \geq T[((\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m).\mu_{\chi_R})(x), (\mu_{\chi_R}.(\mu_1 \cap \mu_2 \cap \ldots \cap \mu_m))(x)]$

Also, $(\nu_1 \cup \nu_2 \cup \ldots \cup \nu_m)(x)$
$= S_{T_m}(\nu_1(x), \nu_2(x), \ldots, \nu_m(x))$
$= S_T(\nu_1(x), S_{T_{m-1}}(\nu_2(x), \nu_3(x), \ldots, \nu_m(x)))$
$\leq S_T\left[S_T(\inf_{x=yz} \nu_1(y), \inf_{x=yz} \nu_1(z)), S_T(\inf_{x=yz} S_{T_{m-1}}(\nu_2(y), \ldots, \nu_m(y)), \inf_{x=yz} S_{T_{m-1}}(\nu_2(z), \ldots, \nu_m(z)))\right]$
(since $A_2 \cup A_3 \cup \ldots \cup A_n$ is an IFQI of R)
$= S_T\left[S_T(\inf_{x=yz} \nu_1(z), \inf_{x=yz} \nu_1(y)), S_T(\inf_{x=yz} S_{T_{m-1}}(\nu_2(y), \ldots, \nu_m(y)), \inf_{x=yz} S_{T_{m-1}}(\nu_2(z), \ldots, \nu_m(z)))\right]$
$= S_T\left[S_T(\inf_{x=yz} \nu_1(z), S_T(\inf_{x=yz} \nu_1(y), \inf_{x=yz} S_{T_{m-1}}(\nu_2(y), \ldots, \nu_m(y)))), \inf_{x=yz} S_{T_{m-1}}(\nu_2(z), \ldots, \nu_m(z)))\right]$
$\leq S_T\left[S_T(\inf_{x=yz} \nu_1(z), \inf_{x=yz} S_T(\nu_1(y), S_{T_{m-1}}(\nu_2(y), \ldots, \nu_m(y)))), \inf_{x=yz} S_{T_{m-1}}(\nu_2(z), \ldots, \nu_m(z)))\right]$
$= S_T\left[S_T(\inf_{x=yz} \nu_1(z), \inf_{x=yz} S_{T_m}(\nu_1(y), \nu_2(y), \ldots, \nu_m(y)), \inf_{x=yz} S_{T_{m-1}}(\nu_2(z), \ldots, \nu_m(z)))\right]$
$= S_T\left[S_T(\inf_{x=yz} \nu_1(z), \inf_{x=yz} S_{T_{m-1}}(\nu_2(z), \ldots, \nu_m(z))), \inf_{x=yz} S_{T_m}(\nu_1(y), \nu_2(y), \ldots, \nu_m(y))\right]$
$\leq S_T\left[\inf_{x=yz} S_T(\nu_1(z), S_{T_{m-1}}(\nu_2(z), \ldots, \nu_m(z))), \inf_{x=yz} S_{T_m}(\nu_1(y), \nu_2(y), \ldots, \nu_m(y))\right]$

$= S_T\big[\inf_{x=yz} S_{T_m}(\nu_1(z),\nu_2(z),\ldots,\nu_m(z))),$
$\quad \inf_{x=yz} S_{T_m}(\nu_1(y),\nu_2(y),\ldots,\nu_m(y))\big]$
$= S_T\big[\inf_{x=yz}(\nu_1\cup\nu_2\cup\ldots,\cup\nu_m)(y),$
$\quad \inf_{x=yz}(\nu_1\cup\nu_2\cup\ldots\cup\nu_m)(z)\big]$
$= S_T\big[\inf_{x=yz} S_T((\nu_1\cup\nu_2\cup\ldots,\cup\nu_m)(y),\nu_{\chi_R}(z)),$
$\quad \inf_{x=yz} S_T(\nu_{\chi_R}(y),(\nu_1\cup\nu_2\cup\ldots\cup\nu_m)(z))\big]$

Thus $(\nu_1\cup\nu_2\cup\ldots\cup\nu_m)(x)\le S_T[((\nu_1\cup\nu_2\cup\ldots\cup\nu_m).\nu_{\chi_R})(x),(\nu_{\chi_R}.(\nu_1\cup\nu_2\cup\ldots\cup\nu_m))(x)]$

Therefore, $[(A_1\cap A_2\cap\ldots\cap A_m).\chi_R]\wedge[\chi_R.(A_1\cap A_2\cap\ldots\cap A_m)]$
$\quad\subseteq (A_1\cap A_2\cap\ldots\cap A_m).$

Hence $A=A_1\cap A_2\cap\ldots\cap A_m$ is an IFQI of R wrt T. □

Theorem 3.8. *Let $\{A_1,A_2,\ldots\}$ here $A_k=(\mu_k,\nu_k), k=1,2,\ldots$ be a collection of IFQIs of R wrt a continuous t-norm T. Then $\bigcap_T A_k$ is also an IFQI of R wrt T.*

Proof. Let $x,y\in R$ and (T,S_T) be dual pair wrt fuzzy complement.

$(\bigcap_T \mu_k)(x-y)=\lim T_m(\mu_1(x-y),\mu_2(x-y),\ldots,\mu_m(x-y))$
$\ge \lim T(T_m(\mu_1(x),\mu_2(x),\ldots,\mu_m(x)),$
$\quad T_m(\mu_1(y),\mu_2(y),\ldots,\mu_m(y)))$ (by theorem 3.4)
$= T(\lim T_m(\mu_1(x),\mu_2(x),\ldots,\mu_m(x)),$
$\quad \lim T_m(\mu_1(y),\mu_2(y),\ldots,\mu_m(y)))$ (since T is continuous)
$= T\big((\bigcap_T\mu_k)(x),(\bigcap_T\mu_k)(y)\big)$ [here limit is taken as $m\to\infty$]

Also, $(\bigcup_{S_T}\nu_i)(x-y)$
$=\lim S_{T_n}(\nu_1(x-y),\nu_2(x-y),\ldots,\nu_n(x-y))$
$\le \lim S_T(S_{T_m}(\nu_1(x),\nu_2(x),\ldots,\nu_m(x)),$
$\quad S_{T_m}(\nu_1(y),\nu_2(y),\ldots,\nu_m(y)))$ (by theorem 3.4)
$= S_T(\lim S_{T_m}(\nu_1(x),\nu_2(x),\ldots,\nu_m(x)),$
$\quad \lim S_{T_n}(\nu_1(y),\nu_2(y),\ldots,\nu_m(y)))$ (since T is continuous)
$= S_T\big((\bigcup_{S_T}\nu_k)(x),(\bigcup_{S_T}\nu_k)(y)\big)$ [here limit is taken as $m\to\infty$]

Again, $(\bigcap_T\mu_k)(x)=\lim T_m(\mu_1(x),\mu_2(x),\ldots,\mu_m(x))$
$\ge \lim T(\sup_{x=yz} T_m(\mu_1(y),\mu_2(y),\ldots,\mu_m(y)),$
$\quad \sup_{x=yz} T_m(\mu_1(z),\mu_2(z),\ldots,\mu_m(z)))$ (by theorem 3.7)
$= T(\lim\sup_{x=yz} T_m(\mu_1(y),\mu_2(y),\ldots,\mu_m(y)),$
$\quad \lim\sup_{x=yz} T_m(\mu_1(z),\mu_2(z),\ldots,\mu_m(z)))$
$= T(\sup_{x=yz}\lim T_m(\mu_1(y),\mu_2(y),\ldots,\mu_m(y)),$
$\quad \sup_{x=yz}\lim T_m(\mu_1(z),\mu_2(z),\ldots,\mu_m(z)))$
$= T(\sup_{x=yz}(\bigcap_T\mu_i)(y),\sup_{x=yz}(\bigcap_T\mu_i)(z))$
$= T(\sup_{x=yz} T((\bigcap_T\mu_k)(y),\mu_{\chi_R}(z))\sup_{x=yz} T(\mu_{\chi_R}(y),(\bigcap_T\mu_k)(z)))$
$= T(((\bigcap_T\mu_k).\mu_{\chi_R})(x),(\mu_{\chi_R}.(\bigcap_T\mu_k)(x)))$
$\quad$ [here limit is taken as $m\to\infty$]

And, $(\bigcup_{S_T}\nu_k)(x)=\lim S_{T_m}(\nu_1(x),\nu_2(x),\ldots,\nu_m(x))$
$\le \lim S_T(\inf_{x=yz} S_{T_m}(\nu_1(y),\nu_2(y),\ldots,\nu_m(y)),$
$\quad \inf_{x=yz} S_{T_m}(\nu_1(z),\nu_2(z),\ldots,\nu_m(z)))$ (by theorem 3.7)
$= S_T(\lim\inf_{x=yz} S_{T_m}(\nu_1(y),\nu_2(y),\ldots,\nu_m(y)),$
$\quad \lim\inf_{x=yz} S_{T_m}(\nu_1(z),\nu_2(z),\ldots,\nu_m(z)))$
$= S_T(\inf_{x=yz}\lim S_{T_m}(\nu_1(y),\nu_2(y),\ldots,\nu_m(y)),$
$\quad \inf_{x=yz}\lim S_{T_m}(\nu_1(z),\nu_2(z),\ldots,\nu_m(z)))$
$= S_T(\inf_{x=yz}(\bigcup_{S_T}\nu_k)(y),\inf_{x=yz}(\bigcup_{S_T}\nu_i)(z))$
$= S_T(\inf_{x=yz} S_T((\bigcup_{S_T}\nu_k)(y),\nu_{\chi_R}(z))\inf_{x=yz} S_T(\nu_{\chi_R}(y),(\bigcup_{S_T}\nu_k)(z)))$
$= S_T(((\bigcup_{S_T}\nu_k).\nu_{\chi_R})(x),(\nu_{\chi_R}.(\bigcup_{S_T}\nu_k)(x)))$
$\quad$ [here limit is taken as $m\to\infty$]

Thus $((\bigcup_T A_k).\chi_R)\wedge(\chi_R.(\bigcup_T A_k))\subseteq(\bigcup_T A_k)$. Hence $(\bigcup_T A_k)$ is an IFQI of R wrt T. □

Theorem 3.9. *Every IFQI of R is an IFQI of R wrt T.*

Proof. Let A be an IFQI of R and $x,y\in R$.
Then $\mu_A(x-y)\ge\min(\mu_A(x),\mu_A(y))\ge T(\mu_A(x),\mu_A(y))$
and $\nu_A(x-y)\le\max(\nu_A(x),\nu_A(y))\le S_T(\nu_A(x),\nu_A(y))$.
Also, $(A.\chi_R)\cap(\chi_R.A)\subseteq A$, therefore for any $x\in R$ we have
$\mu_A(x)\ge\min\left[\mu_{A.\chi_R}(x),\mu_{\chi_R.A}(x)\right]$
$\quad\ge T\left[\mu_{A.\chi_R}(x),\mu_{\chi_R.A}(x)\right]=\mu_{(A.\chi_R)\wedge(\chi_R.A)}(x)$
Also, $\nu_A(x)\le\max\left[\nu_{A.\chi_R}(x),\nu_{\chi_R.A}(x)\right]$
$\quad\le S_T\left[\nu_{A.\chi_R}(x),\nu_{\chi_R.A}(x)\right]=\nu_{(A.\chi_R)\wedge(\chi_R.A)}(x)$
Thus $(A.\chi_R)\wedge(\chi_R.A)\subseteq A$. Hence A is an IFQI of R wrt T. □

Note 3.10. *From the example given above we have ,the reverse of the above theorem is not always true.*

Theorem 3.11. *If A and B are IFQIs of R wrt T, then $A\wedge B$ is also an IFQI of R wrt T.*

Proof. Let A and B are IFQIs of R wrt T and $x,y\in R$.
$\mu_{A\wedge B}(x-y)=T(\mu_A(x-y),\mu_B(x-y))$
$\ge T\big(T(\mu_A(x),\mu_A(y)),T(\mu_B(x),\mu_B(y))\big)$
$=T\big(T(T(\mu_A(x),\mu_A(y)),\mu_B(x)),\mu_B(y)\big)$
$=T\big(T(T(\mu_A(x),\mu_B(x)),\mu_A(y)),\mu_B(y)\big)$
$=T\big(T(\mu_A(x),\mu_B(x)),T(\mu_A(y),\mu_B(y))\big)$
$=T\big(\mu_{A\wedge B}(x),\mu_{A\wedge B}(y)\big)$
And, $\nu_{A\wedge B}(x-y)=S_T(\nu_A(x-y),\nu_B(x-y))$
$\le S_T\big(S_T(\nu_A(x),\nu_A(y)),S_T(\nu_B(x),\nu_B(y))\big)$
$=S_T\big(S_T(S_T(\nu_A(x),\nu_A(y)),\nu_B(x)),\nu_B(y)\big)$
$=S_T\big(S_T(S_T(\nu_A(x),\nu_B(x)),\nu_A(y)),\nu_B(y)\big)$
$=S_T\big(S_T(\nu_A(x),\nu_B(x)),S_T(\nu_A(y),\nu_B(y))\big)$
$=S_T\big(\nu_{A\wedge B}(x),\nu_{A\wedge B}(y)\big)$.
Thus, we have
$\mu_{A\wedge B}(x-y)\ge T(\mu_{A\wedge B}(x),\mu_{A\wedge B}(y))$ and
$\nu_{A\wedge B}(x-y)\le S(\nu_{A\wedge B}(x),\nu_{A\wedge B}(y))$.
Now, $\mu_{A\wedge B}(x)=T(\mu_A(x),\mu_B(y))$

$\geq T\left[T(\sup_{x=yz}\mu_A(y),\sup_{x=yz}\mu_A(z)),T(\sup_{x=yz}\mu_B(y),\sup_{x=yz}\mu_B(z))\right]$
$= T\left[T(T(\sup_{x=yz}\mu_A(y),\sup_{x=yz}\mu_A(z)),\sup_{x=yz}\mu_B(y)),\sup_{x=yz}\mu_B(z))\right]$
$= T\left[T(T(\sup_{x=yz}\mu_A(y),\sup_{x=yz}\mu_B(y)),\sup_{x=yz}\mu_A(z)),\sup_{x=yz}\mu_B(z))\right]$
$= T\left[T(\sup_{x=yz}\mu_A(y),\sup_{x=yz}\mu_B(y)),T(\sup_{x=yz}\mu_A(z),\sup_{x=yz}\mu_B(z))\right]$
$\geq T\left[\sup_{x=yz}T(\mu_A(y),\mu_B(y)),\sup_{x=yz}T(\mu_A(z),\mu_B(z))\right]$
$= T\left[\sup_{x=yz}(\mu_{A\wedge B}(y),\sup_{x=yz}(\mu_{A\wedge B}(z)\right]$
$= T\left[\sup_{x=yz}T((\mu_{A\wedge B}(y),\chi_R(z)),\sup_{x=yz}T(\chi_R(y),\mu_{A\wedge B}(z))\right]$
$= T\left[\mu_{(A\wedge B).\chi_R}(x),\mu_{\chi_R.(A\wedge B)}(x)\right]$.
Also, $\nu_{A\wedge B}(x) = S_T(\nu_A(x),\nu_B(y))$
$\leq S_T\left[S_T(\inf_{x=yz}\nu_A(y),\inf_{x=yz}\nu_A(z)),S_T(\inf_{x=yz}\nu_B(y),\inf_{x=yz}\nu_B(z))\right]$
$= S_T\left[S_T(S_T(\inf_{x=yz}\nu_A(y),\inf_{x=yz}\nu_A(z)),\inf_{x=yz}\nu_B(y)),\inf_{x=yz}\nu_B(z))\right]$
$= S_T\left[S_T(S_T(\inf_{x=yz}\nu_A(y),\inf_{x=yz}\nu_B(y)),\inf_{x=yz}\nu_A(z)),\inf_{x=yz}\nu_B(z))\right]$
$= S_T\left[S_T(\inf_{x=yz}\nu_A(y),\inf_{x=yz}\nu_B(y)),S_T(\inf_{x=yz}\nu_A(z),\inf_{x=yz}\nu_B(z))\right]$
$\leq S_T\left[\inf_{x=yz}S_T(\nu_A(y),\nu_B(y)),\inf_{x=yz}S_T(\nu_A(z),\nu_B(z))\right]$
$= S_T\left[\inf_{x=yz}(\nu_{A\wedge B}(y),\inf_{x=yz}(\nu_{A\wedge B}(z)\right]$
$= S_T\left[\inf_{x=yz}S_T((\nu_{A\wedge B}(y),\chi_R(z)),\sup_{x=yz}T(\chi_R(y),\mu_{A\wedge B}(z))\right]$
$= S_T\left[\nu_{(A\wedge B).\chi_R}(x),\nu_{\chi_R.(A\wedge B)}(x)\right]$.
Therefore $((A\wedge B).\chi_R)\wedge(\chi_R.(A\wedge B))\subseteq A\wedge B$. Hence $A\wedge B$ is an IFQI of R wrt T.

□

4. Conclusion

In this article instuitionistic fuzzy quasi ideal is defined in terms of a triangular norm and some of its properties are discussed as an extension of fuzzy quasi ideals of rings.

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