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Common fixed points in S-metric spaces using EA and CLR-properties

Prasad Kanchanapally[1,*] and V. Naga Raju[2]

Abstract

Two common fixed point theorems for two pairs of weakly compatible mappings satisfying EA property and CLR property are proved in S-metric spaces.Further,examples are provided in support of the theorems.

Keywords

S-metric space, common fixed point, weakly compatibility, E.A.property, CLR property.

AMS Subject Classification

54H25, 47H10.

[1,2] *Department of Mathematics,University College of Science, Osmania University, Hyderabad-500007, Telangana, India.*

*Corresponding author: [1] iitm.prasad@gmail.com;

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Contents

1. Introduction

In 2006, Mustafa and Sims [6] introduced G-metric spaces as a generalization of metric spaces and proved the existence of fixed points under different contractions. In 2012, Sedghi, Shobe and Aliouche [1] introduced a new concept called S-metric space and studied its some properties and they also stated that S-metric space is a generalization of G-metric space.But, in 2014 Dung, Hieu and Radojevic [7] showed by an example that S-metric space is not a generalization of G-metric space and conversely.Thus the class of S-metric spaces and the class of G-metric spaces are distinct.On the other hand, in 1998, Jungck and Rhoades [5] introduced the concept of weakly compatibilty. In 2002, Amari and Moutawakil [4] introduced a new concept, namely E.A. Property. In 2012, Imdad et al. [3] introduced a new concept, namely CLR-property for two pairs of self maps and established some common fixed point theorems under this notion.In this paper, we prove some common fixed point theorems for four self maps of S-metric space.

In the following we provide some basic definitions and preliminaries which we use in this paper.

2. Preliminaries

Definition 2.1. *[1] Let X be a non empty set. Then we say that a function S: $X^3 \to [0,\infty)$ is a S-metric on X iff it satisfies the following for all α,β,γ and $\theta \in X$*

P1) $S(\alpha,\beta,\gamma)=0$ iff $\alpha=\beta=\gamma$

P2) $S(\alpha,\beta,\gamma) \leq S(\alpha,\alpha,\theta)+S(\beta,\beta,\theta)+S(\gamma,\gamma,\theta)$.

Here (X,S) is called a S-metric space.

Example 2.2. *(X,S) is a S-metric space , where $X=[0,1]$ and*

$$S(\alpha,\beta,\gamma)=\begin{cases} 0, \text{ for } \alpha=\beta=\gamma \\ max\{\alpha,\beta,\gamma\}, \text{ otherwise} \end{cases} \quad \text{for } \alpha,\beta,\gamma \in X.$$

Example 2.3. *[2] (X,S) is a S-metric space , where $X=\mathbb{R}$ and $S(\alpha,\beta,\gamma)=|\alpha-\gamma|+|\beta-\gamma|$ for $\alpha,\beta,\gamma \in X$.*

Example 2.4. *(X,S) is a S-metric space , where $X=[0,4]$ and $S(\alpha,\beta,\gamma)=max\{|\alpha-\gamma|,|\beta-\gamma|\}$ for $\alpha,\beta,\gamma \in X$.*

Definition 2.5. *[1] We say that a sequence (α_n) in S-metric space (X,S) converges to some $\alpha \in X$ iff $S(\alpha_n,\alpha_n,\alpha) \to 0$ as $n \to \infty$.*

Definition 2.6. *[1] We say that a sequence (α_n) in S-metric space (X,S) is a Cauchy sequence in X iff $S(\alpha_n,\alpha_n,\alpha_m) \to 0$ as $n,m \to \infty$.*

Definition 2.7. *[1] We say that a S-metric space (X,S) is complete iff every Cauchy sequence in X converges in X.*

Lemma 2.8. *[1] In S-metric space (X,S),we have* $S(\alpha,\alpha,\gamma)=S(\gamma,\gamma,\alpha)$ *for all* $\alpha,\gamma \in X$.

Lemma 2.9. *[1] In a S-metric space (X,S), if there exist sequences* (α_n) *and* (β_n) *in X such that* $\lim_{n\to\infty} \alpha_n = \alpha$ *and* $\lim_{n\to\infty} \beta_n = \beta$*,then* $\lim_{n\to\infty} S(\alpha_n,\alpha_n,\beta_n)=S(\alpha,\alpha,\beta)$.

Definition 2.10. *In a S-metric space (X,S),we say that a point* $\alpha \in X$ *is a common fixed point of two self maps f and R on X iff* $f\alpha=R\alpha=\alpha$.

Definition 2.11. *In a S-metric space (X,S),we say that two self maps f and R defined on X are weakly compatible iff* $f\alpha=R\alpha$ *for* $\alpha \in X$ *implies* $fR\alpha=fR\alpha$.

Definition 2.12. *[4] In a S-metric space (X,S),we say that two self maps f and R on X satisfy E.A.property if there exists a sequence* (α_n) *in X such that* $\lim_{n\to\infty} R(\alpha_n)=\lim_{n\to\infty} f(\alpha_n)=\gamma$ *for some* $\gamma \in X$.

Definition 2.13. *[3] We say that two pairs (f,R) nad (g,T) of self maps of S-metric space (X,S) are said to satisfy common limit range property with respect to R and T if there exist two sequences* (α_n) *and* (β_n) *in X such that*

$$\lim_{n\to\infty} R(\alpha_n)=\lim_{n\to\infty} f(\alpha_n)=\lim_{n\to\infty} g(\beta_n)=\lim_{n\to\infty} T(\beta_n)=\gamma$$

for some $\gamma \in RX \cap TX$ *and it is denoted by* (CLR_{RT}).

3. Main Results

Theorem 3.1. *Suppose in an S-metric space X, there are four self maps f,g,R,T on X satisfying*
i) $S(f\alpha,f\alpha,g\beta)\leq \phi(S(R\alpha,R\alpha,T\beta))$ *for all* $\alpha,\beta \in X$
where $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\phi(0)=0$ *and* $0<\phi(\alpha)<\alpha$ *for every* $\alpha>0$
ii) the pairs (f,R) and (g,T) are weakly compatible
iii)either (f,R) (or) (g,T) satisfies E.A property
iv) $fX \subset TX$ *and* $gX \subset RX$.
If one of the ranges of the mappings f,g,R and T is a complete subset of (X,S), then f,g,R and T have a unique common fixed point.

Proof. We start proof by assuming that the pair (f,R) satisfies E.A property.Then we can find a sequence (α_n) in X such that $\lim_{n\to\infty} R(\alpha_n)=\lim_{n\to\infty} f(\alpha_n)=\gamma$ for some γ.Since $fX \subset TX$ and (α_n) is in X, there is a sequence (β_n) in X such that $\lim_{n\to\infty} f(\alpha_n)=\lim_{n\to\infty} R(\alpha_n)=\lim_{n\to\infty} T(\beta_n)=\gamma$.Without loss of generality,we assume that TX is a complete subset of X.So that $\gamma \in TX$, since $\lim_{n\to\infty} T(\beta_n)=\gamma$.This implies that $\gamma=T\theta_1$ for some $\theta_1 \in X$.Now we show that $g\theta_1=T\theta_1$.For each $n\in\mathbb{N}$,we have $S(f\alpha_n,f\alpha_n,g\theta_1)\leq \phi(S(R\alpha_n,R\alpha_n,T\beta))$. Now letting $n\to\infty$, we have $S(T\theta_1,T\theta_1,g\theta_1)\leq \phi(S(T\theta_1,T\theta_1,T\theta_1))=\phi(0)=0$,since ϕ is continuous.This will imply that $S(T\theta_1,T\theta_1,g\theta_1)\leq 0$.This implies $S(T\theta_1,T\theta_1,g\theta_1)=0$.It follows that $g\theta_1=T\theta_1$.Since $g\theta_1 \in gX$ and $gX \subset RX$, $g\theta_1=R\theta_2$ for some $\theta_2 \in X$.Now let us show that $R\theta_2=f\theta_2$.For this, we consider

$$\begin{aligned} S(f\theta_2,f\theta_2,R\theta_2)&=\phi(S(f\theta_2,f\theta_2,g\theta_1))\\ &\leq \phi(S(R\theta_2,R\theta_2,T\theta_1))\\ &=\phi(S(R\theta_2,R\theta_2,R\theta_2))\\ &=\phi(0)=0.\end{aligned}$$

Therefore, we must have $S(f\theta_2,f\theta_2,R\theta_2)\leq 0$.This will imply that $S(f\theta_2,f\theta_2,R\theta_2)=0$.Thus $f\theta_2=R\theta_2$ and hence $T\theta_1=g\theta_1=R\theta_2=f\theta_2=\gamma$.Now (g,T) is weakly compatible implies that $Tg\theta_1=gT\theta_1$ and hence $T\gamma=g\gamma$.Similarly, since (f,R) is weakly compatible and $R\theta_2=f\theta_2$, we have $Rf\theta_1=fR\theta_1$ and hence $R\gamma=f\gamma$. Now let us show that $g\gamma=\gamma$. Note that $S(f\theta_2,f\theta_2,g\gamma)\leq \phi(S(R\theta_2,R\theta_2,T\gamma))$. Then $S(\gamma,\gamma,g\gamma)\leq \phi(S(\gamma,\gamma,g\gamma))$. If $S(\gamma,\gamma,g\gamma)\neq 0$,then $S(\gamma,\gamma,g\gamma)>0$.By definition of ϕ, we have $\phi(S(\gamma,\gamma,g\gamma))< S(\gamma,\gamma,g\gamma)$. This will imply that $S(\gamma,\gamma,g\gamma)<S(\gamma,\gamma,g\gamma)$ -contradiction. Therefore $S(\gamma,\gamma,g\gamma)=0$ and hence $g\gamma=\gamma$. Now we show that $f\gamma=\gamma$.Note that $S(f\gamma,f\gamma,g\theta_1)\leq \phi(S(R\gamma,R\gamma,T\theta_1))$.Then we have $S(f\gamma,f\gamma,\gamma)\leq \phi(S(f\gamma,f\gamma,\gamma))$.If $S(f\gamma,f\gamma,\gamma)\neq 0$,then $S(f\gamma,f\gamma,\gamma)>0$.By definition of ϕ,we have $S(f\gamma,f\gamma,\gamma)<S(f\gamma,f\gamma,\gamma)$-contradiction. Therefore $S(f\gamma,f\gamma,\gamma)=0$ and hence $f\gamma=\gamma$.Since $R\gamma=f\gamma$ and $T\gamma=g\gamma$,it follows that $f\gamma=g\gamma=R\gamma=T\gamma=\gamma$.This shows that γ is a common fixed point of f,g,R and T.
Similarly one can easily show the result when RX is assumed to be complete subset of X.The cases in which fX or gX is complete subset of X are similar to the cases TX or SX is complete respectively, since $fX \subset TX$ and $gX \subset RX$.
Let us show the uniqueness of common fixed point of f,g,R and T. For this,let δ be another common fixed point of f,g,R and T.Then $f\delta=g\delta=R\delta=T\delta=\delta$ and $f\gamma=g\gamma=R\gamma=T\gamma=\gamma$. Now we consider $S(\delta,\delta,\gamma)=S(f\delta,f\delta,g\gamma)\leq \phi(S(R\delta,R\delta,T\gamma))=\phi(S(\delta,\delta,\gamma))$. If $S(\delta,\delta,\gamma)\neq 0$,then we must have $S(\delta,\delta,\gamma)<S(\delta,\delta,\gamma)$- contradiction. Therefore $\delta=\gamma$ and hence the result proved. □

Corollary 3.2. *Suppose in S-metric space X, there are two self maps f,R on X satisfying*
i) $S(f\alpha,f\alpha,f\beta)\leq \phi(S(R\alpha,R\alpha,R\beta))$ *for all* $\alpha,\beta \in X$*, where* $\phi : [0,\infty) \to [0,\infty)$ *is a continuous function such that* $\phi(0)=0$ *and* $0<\phi(\alpha)<\alpha$ *for every* $\alpha>0$
ii) the pair (f,R) is weakly compatible
iii)(f,R) satisfies E.A property
iv) $fX \subset RX$.
If one of the ranges of the mappings f and R is a complete subset of (X,S), then f and R have a unique common fixed point.

Proof. Let g=f and R=T on X .Then all conditions of the Theorem 3.1 are satisfied and hence the result proved. □

Corollary 3.3. *Suppose in S-metric space X, there are four self maps f,g,R,T on X satisfying*
i) $S(f\alpha,f\alpha,g\beta)\leq q(S(R\alpha,R\alpha,T\beta))$ *for all* $\alpha,\beta \in X$ *and for some* $q \in [0,1)$
ii) the pairs (f,R) and (g,T) are weakly compatible
iii)either (f,R) (or) (g,T) satisfies E.A property

iv) $fX \subset TX$ and $gX \subset RX$.
If one of the ranges of the mappings f,g,R and T is complete subset of (X,S), then f,g,R and T have a unique common fixed point.

Proof. Let $\phi : [0,\infty) \to [0,\infty)$ be a function defined by $\phi(\alpha)$ =qα for $\alpha \in [0,\infty)$.Clearly it is continuous function on $[0,\infty)$ such that $\phi(0) = 0$ and $0 < \phi(\alpha) < \alpha$ for all $\alpha >$ 0.Therefore all the conditions of Theorem 3.1 are satisfied and hence the result proved. □

Corollary 3.4. *Suppose in S-metric space X, there are two self maps f and g on X satisfying*
i) S(fα,fα,gβ)$\leq$ q(S(α,α,β)) for all $\alpha,\beta \in X$ and for some $q \in [0,1)$ and
ii)either (f,I) (or) (g,I) satisfies E.A property,where I is the identity map on X.
If one of the ranges of the mappings f and g is complete subset of (X,S), then f and g have a unique common fixed point.

Proof. Follows from the Theorem 3.1 by setting $\phi\alpha = q\alpha$ for $\alpha \in [0,\infty)$ and R=T=I on X. □

In the next theorem,we use (CLR$_{RT}$) property to relax containment condition and E.A.property.

Theorem 3.5. *Suppose in S-metric space X, there are four self maps f,g,R,T on X satisfying i) S(fα,fα,gβ)$\leq \phi$(S(Rα,Rα,Tβ)) for all $\alpha,\beta \in X$,where $\phi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\phi(0) = 0$ and $0 < \phi(\alpha) < \alpha$ for every $\alpha > 0$*
ii) the pairs (f,R) and (g,T) are weakly compatible
iii)(f,R) and (g,T) satisfy (CLR$_{RT}$) property.
Then f,g,R and T have a unique common fixed point.

Proof. As (f,R) and (g,T) satisfy (CLR$_{RT}$) property, we can find two sequences (α_n) and (β_n) in X such that
$\lim_{n\to\infty} R(\alpha_n) = \lim_{n\to\infty} f(\alpha_n) = \lim_{n\to\infty} g(\beta_n) = \lim_{n\to\infty} T(\beta_n) = \gamma$ for some $\gamma \in RX \cap TX$.Then $\gamma = T\theta_1 = R\theta_2$ for some $\theta_1,\theta_2 \in X$. Now we show that gθ_1=Tθ_1. For each n$\in \mathbb{N}$,we have
S(fα_n,fα_n,gθ_1)$\leq \phi$(S(Rα_n,Rα_n,Tβ)).Now letting $n \to \infty$, we get S(Tθ_1,Tθ_1,gθ_1)$\leq \phi$(S(Tθ_1,Tθ_1,Tθ_1))=$\phi(0) = 0$,since ϕ is continuous.This will imply that S(Tθ_1,Tθ_1,gθ_1)$\leq$ 0 and hence S(Tθ_1,Tθ_1,gθ_1)= 0.It follows that gθ_1=Tθ_1.Therefore we can prove the result as in Theorem 3.1 and hence we omit the rest of the proof. □

Theorem 3.6. *Suppose in S-metric space X, there are four self maps f,g,R,T on X satisfying*
i) S(gβ,gβ,fα)$\leq$ pS(Rα,Rα,fα)+qS(Tβ,Tβ,Rα)
+rS(gβ,gβ,Tβ)+k[S(Tβ,Tβ,fα)
+S(gβ,gβ,Rα)] for all $\alpha,\beta \in X$ and for some p,q,r,k$\in [0,1)$ with p+q+r+2k<1
ii) the pairs (f,R) and (g,T) are weakly compatible
iii)(f,R) and (g,T) satisfy (CLR$_{RT}$) property.
Then f,g,R and T have a unique common fixed point.

Proof. Since (f,R) and (g,T) satisfy (CLR$_{RT}$) property,we can find two sequences (α_n) and (β_n) in X such that
$\lim_{n\to\infty} R(\alpha_n) = \lim_{n\to\infty} f(\alpha_n) = \lim_{n\to\infty} g(\beta_n) = \lim_{n\to\infty} T(\beta_n) = \gamma$ for some $\gamma \in RX \cap TX$.Then γ =Tθ_1=Rθ_2 for some $\theta_1,\theta_2 \in X$.Now we show that gθ_1 =Tθ_1.For each n$\in \mathbb{N}$, we have
S(gθ_1,gθ_1,fα_n)$\leq$ pS(Rα_n,Rα_n,fα_n)+qS(Tθ_1,Tθ_1,Rα_n)
+rS(gθ_1,gθ_1,Tθ_1)+ k[S(Tθ_1,Tθ_1,fα_n)+S(gθ_1,gθ_1,Rα_n)].
i.e S(fα_n,fα_n,gθ_1)$\leq$ pS(Rα_n,Rα_n,fα_n)+qS(Rα_n,Rα_n,Tθ_1)
+rS(gθ_1,gθ_1,Tθ_1)+ k[S(fα_n,fα_n,Tθ_1)+S(Rα_n,Rα_n,gθ_1)].
Now letting n$\to\infty$, we have
S(Tθ_1,Tθ_1,gθ_1)$\leq$ pS(Tθ_1,Tθ_1,Tθ_1)+qS(Tθ_1,Tθ_1,Tθ_1)
+rS(gθ_1,gθ_1,Tθ_1)+ k[S(Tθ_1,Tθ_1,Tθ_1)+S(Tθ_1,Tθ_1,gθ_1)].
=p(0)+q(0)+rS(Tθ_1,Tθ_1,gθ_1)+kS(Tθ_1,Tθ_1,gθ_1)
=(r+k)S(Tθ_1,Tθ_1,gθ_1).
Therefore (1-(r+k))S(Tθ_1,Tθ_1,gθ_1)$\leq$ 0 and hence gθ_1=Tθ_1, since 0 $\leq$r+k$\leq$p+q+r+2k<1. Now let us show that Rθ_2=fθ_2.
For each n$\in \mathbb{N}$, we have
S(gβ_n,gβ_n,fθ_2)$\leq$ pS(Rθ_2,Rθ_2,fθ_2)+qS(Tβ_n,Tβ_n,Rθ_2)
+rS(gβ_n,gβ_n,Tβ_n)+ k[S(Tβ_n,Tβ_n,fθ_2)+S(gβ_n,gβ_n,Rθ_2)].
Letting n$\to\infty$, we have
S(γ,γ,fθ_2)$\leq$ pS(Rθ_2,Rθ_2,fθ_2)+qS(γ,γ,γ)
+rS(γ,γ,γ)+ k[S(γ,γ,fθ_2)+S(γ,γ,γ)].
=pS(γ,γ,fθ_2)+q(0)+r(0)+kS(γ,γ,fθ_2)
=(p+k)S(γ,γ,fθ_2).
Therefore we have (1-(p+k))S(γ,γ,fθ_2) $\leq$ 0.It follows that S(γ,γ,fθ_2)=0, since 0 $\leq$p+k$\leq$p+q+r+2k<1.Thus f$\theta_2 = \gamma$ and and hence fθ_2=Rθ_2= gθ_1=Tθ_1=γ.Since the pair (f,R) is weakly compatible and fθ_2=Rθ_2, fRθ_2=Rfθ_2 and hence fγ=Rγ.Now (g,T) is weakly compatible and gθ_1=Tθ_1 imply that Tgθ_1=Tgθ_1 and hence Tγ=gγ.
Now let us show that γ is a common fixed point of f and R.
For this, we consider
S(gθ_1,gθ_1,fγ)$\leq$ pS(Rγ,Rγ,fγ)+qS(Tθ_1,Tθ_1,Rγ)
+rS(gθ_1,gθ_1,Tθ_1)+ k[S(Tθ_1,Tθ_1,fγ)+S(gθ_1,gθ_1,Tγ)].
=pS(fγ,fγ,fγ)+qS(gθ_1,gθ_1,fγ)
+rS(gθ_1,gθ_1,gθ_1)+ k[S(gθ_1,gθ_1,fγ)+S(gθ_1,gθ_1,fγ)]
=(q+2k) S(gθ_1,gθ_1,fγ).
Therefore we must have (1-(q+2k)) S(gθ_1,gθ_1,fγ)$\leq$ 0. Since 0 $\leq$q+2k$\leq$p+q+r+2k<1, S(gθ_1,gθ_1,fγ)$\leq$ 0.It follows that S(gθ_1,gθ_1,fγ)=0.This will imply that gθ_1=fγ and hence fγ=Rγ=γ.
This shows that γ is a common fixed point of f and R.
Now we show that γ is a common fixed point of g and T.
For this this, we consider
S(gγ,gγ,fθ_2)$\leq$ pS(Rθ_2,Rθ_2,fθ_2)+qS(Tγ,Tγ,Rθ_2)
+rS(gγ,gγ,Tγ)+ k[S(Tγ,Tγ,fθ_2)+S(gγ,gγ,Rθ_2)].
=pS(fθ_2,fθ_2,fθ_2)+qS(gγ,gγ,γ)+rS(gγ,gγ,gγ)
+k[S(gγ,gγ,γ)+S(gγ,gγ,γ)].
=(q+2k)S(gγ,gγ,γ).
S(gγ,gγ,γ)$\leq$(q+2k)S(gγ,gγ,γ).This will imply that (1-(q+2k))S(gγ,gγ,γ)$\leq$ 0.Since 0 $\leq$q+2k$\leq$p+q+r+2k<1,we have S(gγ,gγ,γ)=0.This implies that gγ=γ and hence Tγ= gγ=γ.This shows that γ is a common fixed point of g and T.Hence γ is a common fixed point of f,g,R and T.
Now let us show the uniqueness of common fixed point of f,g,R and T.For this, let δ be another common fixed point of f,g,R and T.Then fδ=gδ=Rδ=Tδ=δ.Now we consider

$S(\gamma,\gamma,\delta)=S(g\gamma,g\gamma,f\delta)\leq pS(R\delta,R\delta,f\delta)+qS(T\gamma,T\gamma,R\delta)$
$+rS(g\gamma,g\gamma,T\gamma)+k[S(T\gamma,T\gamma,f\delta)+S(g\gamma,g\gamma,R\delta)].$
$=pS(\delta,\delta,\delta)+qS(\gamma,\gamma,\delta)+rS(\gamma,\gamma,\gamma)$
$+k[S(\gamma,\gamma,\delta)+S(\gamma,\gamma,\delta)].$
$=(p+2k)S(\gamma,\gamma,\delta).$
Therefore $S(\gamma,\gamma,\delta)\leq (p+2k)S(\gamma,\gamma,\delta)$.This will imply that $(1-(p+2k))S(\gamma,\gamma,\delta)\leq 0$.Since $0\leq p+2k\leq p+q+r+2k<1$, we have $S(\gamma,\gamma,\delta)=0$.This will imply that $\gamma=\delta$ and hence the result proved.

□

The following examples illustrate Theorem 3.1 and Theorem 3.6 respectively.

4. Examples

Example 4.1. *Consider a S-metric space (X,S),* *where* $X=[0,1]$ *and* $S(\alpha,\beta,\gamma)=\begin{cases}0, for\ \alpha=\beta=\gamma\\ max\{\alpha,\beta,\gamma\},\ otherwise\end{cases}$ *for all* $\alpha,\beta,\gamma\in X$
Define four self maps f,g,,R and T on X as follows:
For $\alpha\in X$, $f\alpha=\frac{\alpha}{4}$, $g\alpha=\frac{\alpha}{4}$, $T\alpha=\alpha$ *and* $R\alpha=\frac{\alpha}{2}$.*We also define* $\phi:[0,\infty)\to[0,\infty)$ *by* $\phi(x)=\frac{x}{2}$ *for* $\alpha\in[0,\infty)$.*Clearly* ϕ *is continuous on* $[0,\infty)$ *satisfying* $\phi(0)=0$ *and* $0<\phi(x)<x$ *for all* $x>0$.*Let* $\alpha,\beta\in X$. *Now consider the following cases.*
Case(i): Let $\alpha<\beta$.*Then we have*
$S(f\alpha,f\alpha,g\beta)=max\{\frac{\alpha}{4},\frac{\alpha}{4},\frac{\beta}{4}\}=\frac{1}{4}\ max\ \{\alpha,\beta,\gamma\}=\frac{\beta}{4}$
and $\phi(S(R\alpha,R\alpha,T\beta))=\frac{1}{2}S(R\alpha,R\alpha,T\beta)=\frac{1}{2}\ max\{\frac{\alpha}{2},\frac{\alpha}{2},\beta\}$
$=\frac{\beta}{2}$, *since* $\frac{\alpha}{2}<\frac{\beta}{2}<\beta$.
Therefore $S(f\alpha,f\alpha,g\beta)\leq\phi(S(R\alpha,R\alpha,T\beta))$.
Now consider the case $\alpha>\beta$. *This will imply that*
$S(f\alpha,f\alpha,g\beta)=max\{\frac{\alpha}{4},\frac{\alpha}{4},\frac{\beta}{4}\}=\frac{1}{4}\ max\ \{\alpha,\beta,\gamma\}=\frac{\alpha}{4}$
and $\phi(S(R\alpha,R\alpha,T\beta))=\frac{1}{2}S(R\alpha,R\alpha,T\beta)=\frac{1}{2}\ max\{\frac{\alpha}{2},\frac{\alpha}{2},\beta\}$.
subcase(i) : *Let* $\frac{\alpha}{2}>\beta$.*Then we must have* $\phi(S(R\alpha,R\alpha,T\beta))=\frac{1}{2}(\frac{\alpha}{2})=\frac{\alpha}{4}\geq S(R\alpha,R\alpha,T\beta)$.
subcase(ii) : *Let* $\frac{\alpha}{2}<\beta$. *Then we have* $\phi(S(R\alpha,R\alpha,T\beta))=\frac{\beta}{2}>\frac{\alpha}{4}=S(R\alpha,R\alpha,T\beta)$.*From both cases, we conclude that* $S(f\alpha,f\alpha,g\beta)\leq\phi(S(R\alpha,R\alpha,T\beta))$ *for all* $\alpha,\beta\in X$.
Case(ii): Note that $fX=[0,\frac{1}{4}]$,$gX=[0,\frac{1}{4}]$,$RX=[0,\frac{1}{2}]$ *and* $TX=X$. *This will imply that* $fX\subset TX$ *and* $gX\subset RX$.
Case(iii): Now let us show that the pairs (f,R) and (g,T) are weakly compatible.For this, let $T\alpha=g\alpha$ *for* $\alpha\in X$.*Then* $\alpha=\frac{\alpha}{4}$.*It follows that* $\alpha=0$.*Now we consider,* $Tg(\alpha)=T(g\alpha)=T(0)=0$ *and* $gT(\alpha)=g(T\alpha)=g(0)=0$.*Therefore the pair (g,T) is weakly compatible.*
Now let $R\alpha=f\alpha$ *for* $\alpha\in X$.*This implies that* $\frac{\alpha}{2}=\frac{\alpha}{4}$ *and hence* $\alpha=0$.*Now we consider,* $fR(\alpha)=f(R\alpha)=f(0)=0=Rf(\alpha)$.*It follows that the pair (f,R) is also weakly compatible.*
Case(iv): Now we show that the pair (g,T) satisfies E.A property.For this, Consider $\alpha_n=\frac{1}{2n+1}$ *for* $n\in\mathbb{N}$.*Clearly* (α_n) *is in X and note that* $T\alpha_n=\alpha_n=\frac{1}{2n+1}$ *and* $g\alpha_n=\frac{\alpha_n}{4}=\frac{1}{4(2n+1)}$ *for all* $n\in\mathbb{N}$.*This will imply that* $S(T\alpha_n,T\alpha_n,0)=S(\frac{1}{2n+1},\frac{1}{2n+1},0)$
$=max\{\frac{1}{2n+1},\frac{1}{2n+1},0\}=\frac{1}{2n+1}\to 0$.*This shows that* $T\alpha_n\to 0$, *as* $n\to\infty$.
Also note that $S(g\alpha_n,g\alpha_n,0)=S(\frac{1}{4(2n+1)},\frac{1}{4(2n+1)},0)$
$=max\{\frac{1}{4(2n+1)},\frac{1}{4(2n+1)},0\}\to 0.$
This shows that $g\alpha_n\to 0$, *as* $n\to\infty$.*Thus there exists a sequence* (α_n) *in X such that* $g\alpha_n\to 0$ *and* $T\alpha_n\to 0$.*Hence the pair (g,T) satisfies E.A property.*
Case(v): As $fX=[0,\frac{1}{4}]$,*then fX is complete subeset of X.*
Therefore all conditions of Theorem 3.1 are satisfied and f,g,R and T have a unique common fixed point,namely zero.

Example 4.2. *Consider a S-metric space (X,S),* *where* $X=[0,4]$ *and* $S(\alpha,\beta,\gamma)=\begin{cases}0, for\ \alpha=\beta=\gamma\\ max\{\alpha,\beta,\gamma\},\ otherwise\end{cases}$ *for all* $\alpha,\beta,\gamma\in X$
We define four self maps f,g,,R and T on X by $\alpha\in X$, $f\alpha=\frac{\alpha}{2}$, $g\alpha=\frac{\alpha}{2}$, $T\alpha=\alpha$ *and* $R\alpha=\alpha$ *for* $\alpha\in X$.
Case(i): We show that (f,R) and (g,T) satisfy (CLR_{RT}) *property.For this, we choose* $\alpha_n=\frac{1}{n}$ *and* $\beta_n=\frac{1}{2n+3}$ *for* $n\in\mathbb{N}$.*Clearly* (α_n) *and* (β_n) *are in X.Then we have*
$S(R\alpha_n,R\alpha_n,0)=S(\frac{1}{n},\frac{1}{n},0)=max\{\frac{1}{n},\frac{1}{n},0\}=\frac{1}{n}\to 0$, *as* $n\to\infty$.*Also we have* $S(f\alpha_n,f\alpha_n,0)=S(\frac{1}{2n},\frac{1}{2n},0)=max\{\frac{1}{2n},\frac{1}{2n},0\}=\frac{1}{2n}\to 0$, *as* $n\to\infty$. *Similarly, we get that*
$S(g\beta_n,g\beta_n,0)=S(\frac{1}{2(2n+3)},\frac{1}{2(2n+3)},0)=max\{\frac{1}{2(2n+3)},\frac{1}{2(2n+3)},0\}$
$=\frac{1}{2(2n+3)}\to 0$ *and*
$S(T\beta_n,T\beta_n,0)=S(\frac{1}{2n+3},\frac{1}{2n+3},0)=max\{\frac{1}{2n+3},\frac{1}{2n+3},0\}$
$=\frac{1}{2n+3}\to 0$, *as* $n\to\infty$.
Since R0=0=T0,we have $0\in RX\cap TX$.
Therefore there exist sequences (α_n) *and* (β_n) *in X such that* $\lim_{n\to\infty}R(\alpha_n)=\lim_{n\to\infty}f(\alpha_n)=\lim_{n\to\infty}T(\beta_n)=\lim_{n\to\infty}g(\beta_n)$.*Therefore (f,R) and (g,T) satisfy* (CLR_{RT}) *property.*
Case(ii): Let us compute the following.
$S(g\beta,g\beta,f\alpha)=S(\frac{\beta}{2},\frac{\beta}{2},\frac{\alpha}{2})=max\{\frac{\beta}{2},\frac{\beta}{2},\frac{\alpha}{2}\}=\frac{1}{2}\ max\{\beta,\beta,\alpha\}.$
$S(R\alpha,R\alpha,f\alpha)=S(\alpha,\alpha,\frac{\alpha}{2})=max\{\alpha,\alpha,\frac{\alpha}{2}\}=max\{\alpha,\alpha,\frac{\alpha}{2}\}=\alpha.$
$S(T\beta,T\beta,R\alpha)=S(\beta,\beta,\alpha)=max\{\beta,\beta,\alpha\}.$
$S(g\beta,g\beta,T\beta)=S(\frac{\beta}{2},\frac{\beta}{2},\beta)=max\{\alpha,\alpha,\frac{\alpha}{2}\}=max\{\frac{\beta}{2},\frac{\beta}{2},\beta\}=\beta.$
$S(T\beta,T\beta,f\alpha)+S(g\beta,g\beta,R\alpha)=max\{\beta,\beta,\frac{\alpha}{2}\}+max\{\frac{\beta}{2},\frac{\beta}{2},\alpha\}.$
Now consider the following cases.We start with $\alpha>\beta$.
Let us consider
$S(g\beta,g\beta,f\alpha)=\frac{1}{2}\alpha\leq\frac{5}{8}\alpha\leq\frac{2}{8}\ max\{\alpha,\alpha,\frac{\alpha}{2}\}+\frac{2}{8}\ max\{\beta,\beta,\alpha\}$
$+\frac{1}{8}max\{\frac{\beta}{2},\frac{\beta}{2},\beta\}+\frac{1}{8}[max\{\beta,\beta,\frac{\alpha}{2}\}+max\{\frac{\beta}{2},\frac{\beta}{2},\alpha\}]$
$\leq\frac{2}{8}\ S(R\alpha,R\alpha,f\alpha)+\frac{2}{8}S(T\beta,T\beta,R\alpha)$
$+\frac{1}{8}S(g\beta,g\beta,T\beta)+\frac{1}{8}[S(T\beta,T\beta,f\alpha)+S(g\beta,g\beta,R\alpha)]$
Therefore $S(g\beta,g\beta,f\alpha)\leq\frac{2}{8}\ S(R\alpha,R\alpha,f\alpha)+\frac{2}{8}S(T\beta,T\beta,R\alpha)+\frac{1}{8}S(g\beta,g\beta,T\beta)+\frac{1}{8}[S(T\beta,T\beta,f\alpha)+S(g\beta,g\beta,R\alpha)]$
By choosing $p=\frac{2}{8}$,$q=\frac{2}{8}$,$r=\frac{1}{8}$,$k=\frac{1}{8}$, *we have* $p,q,r,k\in[0,1)$ *and* $0\leq p+q+r+2k=\frac{2}{8}+\frac{2}{8}+\frac{1}{8}+\frac{2}{8}=\frac{7}{8}<1$.*Hence*
$S(g\beta,g\beta,f\alpha)\leq pS(R\alpha,R\alpha,f\alpha)+qS(T\beta,T\beta,R\alpha)$
$+rS(g\beta,g\beta,T\beta)+k[S(T\beta,T\beta,f\alpha)+S(g\beta,g\beta,R\alpha)]$
Let $\alpha<\beta$.*Now we consider*
$S(g\beta,g\beta,f\alpha)=\frac{1}{2}\ max\{\beta,\beta,\alpha\}=\frac{\beta}{2}$

$\leq \frac{1}{8}$ *max*$\{\alpha, \alpha, \frac{\alpha}{2}\}$+$\frac{2}{8}$ *max*$\{\beta, \beta, \alpha\}$+
$\frac{2}{8}$*max*$\{\frac{\beta}{2}, \frac{\beta}{2}, \beta\}$+ $\frac{1}{8}$[*max*$\{\beta, \beta, \frac{\alpha}{2}\}$+*max*$\{\frac{\beta}{2}, \frac{\beta}{2}, \alpha\}$]
$\leq \frac{1}{8}$ $S(R\alpha,R\alpha,f\alpha)+\frac{2}{8}S(T\beta,T\beta,R\alpha)+\frac{2}{8}$ $S(g\beta,g\beta,T\beta)$
$+\frac{1}{8}[S(T\beta,T\beta,f\alpha)+S(g\beta,g\beta,R\alpha)]$

Therefore $S(g\beta,g\beta,f\alpha)\leq \frac{1}{8}$ $S(R\alpha,R\alpha,f\alpha)+\frac{2}{8}S(T\beta,T\beta,R\alpha)$
$+\frac{2}{8}S(g\beta,g\beta,T\beta)+$ $\frac{1}{8}[S(T\beta,T\beta,f\alpha)+S(g\beta,g\beta,R\alpha)]$.
By taking $p=\frac{1}{8},q=\frac{2}{8},r=\frac{2}{8},k=\frac{1}{8}$, *we must have*
$p,q,r,k\in [0,1)$ *and* $0 \leq p+q+r+2k=\frac{1}{8}+\frac{2}{8}+\frac{2}{8}+\frac{2}{8}=\frac{7}{8}<1$.
such that $S(g\beta,g\beta,f\alpha)\leq pS(R\alpha,R\alpha,f\alpha)+qS(T\beta,T\beta,R\alpha)$
$+rS(g\beta,g\beta,T\beta)+$ $k[S(T\beta,T\beta,f\alpha)+S(g\beta,g\beta,R\alpha)]$.
From both cases, we conclude that there exist p,q,r and k in $[0,1)$ *with* $0 \leq p+q+r+2k<1$ *such that*
$S(g\beta,g\beta,f\alpha)\leq pS(R\alpha,R\alpha,f\alpha)+qS(T\beta,T\beta,R\alpha)$
$+r\, S(g\beta,g\beta,T\beta)+$ $k[S(T\beta,T\beta,f\alpha)+S(g\beta,g\beta,R\alpha)]$ *for all* $\alpha,\beta\in X$.

Case(iii): Now we show that the pairs (f,R) and (g,T) are weakly compatible.For this, let $f\alpha=R\alpha$ *for* $\alpha\in X$.*Then* $\frac{\alpha}{2}=\alpha$.*It follows that* $\alpha=0$.*This will imply that* $Rf\alpha=R(f\alpha)=R(0)=0$ *and* $fR\alpha=0$.*Thus* $Rf\alpha=fR\alpha$ *whenever* $f\alpha=R\alpha$ *for* $\alpha\in X$.*This shows that the (f,R) is weakly compatible.Similarly, one can easily show that the pair (g,T) is also weakly compatible.*

Therefore the hypothesis of Theorem 3.6 is satisfied and f,g,R and T have a common unique fixed point,namely zero.

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