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Simpson type Katugampola fractional integral inequalities via Harmonic convex functions

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Abstract. In this paper, we attain the new lemma of Simpson type Katugampola fractional integral equality for harmonically convex functions. With the help of this equality, we obtain some new results related to Simpson-like type Katugampola fractional integral inequalities using some inequalities for example power mean inequality and Hölder inequality. Then, we give some conclusions for some special cases of Katugampola fractional integrals when $\rho \rightarrow 1$.

AMS Subject Classifications: 26D15, 26D10, 34A08.

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Contents

1. Introduction

In mathematics, an inequality is a relationship that makes an unequal comparison between two numbers or other mathematical expressions. Inequalities are used in many different areas in real life to facilitate the complexity. For example, businesses use inequalities to control inventory, plan production lines, create pricing models, and move store goods and materials. On the other hand, inequalities are used in engineering and production quality assurance. Therefore, almost all higher mathematical science makes extensive use of inequalities. In the literature, there are some inequalities such as Hermite-Hadamard type inequality, Simpson's type inequality. Simpson's inequality are significantly studied by many mathematicians. It is adapted some kinds of functions for example, convex functions, s-convex functions, harmonic convex functions, readers can see in [1, 2, 7, 9, 10, 12–15, 18, 21].

Now, let we give the following Simpson's inequality we inspire.

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Theorem 1.1. *Let $\Psi : [\varepsilon, \eta] \to \mathbb{R}$ be a four times continuously differentiable mapping on (ε, η) and $\left\|\Psi^{(4)}\right\|_{\infty} = \sup\left|\Psi^{(4)}(x)\right| < \infty$. Then, the following inequality holds:*

$$\left|\int_{\varepsilon}^{\eta} \Psi(w)\, dw - \frac{\eta-\varepsilon}{3}\left[\frac{\Psi(\varepsilon)+\Psi(\eta)}{2} + 2\Psi\left(\frac{\varepsilon+\eta}{2}\right)\right]\right| \leq \frac{1}{2880}\left\|\Psi^{(4)}\right\|_{\infty} \cdot (\eta-\varepsilon)^4 . \tag{1.1}$$

In the following definition readers can find the definition of harmonically convex functions.

Definition 1.2. *[3].Let $A \subset \mathbb{R}\backslash\{0\}$ and $\Psi : A \to \mathbb{R}$ be a function. Ψ is said to be harmonically convex, if*

$$\Psi\left(\frac{uv}{tu+(1-t)v}\right) \leq t\Psi(v) + (1-t)\Psi(u) \tag{1.2}$$

for all $u, v \in A$ and $t \in [0,1]$. Otherwise, Ψ is said to be harmonically concave.

Using the above definition, many authors obtained several inequalities for harmonic convex functions [3, 8, 16]. In the literature, one of the most studied inequalities for harmonic convex functions is Hermite-Hadamard, which is stated as follows:

Theorem 1.3. *[3]Let $\Psi : A \subset \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a harmonically convex function and $u, v \in A$ with $u < v$. If $\Psi \in L[u,v]$ then the following inequalities hold:*

$$\Psi\left(\frac{2uv}{u+v}\right) \leq \frac{uv}{v-u}\int_{u}^{v} \frac{\Psi(\eta)}{\eta^2} d\eta \leq \frac{\Psi(u)+\Psi(v)}{2}. \tag{1.3}$$

The main aim of this paper is to establish Simpson type Katugampola fractional integral inequalities for harmonic convex functions.

2. Preliminaries

In this section, we give some definitions and fundamental results we use in our results.

Definition 2.1. *Let $u, v \in \mathbb{R}$ with $u < v$ and $\Psi \in L[u,v]$. The left and right Riemann- Liouville fractional integrals $J_{u+}^{\alpha}\Psi$ and $J_{v-}^{\alpha}\Psi$ of order $\alpha > 0$ are defined by*

$$J_{u+}^{\alpha}\Psi(\varepsilon) = \frac{1}{\Gamma(\alpha)}\int_{u}^{\varepsilon} (\varepsilon-\eta)^{\alpha-1}\Psi(\eta)\, d\eta, \;\; \varepsilon > u$$

and

$$J_{v-}^{\alpha}\Psi(\varepsilon) = \frac{1}{\Gamma(\alpha)}\int_{\varepsilon}^{v} (t-\varepsilon)^{\alpha-1}\Psi(\eta)\, d\eta, \;\; \varepsilon < v$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_{0}^{\infty} e^{-t}t^{\alpha-1}dt$ (see [6], p. 69).

In 2011, Katugampola [4] introduced a new fractional integral operator which generalizes the Riemann-Liouville and Hadamard fractional integrals as follows.

Definition 2.2. *Let* $[u,v] \subset \mathbb{R}$ *be a finite interval. Then the left and right-side Katugampola fractional integrals of order* $\alpha > 0$ *of* $\Psi \in X_c^p(u,v)$ *are defined by*

$$^{\rho}I_{a+}^{\alpha}\Psi(\varepsilon) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\varepsilon}\frac{\eta^{\rho-1}}{(\varepsilon^{\rho}-\eta^{\rho})^{1-\alpha}}\Psi(\eta)\,d\eta,$$

and

$$^{\rho}I_{b-}^{\alpha}\Psi(\varepsilon) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\varepsilon}^{v}\frac{\eta^{\rho-1}}{(\eta^{\rho}-\varepsilon^{\rho})^{1-\alpha}}\Psi(\eta)\,d\eta,$$

with $u < \varepsilon < v$ *and* $\rho > 0$, *respectively.*

If we take $\rho \to 1$ in the Definition 2.2, we obtain the Definition 2.1. For more information about the Katugampola fractional integrals, readers can see the papers [5, 11, 16, 19].

3. Main Results

Along this study, we will use the following notations to make the article easier to read and to avoid the complexity of the calculations.

$$u_1(t) = \frac{2a^{\rho}b^{\rho}}{(1-t^{\rho})a^{\rho}+(1+t^{\rho})b^{\rho}},$$

$$u_2(t) = \frac{2a^{\rho}b^{\rho}}{(1+t^{\rho})a^{\rho}+(1-t^{\rho})b^{\rho}},$$

$$H = \frac{2a^{\rho}b^{\rho}}{a^{\rho}+b^{\rho}}.$$

Let's start the following Lemma which helps us to obtain the main results:

Lemma 3.1. *Let* $\varphi : I \subset (0,\infty) \to \mathbb{R}$, *be a differentiable function on* $I^{\circ}, a^{\rho}, b^{\rho} \in I^{\circ}$ *and* $a < b$. *If* $\varphi' \in L[a^{\rho}, b^{\rho}]$, *then the following equality holds:*

$$\frac{1}{6}[\varphi(a^{\rho})+4\varphi(H)+\varphi(b^{\rho})] - 2^{\alpha-1}\rho^{\alpha}\left(\frac{a^{\rho}b^{\rho}}{b^{\rho}-a^{\rho}}\right)^{\alpha}\Gamma(\alpha+1)\begin{bmatrix} {}^{\rho}I_{\frac{1}{b}+}^{\alpha}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ +{}^{\rho}I_{\frac{1}{a}-}^{\alpha}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{bmatrix} \tag{3.1}$$

$$= \rho\frac{b^{\rho}-a^{\rho}}{2a^{\rho}b^{\rho}}\int_0^1\left(\tfrac{1}{3}-\tfrac{t^{\alpha\rho}}{2}\right)t^{\rho-1}\left[(u_1(t))^2\,\varphi'(u_1(t)) - (u_2(t))^2\,\varphi'(u_2(t))\right]dt$$

and $\phi(x) = \frac{1}{x}, \alpha > 0$.

Proof. We start by considering the following computations which follows from change of variables and using the definition of the Katugampola fractional integrals.

$$\begin{aligned}
I_1 &= \int_0^1 \left(\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right) t^{\rho-1}\left(u_1(t)\right)^2 \varphi'\left(u_1(t)\right) dt \\
&= \frac{1}{3\rho}\frac{2a^\rho b^\rho}{a^\rho-b^\rho}\varphi\left(u_1(t)\right)|_0^1 \\
&- \frac{1}{2\rho}\frac{2a^\rho b^\rho}{a^\rho-b^\rho}\left(t^{\alpha\rho}\varphi\left(u_1(t)\right)|_0^1-\alpha\rho\int_0^1 t^{\alpha\rho-1}\varphi\left(u_1(t)\right)dt\right) \\
&= \frac{1}{3\rho}\frac{2a^\rho b^\rho}{a^\rho-b^\rho}\left(\varphi\left(a^\rho\right)-\varphi\left(H\right)\right) \\
&- \frac{1}{2\rho}\frac{2a^\rho b^\rho}{a^\rho-b^\rho}\varphi\left(a^\rho\right)+\frac{\rho^{\alpha-1}}{2}\left(\frac{2a^\rho b^\rho}{a^\rho-b^\rho}\right)^{\alpha+1}\Gamma\left(\alpha+1\right)\ {}^\rho I^\alpha_{\frac{1}{a}-}\left(\varphi\circ\phi\right)\left(\tfrac{1}{H}\right) \\
&= \frac{1}{\rho}\frac{2a^\rho b^\rho}{a^\rho-b^\rho}\left(\frac{1}{6}\varphi\left(a^\rho\right)+\frac{1}{3}\varphi\left(H\right)\right)-\frac{\rho^{\alpha-1}}{2}\left(\frac{2a^\rho b^\rho}{a^\rho-b^\rho}\right)^{\alpha+1}\Gamma\left(\alpha+1\right)\ {}^\rho I^\alpha_{\frac{1}{a}-}\left(\varphi\circ\phi\right)\left(\tfrac{1}{H}\right)
\end{aligned}$$

and similarly

$$\begin{aligned}
I_2 &= \int_0^1 \left(\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right) t^{\rho-1}\left(u_2(t)\right)^2 \varphi'\left(u_2(t)\right) dt \\
&= \frac{1}{\rho}\frac{2a^\rho b^\rho}{b^\rho-a^\rho}\left(-\frac{1}{6}\varphi\left(b^\rho\right)-\frac{1}{3}\varphi\left(H\right)\right)+\frac{\rho^{\alpha-1}}{2}\left(\frac{2a^\rho b^\rho}{b^\rho-a^\rho}\right)^{\alpha+1}\Gamma\left(\alpha+1\right)\ {}^\rho I^\alpha_{\frac{1}{b}+}\left(\varphi\circ\phi\right)\left(\tfrac{1}{H}\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left(I_1-I_2\right) \\
&= \frac{1}{6}\left[\varphi\left(a^\rho\right)+4\varphi\left(H\right)+\varphi\left(b^\rho\right)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma\left(\alpha+1\right)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}\left(\varphi\circ\phi\right)\left(\frac{1}{H}\right)\\ +{}^\rho I^\alpha_{\frac{1}{a}-}\left(\varphi\circ\phi\right)\left(\frac{1}{H}\right)\end{array}\right].
\end{aligned}$$

■

Remark 3.2. *If we take $\rho\to 1$ in Lemma 3.1, we have the following equality*

$$\begin{aligned}
&\frac{1}{6}\left[\varphi\left(a\right)+4\varphi\left(\frac{2ab}{a+b}\right)+\varphi\left(b\right)\right]-2^{\alpha-1}\left(\frac{ab}{b-a}\right)^\alpha\Gamma\left(\alpha+1\right)\left[\begin{array}{c}J^\alpha_{1/b^+}\left(\varphi\circ\phi\right)\left(\frac{a+b}{2ab}\right)\\ +I^\alpha_{1/a^-}\left(\varphi\circ\phi\right)\left(\frac{a+b}{2ab}\right)\end{array}\right] \\
&= \frac{b-a}{2ab}\int_0^1\left(\frac{1}{3}-\frac{t^\alpha}{2}\right)\left[\begin{array}{c}\left(\frac{2ab}{(1-t)a+(1+t)b}\right)^2\varphi'\left(\frac{2ab}{(1-t)a+(1+t)b}\right)\\ -\left(\frac{2ab}{(1+t)a+(1-t)b}\right)^2\varphi'\left(\frac{2ab}{(1+t)a+(1-t)b}\right)\end{array}\right]dt.
\end{aligned} \tag{3.2}$$

Remark 3.3. *If we take $\alpha=1$ in Remark 3.2, we have the equality [[17], Remark 1].*

Theorem 3.4. *Let $\varphi: I\subset(0,\infty)\to\mathbb{R}$, be a differentiable function on $I^\circ, a^\rho, b^\rho\in I^\circ$ and $a<b$. If $\varphi'\in L\left[a^\rho,b^\rho\right]$ and $\left|\varphi'\right|$ is a harmonic convex function on $\left[a^\rho,b^\rho\right]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi\left(a^\rho\right)+4\varphi\left(H\right)+\varphi\left(b^\rho\right)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma\left(\alpha+1\right)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}\left(\varphi\circ\phi\right)\left(\frac{1}{H}\right)\\ +{}^\rho I^\alpha_{\frac{1}{a}-}\left(\varphi\circ\phi\right)\left(\frac{1}{H}\right)\end{array}\right]\right| \\
&\le \frac{b^\rho-a^\rho}{6a^\rho b^\rho}\left(\left|\varphi'\left(a^\rho\right)\right|\left(K_1\left(w;\alpha\right)+K_2\left(w;\alpha\right)\right)+\left|\varphi'\left(b^\rho\right)\right|\left(K_3\left(w;\alpha\right)+K_4\left(w;\alpha\right)\right)\right)
\end{aligned} \tag{3.3}$$

where $\alpha>0$ and $K_1(w;\alpha), K_2(w;\alpha), K_3(w;\alpha), K_4(w;\alpha)$ are the same as in [[17], Theorem 3.]

Proof. Using Lemma 3.1 and harmonic convexity of $|\varphi'|$, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right|\\
&\leq\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left((u_1(t))^2\left|\varphi'(u_1(t))\right|+(u_2(t))^2\left|\varphi'(u_2(t))\right|\right)dt\\
&\leq\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(\begin{array}{c}(u_1(t))^2\left(\frac{1+t^\rho}{2}\left|\varphi'(a^\rho)\right|+\frac{1-t^\rho}{2}\left|\varphi'(b^\rho)\right|\right)\\+(u_2(t))^2\left(\frac{1-t^\rho}{2}\left|\varphi'(a^\rho)\right|+\frac{1+t^\rho}{2}\left|\varphi'(b^\rho)\right|\right)\end{array}\right)dt\\
&=\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{w^\alpha}{2}\right|\left(\begin{array}{c}(u_1(w))^2\left(\frac{1+w}{2}\left|\varphi'(a^\rho)\right|+\frac{1-w}{2}\left|\varphi'(b^\rho)\right|\right)\\+(u_2(w))^2\left(\frac{1-w}{2}\left|\varphi'(a^\rho)\right|+\frac{1+w}{2}\left|\varphi'(b^\rho)\right|\right)\end{array}\right)dw\\
&\leq\frac{b^\rho-a^\rho}{6a^\rho b^\rho}\left(\left|\varphi'(a^\rho)\right|\left(K_1(w;\alpha)+K_2(w;\alpha)\right)+\left|\varphi'(b^\rho)\right|\left(K_3(w;\alpha)+K_4(w;\alpha)\right)\right).
\end{aligned}$$

The last inequality is obtained using where $\left|\frac{1}{3}-\frac{w^\alpha}{2}\right|\leq\frac{1}{3}$ for all $u\in[0,1]$. This completes the proof. ■

Remark 3.5. *If we take $\rho\to1$, we have the inequality [[17], Theorem 3].*

Theorem 3.6. *Let $\varphi:I\subset(0,\infty)\to\mathbb{R}$, be a differentiable function on $I^\circ, a^\rho, b^\rho\in I^\circ$ and $a<b$. If $\varphi'\in L[a^\rho,b^\rho]$ and $|\varphi'|^q$ is a harmonic convex function on $[a^\rho,b^\rho]$ for $q>1$ and $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{array}\right]\right| \quad (3.4)\\
&\leq\frac{b^\rho-a^\rho}{6a^\rho b^\rho}\left[\begin{array}{c}\left(X_1(q;a,b)\left|\varphi'(a)\right|^q+X_2(q;a,b)\left|\varphi'(b)\right|^q\right)^{\frac{1}{q}}\\+\left(X_3(q;a,b)\left|\varphi'(a)\right|^q+X_4(q;a,b)\left|\phi'(b)\right|^q\right)^{\frac{1}{q}}\end{array}\right]
\end{aligned}$$

where $\alpha>1$ and $X_1(q;a,b), X_2(q;a,b), X_3(q;a,b), X_4(q;a,b)$ are the same as in [[17], Theorem 4].

Proof. From Lemma 3.1 and using the Hölder's integral inequality and the harmonic convexity of $|\varphi'|^q$, we have

$$\left|\frac{1}{6}\left[\varphi\left(a^\rho\right)+4\varphi\left(H\right)+\varphi\left(b^\rho\right)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma\left(\alpha+1\right)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}\left(\varphi\circ\phi\right)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}\left(\varphi\circ\phi\right)\left(\frac{1}{H}\right)\end{array}\right]\right|$$
$$\leq\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(\left(u_1\left(t\right)\right)^2\left|\varphi'\left(u_1\left(t\right)\right)\right|+\left(u_2\left(t\right)\right)^2\left|\varphi'\left(u_2\left(t\right)\right)\right|\right)dt$$
$$=\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\frac{1}{3}-\frac{w^\rho}{2}\right|\left(\left(u_1\left(w\right)\right)^2\left|\varphi'\left(u_1\left(w\right)\right)\right|+\left(u_2\left(w\right)\right)^2\left|\varphi'\left(u_2\left(w\right)\right)\right|\right)$$
$$\leq\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left(\int_0^1\left|\frac{1}{3}-\frac{w^\rho}{2}\right|^p dw\right)^{\frac{1}{p}}\left\{\begin{array}{c}\left(\int_0^1\left(u_1\left(w\right)\right)^{2q}\left|\varphi'\left(u_1\left(w\right)\right)\right|^q dt\right)^{\frac{1}{q}}\\+\left(\int_0^1\left(u_2\left(w\right)\right)^{2q}\left|\varphi'\left(u_2\left(w\right)\right)\right|^q dt\right)^{\frac{1}{q}}\end{array}\right\}dw$$
$$\leq\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left\{\begin{array}{c}\left(\int_0^1\left|\frac{1}{3}-\frac{w^\rho}{2}\right|^p dw\right)^{\frac{1}{p}}\\\times\left[\begin{array}{c}\left(\int_0^1\left(u_1\left(w\right)\right)^{2q}\left[\left|\varphi'\left(a^\rho\right)\right|^q\left(\frac{1+w}{2}\right)+\left|\varphi'\left(b^\rho\right)\right|^q\left(\frac{1-w}{2}\right)\right]dt\right)^{\frac{1}{q}}\\+\left(\int_0^1\left(u_2\left(t\right)\right)^{2q}\left[\left|\varphi'\left(a^\rho\right)\right|^q\left(\frac{1-w}{2}\right)+\left|\varphi'\left(b^\rho\right)\right|^q\left(\frac{1+w}{2}\right)\right]dt\right)^{\frac{1}{q}}\end{array}\right]\end{array}\right\}$$
$$\leq\frac{b^\rho-a^\rho}{6a^\rho b^\rho}\left[\begin{array}{c}\left(X_1\left(q;a,b\right)\left|\varphi'\left(a^\rho\right)\right|^q+X_2\left(q;a,b\right)\left|\varphi'\left(b^\rho\right)\right|^q\right)^{\frac{1}{q}}\\+\left(X_3\left(q;a,b\right)\left|\varphi'\left(a^\rho\right)\right|^q+X_4\left(q;a,b\right)\left|\phi'\left(b^\rho\right)\right|^q\right)^{\frac{1}{q}}\end{array}\right].$$

The last inequality is obtained using where $\left|\frac{1}{3}-\frac{w^\alpha}{2}\right|\leq\frac{1}{3}$ for all $u\in[0,1]$. This completes the proof. ■

Remark 3.7. *If wetake $\rho\to1$, we have the inequality [[17], Theorem 4].*

Theorem 3.8. *Let $\varphi: I\subset(0,\infty)\to\mathbb{R}$, be a differentiable function on I°, $a^\rho, b^\rho\in I^\circ$ and $a<b$. If $\varphi'\in$ If φ' $\in L\left[a^\rho,b^\rho\right]$ and $|\varphi'|^q$ is a harmonic convex function on $\left[a^\rho,b^\rho\right]$ for $q\geq1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[\varphi\left(a^\rho\right)+4\varphi\left(H\right)+\varphi\left(b^\rho\right)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma\left(\alpha+1\right)\left[\begin{array}{c}{}^\rho I^\alpha_{\frac{1}{b}+}\left(\varphi\circ\phi\right)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}\left(\varphi\circ\phi\right)\left(\frac{1}{H}\right)\end{array}\right]\right| \tag{3.5}$$
$$\leq\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left(\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}dt\right)^{1-\frac{1}{q}}\left[\begin{array}{c}\left(\zeta_1\left(q,t;\alpha,n\right)\left|\varphi'\left(a\right)\right|^q+\zeta_2\left(q,t;\alpha,n\right)\left|\varphi'\left(b\right)\right|^q\right)^{\frac{1}{q}}\\+\left(\zeta_3\left(q,t;\alpha,n\right)\left|\varphi'\left(a\right)\right|^q+\zeta_4\left(q,t;\alpha,n\right)\left|\varphi'\left(b\right)\right|^q\right)^{\frac{1}{q}}\end{array}\right]$$

where

$$\zeta_1\left(q,t;\alpha,\rho\right)=\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_1\left(t\right)\right)^{2q}\frac{1+t}{2}dt,$$

$$\zeta_2(q,t;\alpha,\rho)=\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\frac{1-t}{2}dt,$$

$$\zeta_3(q,t;\alpha,\rho)=\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_2(t))^{2q}\frac{1-t}{2}dt,$$

$$\zeta_4(q,t;\alpha,\rho)=\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_2(t))^{2q}\frac{1-t}{2}dt,$$

$\alpha>1$ *and* $\zeta_1(q,t;\alpha,\rho),\zeta_2(q,t;\alpha,\rho),\zeta_3(q,t;\alpha,\rho),\zeta_4(q,t;\alpha,\rho)$ *are the same as in [[17], Theorem 5].*

Proof. From Lemma 3.1 and using the power mean inequality, we have that the following inequality holds:

$$\begin{aligned}&\left|\frac{1}{6}[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\begin{bmatrix}{}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right)\\+{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{bmatrix}\right|\\
&\leq\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\int_0^1\left|\left(\tfrac{1}{3}-\tfrac{t^{\alpha\rho}}{2}\right)\right|t^{\rho-1}\left[(u_1(t))^2\left|\varphi'(u_1(t))\right|+(u_2(t))^2\left|\varphi'(u_2(t))\right|\right]dt\\
&\leq\rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left\{\begin{array}{c}\left(\int_0^1\left|\tfrac{1}{3}-\tfrac{t^{\alpha\rho}}{2}\right|t^{\rho-1}dt\right)^{1-\frac{1}{q}}\\
\times\begin{bmatrix}\left(\int_0^1\left|\tfrac{1}{3}-\tfrac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\left|\varphi'(u_1(t))\right|^q dt\right)^{\frac{1}{q}}\\
+\left(\int_0^1\left|\tfrac{1}{3}-\tfrac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_2(t))^{2q}\left|\varphi'(u_1(t))\right|^q dt\right)^{\frac{1}{q}}\end{bmatrix}\end{array}\right\}.\end{aligned}$$

By the harmonic convexity of $|\varphi'|^q$

$$\begin{aligned}&\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\left|\varphi'(u_1(t))\right|^q dt\\
&\leq\left|\varphi'(a)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\frac{1+t}{2}dt\\
&+\left|\varphi'(b)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}(u_1(t))^{2q}\frac{1-t}{2}dt\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right| t^{\rho-1}\left(u_2(t)\right)^{2q}\left|\varphi'\left(u_2(t)\right)\right|^q dt \\
&\leq \left|\varphi'(a)\right|^q \int_0^1 \left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right| t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1-t}{2}dt \\
&+\left|\varphi'(b)\right|^q \int_0^1 \left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right| t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1+t}{2}dt.
\end{aligned}$$

Using the last two inequalities we obtain

$$\begin{aligned}
&\left|\frac{1}{6}\left[\varphi(a^\rho)+4\varphi(H)+\varphi(b^\rho)\right]-2^{\alpha-1}\rho^\alpha\left(\frac{a^\rho b^\rho}{b^\rho-a^\rho}\right)^\alpha\Gamma(\alpha+1)\begin{bmatrix} {}^\rho I^\alpha_{\frac{1}{b}+}(\varphi\circ\phi)\left(\frac{1}{H}\right) \\ +{}^\rho I^\alpha_{\frac{1}{a}-}(\varphi\circ\phi)\left(\frac{1}{H}\right)\end{bmatrix}\right| \\
&\leq \rho\frac{b^\rho-a^\rho}{2a^\rho b^\rho}\left(\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}dt\right)^{1-\frac{1}{q}} \\
&\times\left[\begin{array}{c}\left(\begin{array}{c}\left|\varphi'(a)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_1(t)\right)^{2q}\frac{1+t}{2}dt \\ +\left|\varphi'(b)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_1(t)\right)^{2q}\frac{1-t}{2}dt\end{array}\right)^{\frac{1}{q}} \\ +\left(\begin{array}{c}\left|\varphi'(a)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1-t}{2}dt \\ +\left|\varphi'(b)\right|^q\int_0^1\left|\frac{1}{3}-\frac{t^{\alpha\rho}}{2}\right|t^{\rho-1}\left(u_2(t)\right)^{2q}\frac{1+t}{2}dt\end{array}\right)^{\frac{1}{q}}\end{array}\right].
\end{aligned}$$

■

Remark 3.9. *If wetake $\rho \to 1$, we have the inequality [[17], Theorem 5].*

4. Conclusion

In this paper, using a new identity of Simpson-like type for Katugampola fractional integral for harmonic convex functions, we obtained some new integral inequalities related to Simpson inequalities. Furthermore, some interesting conclusions were obtained for some special values of ρ. This study generalizes the paper [17].

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