Malaya Journal of Matematik, Vol. 8, No. 3, 1284-1290, 2020

https://doi.org/10.26637/MJM0803/0097

A type of strongly regular gamma rings

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Abstract

In this study, using Yuan and Lee's [13] explanation of fuzzy group founded on fuzzy binary operation and Aktas and Cagman [2] definition of fuzzy ring, we give a innovative caring of explanation to $(A:B)$. The idea of fuzzy regular and fuzzy left strongly regular are presented and we make a hypothetical learning on their elementary belongings equivalent to those of ordinary rings. We have presented that if (R,G,H) is strongly regular, then for any a in R, left annihilator of "a" is an ideal.

Keywords

Ring theory, regular rings, ideal in associative algebras, fuzzy algebraic structures.

AMS Subject Classification

26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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Article History: Received **12** June **2020**; Accepted **29** July **2020**

1. Introduction

The study of fuzzy algebraic structure has been followed in many directions such as groups, rings, modulus, vector spaces, topology and so on. In 1965, the American cyberneticist L. A. Zadeh [14] familiarized the concept of fuzzy subsets and studied their properties on the similar lines to set theory. In 1967, Rosenfeld [10] clear the fuzzy subgroup and gave some of its properties. Rosenfeld's definition of a fuzzy group is a revolving argument for pure Mathematicians. In the definition of fuzzy subgroups, Rosenfeld has expected that the subsets of a group G are fuzzy and the binary operation on G is nonfuzzy in the traditional sense. Additional tactic is to accept that the set is non fuzzy or traditional and the binary operation is fuzzy in the fuzzy sense. Conflicting a slight gone from this approach, Demirci [4, 5] presented the concept of smooth group by the use of fuzzy binary operation and the concept of fuzzy equality, and Aktas [1] studied its application using Demirci's concept. Yuan and Lee [10] introduced a new kind of fuzzy group based on fuzzy binary operation and Aktas and Cagman's [2] definition of fuzzy ring, we give a different definition to $(A:B)$. The concept of fuzzy regular and fuzzy left strongly regular are introduced and we obtain similar results for rings analogous to ordinary ring theory. In this paper our intention is to present the well-known explanation of $(A:B)$ in a fuzzy gamma ring R. In a gamma ring (M,Γ,R,S) and if A and B are subsets of R then

$$(A:B)=\{x\in M/\bigvee s(x,\alpha,b,\beta,y)>\theta\Rightarrow y\in A\}$$

It has been show that (A:B) is an ideal of ring M if A and B are fuzzy ideal of gamma ring M. We extend all the results obtained for $(A:B)$ for classical ring to $(A:B)$ in a fuzzy gamma ring M.

2. Preliminaries

Let $\theta\in[0,1)$. In this section the primary explanation is brief for understanding the ideas better, that is essential in this paper. Most of the insides of this section are contained in [2], [3], [6], [10].

Definition 2.1. *[3, Definition 2] If $M=\{a,b,c...\}$ and $\Gamma=\{\alpha,\beta,\gamma...\}$ are additive abelian groups and for all a,b, $c\in M$ and for all $\alpha,\beta\in\Gamma$ the following conditions are satisfied*

1. $a\alpha b\in M$.
2. $(a+b)\alpha c=a\alpha c+b\alpha c, a(\alpha+\beta)b=a\alpha b+a\beta b$, $a\alpha(b+c)=a\alpha b+a\alpha c$.
3. $(a\alpha b)\beta c=a\alpha(b\beta c)$.

Then M is called a $\Gamma-$ring.

Definition 2.2. [13, Definition 2.2] *Let G be a nonempty set and R be a fuzzy subset of $G\times G\times G(R:G\times G\times G\longrightarrow[0,1])$. R is called a fuzzy binary operation on G if*

1. $\forall\, a,b \in G, \exists c \in G$ *such that* $R(a,b,c) > \theta$
2. $\forall a,b,c_1,c_2 \in G, R(a,b,c_1) > \theta$ *and* $R(a,b,c_2) > \theta$ *implies* $c_1 = c_2$.

Let R be a fuzzy binary operation on G, then we have a mapping

$$R : F(G) \times F(G) \to F(G)$$
$$(A : B) \mapsto R(A,B)$$

where $F(G) = \{A|A : R \to [0,1] \text{is a mapping}\}$ and for $A,B \in F(G), R(A,B)$ is defined by

$$R(A,B)(c) = \bigvee_{a,b\in G} (A(a) \wedge B(b) \wedge R(a,b,c)), \forall C \in G \quad (2.1)$$

Let $A = \{a\}$ and $B = \{b\}$, and let $R(A,B)$ be denoted as $a \circ b$, then

$$(a \circ b)(c) = R(a,b,c), \forall c \in G \quad (2.2)$$

$$((a \circ b) \circ c)(z) = \bigvee_{d\in G} (R(a,b,d) \wedge R(d,c,z)), \forall z \in G \quad (2.3)$$

$$(a \circ (b \circ c))(z) = \bigvee_{d\in G} (R(b,c,d) \wedge R(a,d,z)), \forall z \in G \quad (2.4)$$

Definition 2.3. [13, Definition 2.3] *Let G be a nonempty set and R be a fuzzy binary operation on G. Then* (G,R) *is called a fuzzy abelian group if the following conditions are true:*

(G1) $\forall\, a,b,c,z_1,z_2 \in G, ((a \circ b) \circ c)(z_1) > \theta$ *and* $(a \circ (b \circ c))(z_2) > \theta$ *implies* $z_1 = z_2$*;*

(G2) $\exists e_0 \in G$ *such that* $(e_0 \circ a)(a) > \theta$ *and* $(a \circ e_0)(a) > \theta$ *for any* $a \in G$ *(*e_0 *is called an identity element of G);*

(G3) $\forall\, a \in G, \exists b \in G$ *such that* $(a \circ b)(e_0) > \theta$ *and* $(b \circ a)(e_0) > \theta$ *(b is called an inverse element of a and is denoted as* a^{-1}*).*

(G4) $\forall\, a,b \in G, (a \circ b)(z_1) > \theta$ *and* $(b \circ a)(z_2) > \theta$ *implies* $z_1 = z_2$ *.*

Results: Let M and Γ be a nonempty sets, R_M a fuzzy binary operation on M and R_Γ on Γ. Hence, R_M is a fuzzy subset of $M \times M \times M$ and R_Γ a fuzzy subset of $\Gamma \times \Gamma \times \Gamma$. We assume throughout that the value of θ is same for R_M and R_Γ.

Let (M,R_M) and (Γ,R_Γ) be fuzzy groups. We now define a new fuzzy binary operation S on (M,Γ) which is a fuzzy subset of $M \times \Gamma \times M \times \Gamma \times M$.

Definition 2.4. [6, Definition 3.1] *Let M and* Γ *be two nonempty sets and S a fuzzy subset of* $M \times \Gamma \times M \times \Gamma \times M$*. Then S is called a fuzzy binary operation on* (M,Γ) *if*

1. $\forall\, a,b \in M, \forall\, \alpha,\beta \in \Gamma, \exists c \in M$ *such that* $S(a,\alpha,b,\beta,c) > \theta$
2. $\forall\, a,b,c_1,c_2 \in M, \forall \gamma \in \Gamma,$

$$\bigvee_{\beta\in\Gamma} S(a,\gamma,b,\beta,c_1) > \theta$$

and

$$\bigvee_{\beta\in\Gamma} S(a,\gamma,b,\beta,c_2) > \theta$$

implies $c_1 = c_2$*, where* $\theta \in [0,1)$ *is as above for* R_M *and* R_Γ*.*

Let S be a fuzzy binary operation on (M,Γ)*. Then we may regard S as the mapping*

$$S : F(M) \times F(\Gamma) \times F(M) \to F(M), (A,G,B) \to S(A,G,B)$$

where
$F(M) = \{A|A : M \to [0,1] \textit{is a mapping}\}$
$F(\Gamma) = \{G|G : \Gamma \to [0,1] \textit{is a mapping}\}$*and*

$$S(A,G,B)(c) = \bigvee_{a,b\in M} \bigvee_{\alpha,\beta\in\Gamma} (A(a) \wedge G(\alpha) \wedge B(b) \wedge S(a,\alpha,b,\beta,c)), \forall c \in M.$$

Let $A = \{a\}, B = \{b\}, G = \{\alpha\}$ *and* $G' = \{\alpha'\}$*. Let* $R_M(A,B)$*,* $R_\Gamma(G,G')$ *and* $S(A,G,B)$ *be denoted by* $a \circ b, \alpha \circ \alpha'$ *and* $a * \alpha * b$*, respectively. We will use the following notation to simplify the calculation*

$$(a * \alpha * b)(c) = \bigvee_{\alpha,\alpha'\in\Gamma} (S(a,\alpha,b,\alpha',c)), \forall c \in M \quad (2.5)$$

$$((a * \alpha * b) * \beta * c)(z) = \bigvee_{d\in M,\alpha',\beta'\in\Gamma} (S(a,\alpha,b,\alpha',d) \wedge S(d,\beta,c,\beta',z)) \quad (2.6)$$

$$(a * \alpha * (b * \beta * c))(z) = \bigvee_{d\in M,\alpha',\beta'\in\Gamma} (S(b,\beta,c,\alpha',d) \wedge S(a,\alpha,d,\beta',z)) \quad (2.7)$$

$$(a * \alpha * (b \circ c))(z) = \bigvee_{d\in M,\alpha',\in\Gamma} (R_M(b,c,d) \wedge S(a,\alpha,d,\alpha',z)) \quad (2.8)$$

$$((a * \alpha * b) \circ (a * \alpha * c))(z) = \bigvee_{d,e\in M,\alpha',\beta'\in\Gamma} (S(a,\alpha,b,\alpha',d) \wedge S(a,\alpha,c,\beta',e) \wedge (R_M(d,e,z)) \quad (2.9)$$

$$(a * (\alpha \circ \beta) * b)(c) = \bigvee_{\gamma,\alpha',\in\Gamma} (R_\Gamma(\alpha,\beta,\gamma) \wedge S(a,\gamma,b,\alpha',c)) \quad (2.10)$$

$$((a*\alpha*b)\circ(a*\beta*b))(c) = \bigvee_{d,e\in M,\alpha',\beta'\in\Gamma}(S(a,\alpha,b,\alpha',d) \wedge S(a,\beta,b,\beta',e) \wedge (R_M(d,e,c)) \quad (2.11)$$

$$((a\circ b)*\alpha*c)(z) = \bigvee_{d\in M,\alpha',\in\Gamma}(R_M(a,b,d)\wedge S(d,\alpha,c,\alpha',z)) \quad (2.12)$$

$$((a*\alpha*c)\circ(b*\alpha*c))(z) = \bigvee_{d,e\in M,\alpha',\beta'\in\Gamma}(S(a,\alpha,c,\alpha',d) \wedge S(b,\alpha,c,\beta',e) \wedge (R_M(d,e,z)) \quad (2.13)$$

Definition 2.5. [6, Definition 3.2] *Let M and Γ be nonempty sets, R_M, R_Γ and S fuzzy binary operations on M, Γ and (M,Γ), respectively, all with the same value of θ. To simplify the notation, from now on we denote both R_M, R_Γ by R. Then (M,Γ,R,S) is a fuzzy gamma ring if the following condition hold.*

1. (M,R) *and* (Γ,R) *are abelian fuzzy groups,*
2. $\forall a,b,c,z_1,z_2 \in M, \forall\ \gamma,\beta \in \Gamma, ((a*\gamma*b)*\beta*c)(z_1) > \theta$ *and* $((a*\gamma*(b*\beta*c))(z_2) > \theta$ *implies* $z_1 = z_2$.
3. $\forall a,b,c,z_1,z_2 \in M, \forall\ \gamma,\beta \in \Gamma, (a*\gamma*(b\circ c))(z_1) > \theta$ *and* $((a*\gamma*b)\circ(a*\gamma*c))(z_2) > \theta$ *implies* $z_1 = z_2$.
4. $\forall a,b,c,z_1,z_2 \in M, \forall\alpha,\beta \in \Gamma, (a*(\gamma\circ\beta)*b)(z_1) > \theta$ *and* $((a*\gamma*b)\circ(a*\beta*b))(z_2) > \theta$ *implies* $z_1 = z_2$.
5. $\forall a,b,c,z_1,z_2 \in M, \forall\alpha,\beta \in \Gamma, ((a\circ b)*\gamma*c)(z_1) > \theta$ *and* $((a*\gamma*c)\circ(b*\gamma*c))(z_2) > \theta$ *implies* $z_1 = z_2$.

The identity element of the fuzzy group (M,R) is called the zero element of (M,Γ,R,S), and it is denoted by e_0.

Proposition 2.6. [13, Proposition 2.1] *Let (G,R) be a fuzzy group, then*

1. *the identity element of G is unique;*
2. $(a\circ a)(a) > \theta$ *implies* $a = e$;
3. $(a\circ b)(d) > \theta$ *and* $(a\circ c)(d) > \theta$ *implies* $b = c$;
4. $(b\circ a)(d) > \theta$ *and* $(c\circ a)(d) > \theta$ *implies* $b = c$;
5. *for each $a \in G$, the inverse element of a is unique.*
6. $(a^{-1})^{-1} = a$;
7. $(b^{-1}\circ a^{-1})(c) > \theta$ *and* $(a\circ b)(d) > \theta$ *implies* $c = d^{-1}$.

Theorem 2.7. *If e_0 is the zero element of (M,R) in the fuzzy ring (M,Γ,R,S) then for any $a \in M, \gamma \in \Gamma, (a*\gamma*e_0)(x) > \theta$ implies $x = e_0$.*

Proof. Let $a,z \in M, \gamma \in \Gamma$.
Now

$$(a*\gamma*(e_0\circ e_0)(z) = \bigvee_{d\in M,\alpha'\in\Gamma}(R_e(e_0,e_0,d))\wedge S(a,\gamma,d,\gamma',z))$$

$$((a*\gamma*e_0)\circ(a*\gamma*e_0))(z) = \bigvee_{d\in M,\alpha',\beta'\in\Gamma}(S(a,\gamma,e_0,\alpha',d) \wedge S(a,\gamma,e_0,\beta',e) \wedge R(d,e,z))$$

Let $x \in M$ such that $S(a,\gamma,e_0,\alpha',x) > \theta$. Let us show that $x = e_0$.

Now $(a*\gamma*(e_0\circ e_0))(x) > \theta$, since $S(a,\gamma,e_0,\alpha',x) > \theta$ and $R(e_0,e_0,e_0) > \theta$.
Hence $((a*\gamma*e_0))\circ(a*\gamma*e_0)(x) > \theta$.

Thus $S(a,\gamma,e_0,\alpha',x)\wedge S(a,\gamma,e_0,\beta',x)\wedge R(x,x,x) > \theta$. Hence $R(x,x,x) > \theta$. □

Theorem 2.8. [6, Definition 3.4] *Let (M,Γ,R,S) be a fuzzy gamma ring, $a,b,c \in M$ and $\gamma \in \Gamma$. Then*

1. (i) $(a*\gamma*b)(b) > \theta$ *and* $(a*\gamma*b)(e_0) > \theta$ *implies* $b = e_0$.
   (ii) $(b*\gamma*a)(a) > \theta$ *and* $(b*\gamma*a)(e_0) > \theta$ *implies* $a = e_0$.
2. *Let b^{-1} be the inverse of b in (M,R), then*
   (i) $(a*\gamma*b^{-1})(v) > \theta$ *and* $(a*\gamma*b)(w) > \theta$ *implies* $v = w^{-1}$.
   (ii) $(a^{-1}*\gamma*b)(u) > \theta$ *and* $(a*\gamma*b)(s) > \theta$ *implies* $u = s^{-1}$.
   (iii) $(a^{-1}*\gamma*b^{-1})(t) > \theta$ *and* $(a*\gamma*b)(r) > \theta$ *implies* $t = r$.
3. (i) $(a*\gamma*(b\circ c^{-1})(z_1) > \theta$ *and* $((a*\gamma*b)\circ(a*\gamma*c^{-1}))(z_2) > \theta$ *implies* $z_1 = z_2$.
   (ii) $((a\circ b^{-1})*\gamma*c)(z_1) > \theta$ *and* $((a*\gamma*c)\circ(b^{-1}*\gamma*c))(z_2) > \theta$ *implies* $z_1 = z_2$.
   (iii) $(a*(\gamma\circ\beta^{-1})*b)(z_1) > \theta$ *and* $((a*\gamma*b)\circ(a*\beta^{-1}*b))(z_2) > \theta$ *implies* $z_1 = z_2$.

Definition 2.9. [6, Definition 3.12] *Let (M,Γ,R,S) be a fuzzy gamma ring. A nonempty subset I of M is called a left(right) fuzzy ideal of M, if for all $,b \in I$, all $n,m \in M$, and all $\gamma \in \Gamma$*

1. $(a\circ b)(m) > \theta$ *implies* $m \in I, a^{-1} \in I$.
2. $(n*\gamma*a)(m) > \theta$ *implies* $m \in I$
3. $((a*\gamma*n)(m) > \theta$ *implies* $m \in I$.

A nonempty set I of a fuzzy gamma ring (M,Γ,R,S) is called a fuzzy (two-sided) ideal of (M,Γ,R,S), if I is both a left and right ideal of (M,Γ,R,S).

3. Main Result

Definition 3.1. *Let A and B be any two subset of M then*

$$(A:B)=\{x\in M/\bigvee_{b\in B,\alpha,\beta\in\Gamma} S(x,\alpha,b,\beta,y)>\theta\Rightarrow y\in A\}$$

Now we are going to prove our main theorem:

Theorem 3.2. *Let (M,Γ,R,S) be a fuzzy gamma ring and A be any fuzzy ideal of M and B be any subset of M then $(A:B)$ is a fuzzy left ideal of (M,Γ,R,S).*

Proof. 1. Let $x_1,x_2\in(A:B)$. Suppose $(x_1\circ x_2)(z)>\theta$, Let us show that $z\in(A:B)$. Since $(x_1\circ x_2)(z)>\theta$ implies

$$R(x_1,x_2,z)>\theta. \tag{3.1}$$

Now, suppose

$$\bigvee_{b\in B,\alpha,\beta\in\Gamma}(S(z,\alpha,b,\beta,y))>\theta$$

Hence there exists a $b_1\in B$ such that

$$S(z,\alpha,b_1,\beta,y)>\theta \tag{3.2}$$

Now, consider

$$\begin{aligned}((x_1\circ x_2)*\alpha*b_1)(y)=&\bigvee_{d\in M,\beta\in\Gamma}(R(x_1,x_2,d)\\ &\wedge S(d,\alpha,b_1,\beta,y))\\ &\geq R(x_1,x_2,z)\\ &\wedge S(z,\alpha,b_1,\beta,y)>\theta\end{aligned}$$

(from (3.1) and (3.2))

Therefore $((x_1\circ x_2)*\alpha*b_1)(y)>\theta$.

Thus $((x_1*\alpha*b_1)\circ(x_2*\alpha_1b_1))(y)>\theta$.

Hence $(x_2*\alpha*b_1)$

$$\bigvee_{d,e\in M,\alpha',\beta'\in\Gamma}(S(x_1,\alpha,b_1,\alpha',d)\wedge S(x_2,\alpha,b_1,\beta',e)\wedge R(d,e,y))>\theta$$

By definition there exists unique $d_1\in R$ and $e_1\in R$ such that $S(x_1,\alpha,b_1,\alpha',$ $d_1>\theta, S(x_2,\alpha,b_1,\beta',e_1)>\theta$. Hence $R(d_1,e_1,y)>\theta$.

Since $x_1\in(A:B)$ and $S(x_1,\alpha,b_1,\alpha',d_1)>\theta$ implies $d_1\in A$. Since $x_2\in(A:B)$ and $S(x_2,\alpha,b_1,\beta',e_1)>\theta$ implies $e_1\in A$. Since $d_1,e_1\in A$ and A is a fuzzy ideal of M and $R(d_1,e_1,y)>\theta$ implies $y\in A$.

Thus

$$\bigvee_{b\in B,\alpha,\beta\in\Gamma} S(z,\alpha,b,\beta,y)>\theta\Rightarrow y\in A$$

Therefore $z\in(A:B)$

2. Let $x\in(A:B)$. Let us show that $x^{-1}\in(A:B)$. Suppose

$$\bigvee_{b\in B,\alpha,\beta\in\Gamma} S(x^{-1},\alpha,b,\beta,y)>\theta$$

Hence there exists $b_1\in B$ such that $S(x^{-1},\alpha,b_1,\beta,y)>\theta$. By Theorem 2.8, $S(x,\alpha,b_1,\beta,y^{-1})>\theta$ implies $y^{-1}\in A$. Since A is an ideal, $y\in A$. Therefore $x^{-1}\in(A:B)$.

3. Let $x\in(A:B)$ and let $n\in M$. Suppose $(n*\alpha*x)(z)>\theta$,

$$\bigvee_{\alpha'\in\Gamma} S(n,\alpha,x,\alpha',z)>\theta \tag{3.3}$$

Let us show that $z\in(A:B)$.

Suppose

$$\bigvee_{b\in B,\beta,\beta'\in\Gamma}(S(z,\alpha,b,\beta',y)>\theta)$$

Hence there exists $b_1\in B$ such that

$$S(z,\beta,b_1,\beta',y)>\theta \tag{3.4}$$

Now

$$\begin{aligned}[(n*\alpha*x)*\beta*b_1](y)=&\bigvee_{d\in M,\alpha',\ \beta'\in\Gamma}(S(n,\alpha,x,\alpha',d)\\ &\wedge S(d,\beta,b_1,\beta',y))\\ &\geq((S(n,\alpha,x,\alpha',z)\\ &\wedge S(z,\beta,b_1,\beta',y))>\theta\end{aligned}$$

(from (3.3) and (3.4))

Hence $[n*\alpha*(x*\beta*b_1)](y)>\theta$

Therefore

$$\bigvee_{d\in M,\alpha',\beta'\in\Gamma}(S(x,\beta,b_1,\alpha',d)\wedge S(n,\alpha,d,\beta',y))>\theta$$

By definition there exists a unique $d_1\in M$ such that $S(x,\beta,b_1,\alpha',d_1)>\theta$ and hence $S(n,\alpha,d_1,\beta',y)>\theta$.

Since $x\in(A:B)$ and $S(x,\beta,b_1,\alpha',d_1)>\theta$ implies $d_1\in A$.

Since $S(n,\alpha,d_1,\beta',y)>\theta$ implies $y\in A$. Therefore $z\in(A:B)$.

$(n*\alpha*x)(z)>\theta\Rightarrow z\in(A:B)$.

Therefore $(A:B)$ is a fuzzy left ideal.

□

Theorem 3.3. *If A and B are fuzzy ideals of (M,Γ,R,S) then $(A:B)$ is an ideal of (M,Γ,R,S).*

Proof. By theorem 3.1, $(A:B)$ is a left ideal of (M,Γ,R,S). Let us show that $(A:B)$ is a right ideal of (M,Γ,R,S). Let $x\in(A:B)$ and let $n\in M$. Suppose $(x*\alpha*n)(z)>\theta$.
Hence

$$\bigvee_{\alpha'\in\Gamma} S(x,\alpha,n,\alpha',z)>\theta \tag{3.5}$$

Let us show that $z\in(A:B)$. Suppose $\bigvee_{b\in B,\beta,\beta'\in\Gamma} S(z,\beta,b,\beta',y)>\theta$. Hence there exists $b_1\in B$ such that

$$S(z,\beta,b_1,\beta',y)>\theta \tag{3.6}$$

Now

$$\begin{aligned}&(x*\alpha*n)*\beta*b_1](y)\\&=\bigvee_{d\in M,\alpha',\beta'\in\Gamma}(S(x,\alpha,n,\alpha',d))\wedge S(d,\beta,b_1,\beta',y)\\&\geq(S(x,\alpha,n,\alpha',z)\wedge S(z,\beta,b_1,\beta',y)>\theta\end{aligned}$$

(from (3.5) and (3.6))

Hence $[x*\alpha*(n*\beta*b_1)](y)>\theta$

Therefore

$$\bigvee_{d\in M,\alpha',\beta'\in\Gamma}(S(n,\beta,b_1,\beta',d)\wedge S(x,\alpha,d,\alpha',y)>\theta$$

By definition there exists $d_1\in M$ such that $S(n,\beta,b_1,\beta',d_1)>\theta$ and $S(x,\alpha,d_1,\alpha',y)>\theta$

Since B is an ideal and $S(n,\beta,b_1,\beta',d_1)>\theta$ implies $d_1\in B$.

Since $d_1\in B$ and $S(x,\alpha,d,\alpha',y)>\theta$ implies $y\in A$.

Thus $z\in(A:B)$. $(x*\alpha*n)(z)>\theta$ implies $z\in(A:B)$.

Therefore $(A:B)$ is a fuzzy ideal. □

Definition 3.4. *For any $a\in M$, the left annihilator of $a\in M$ is a*

$$l(a)=\{x\in M/S(x,\alpha,a,\alpha',y)>\theta\Rightarrow y=e_0\}$$

Theorem 3.5. *For any $a\in M$ the left annihilator of "a" is a left ideal*

Proof. The proof can be done taking $A=\{e_0\}$ and $B=\{a\}$ in Theorem 3.1. □

4. Strongly regular rings

Definition 4.1. *M is said to be regular if for each element $a\in M$, there exists a $m\in M$ $\gamma_1,\gamma_2\in\Gamma$ such that*

$$((a*\gamma_1*m)*\gamma_2*a)(a)>\theta$$

.

Theorem 4.2. *Let (M,Γ,R,S) be a fuzzy gamma ring and I be a fuzzy ideal of (M,Γ,R,S). Then for any $a\in I$, we can find $x,y\in I$ such that $(x*\alpha*y)(a)\theta$.*

Proof. Let I be a fuzzy ideal of R and let $a\in I$. Since R is regular there exists $x\in R$ such that $((a*\gamma_1*m)*\gamma_2*a)(a)>\theta$.

Hence,

$$\bigvee(S(a,\gamma_1,m,\gamma_1',d)\wedge S(d,\gamma_2,a,\gamma_2',a))>\theta$$

By definition of H there exists a unique $d_1\in R$ such that $S(a,\gamma_1,m,\gamma_1',d_1)>\theta$ and $a\in I$ implies $d_1\in I$. Hence $S(d,\gamma_2,a,\gamma_2',a)>\theta$ where $d_1\in I$.
Therefore $(d_1*\gamma_2*a)(a)>\theta$ where $d_1,a\in I$. □

Definition 4.3. *M is said to be strongly regular if for each element $a,b\in M$, there exists a $m\in M$ $\alpha,\gamma_1,\gamma_2\in\Gamma$ such that $(a*\alpha*b)(z)>\theta\Leftrightarrow(m*\gamma_1*(a*\alpha*b)*\gamma_2*(a*\alpha*b))(z)>\theta$.*

Theorem 4.4. *Let (M,Γ,R,S) be a fuzzy gamma ring and e_* be the identity element of (M,Γ,S) and for any $a,b\in M$ and $\gamma_1,\gamma_2\in\Gamma$
$((a*\gamma_1*e_*)*\gamma_2*b)(z)>\theta$ implies $(a*\gamma_2*b)(z)>\theta$.*

Proof. Assume $((a*\gamma_1*e_*)*\gamma_2*b)(z)>\theta$.

Hence

$$\bigvee_{f\in M,\gamma_1',\gamma_2'}(S(a,\gamma_1,e_*,\gamma_1',f)\wedge S(f,\gamma_2,b\gamma_2',z))>\theta$$

We must have $f=a$ and $S(a,\gamma_2,b,\gamma_2',z)>\theta$.
Therefore $(a*\gamma_2*b)(z)>\theta$ □

Theorem 4.5. *If M is a left strongly regular then given $a\in(M,\Gamma,R,S)$ and $\alpha,\gamma_1,\gamma_2,\exists m\in M$ such that $(m*\gamma_1*(a*\gamma_2*a))(a)>\theta$.*

Proof. Assume M is left strongly regular. Let $a\in(M,\Gamma,R,S)$ and e_* be the identity element of (M,Γ,S). Now take b as e_*. Then there exists a $m\in M$ such that
$(a*\alpha*e_*)(z)>\theta\Leftrightarrow(m*\gamma_1*(a*\alpha*e_*)*\gamma_2*(a*\alpha*e_*))(z)>\theta$.
Since $(a*\alpha*e_*)(a)>\theta$ we have $(m*\gamma_1*(a*\alpha*e_*)*\gamma_2*(a*\alpha*e_*))(a)>\theta$ by Theorem 4.4, $(m*\gamma_1*a*\gamma_2*a)(a)>\theta$. □

Theorem 4.6. *Let I be a fuzzy ideal of a strongly regular ring (M,Γ,R,S) . For any $a\in M$, if $S(a,\gamma_1,a,\gamma_2,y)>\theta$ implies $y\in I$ then $a\in I$.*

Proof. Let I be a fuzzy ideal of M and let $a\in I$. Since M is a left strongly regular by theorem 4.3 there exists a $x\in M$ such

that $(x*\gamma_1,*(a*\gamma_2*a))(a)>\theta$.
Hence

$$\bigvee_{d\in R}(S(a,\gamma_2,a,\gamma_2',d)\wedge S(x,\gamma_1,x,\gamma_1',a))>\theta$$

Hence there exists a unique $d_1\in M$ such that $S(a,\gamma_2,a,\gamma_2',d_1)\theta$ and $S(x,\gamma_1,a,\gamma_1',a)>\theta$ implies $d_1\in I$.
Thus $S(x,\gamma_1,a,\gamma_1',a)>\theta$ with $d_1\in I$ implies $a\in I$. □

Theorem 4.7. *Let I be a fuzzy ideal of a fuzzy ring (M,Γ,R,S). For any $a,b,c\in M$, if $(a*\alpha*b)(y)>\theta\Rightarrow y\in I$ then $((a*\alpha*b)*\beta*c)(z)>\theta$ implies $z\in I$.*

Proof. Suppose $(a*\alpha*b)(y)>\theta$ implies $y\in I$.
Thus

$$\bigvee_{\alpha'\in\Gamma}(S(a,\alpha,b,\alpha',y)>\theta\Rightarrow y\in I.$$

Assume $((a*\alpha*b)*\beta*c)(z)>\theta$. Let us show that $z\in I$.

Now

$$\bigvee_{f\in M}(S(a,\alpha,b,\alpha',f)\wedge S(f,\beta,c,\beta',z)>\theta$$

By definition of S there exists a unique $f\in M$ such that $S(a,\alpha,b,\alpha',d)>\theta$ and hence we have $S(f,\beta,c,\beta',z)>\theta$ Since $S(a,\alpha,b,\alpha',d)>\theta\Rightarrow d\in I$.

Since $d\in I$ and $S(f,\beta,c,\beta',z)>\theta$ implies $z\in I$.

Therefore $((a*\alpha*b)*\beta*c)(z)>\theta$ implies $z\in I$. □

Theorem 4.8. *Let I be a fuzzy ideal of a fuzzy gamma ring (M,Γ,R,S) and for any $a,b,c\in M$, if $(a*\alpha*b)(y)\theta\Rightarrow y\in I$ then $(c*\alpha*(a*\beta*b))(z)>\theta$ implies $z\in I$.*

Proof. Given $(a*\alpha*b)(y)>\theta$ implies $y\in I$.

Thus

$$\bigvee_{\alpha,\alpha'\in\Gamma}(S(a,\alpha,b,\alpha',y)>\theta$$

implies $y\in I$.
Assume $((c*\alpha*(a*\beta*b))(z)>\theta$.
Let us show that $z\in I$.
Now

$$\bigvee_{f\in M,\alpha',\beta'\in M}(S(a,\beta,b,\alpha',f)\wedge S(c,\alpha,f,\beta',z))>\theta$$

By definition of S there exists a unique $d\in M$ such that $S(a,\beta,b,\alpha',d)>\theta$ and hence $S(c,\alpha,\beta',z)>\theta$.

Since $(S(a,\beta,b,\alpha',d)>\theta$ implies $d\in I$.

Since $d\in I$ and $S(c,\alpha,d,\beta',z)>\theta$ implies $z\in I$.

Therefore $(c*\alpha*(a*\beta*b))(z)>\theta$ implies $z\in I$. □

Theorem 4.9. *Let I be a fuzzy ideal of a strongly regular fuzzy gamma ring $(M,\Gamma,R,S)$and let $x\in M$. For any $a,b\in I$ such that $S(a,\alpha,b,\beta,x)>\theta$ implies $x\in I$ then $S(b,\alpha,a,\beta,y)>\theta$ implies $y\in I$.*

Proof. Since M is a left strongly regular, there exists a $x\in M$ such that $(a*\alpha*b)(z)>\theta\Leftrightarrow(x*\gamma_1*(a*\alpha*b)*\gamma_2*(a*\alpha*b))(z)>\theta$.

Let $S(b,\alpha,a,\beta,y)>\theta$.

Hence $(b*\alpha*a)(y)>\theta$ using Theorem 4.5 and 4.6 we have $((b*\alpha*a)*\alpha*b)(y)>\theta$ and hence $(x*\gamma*(a*\alpha*b)*\alpha*(a*\alpha*b))(y)>\theta$.

Hence $(a*\alpha*b)(y)>\theta$ implies $y\in I$. □

Theorem 4.10. *If e_0 is the zero element of (R,G) in the fuzzy gamma ring (M,Γ,R,S) then for any $a,b,c\in M,\alpha\in\Gamma$, $(a*\alpha*b)(e_0)>\theta$ implies $(c*\beta*(a*\alpha*b))(e_0)>\theta$ and $((a*\alpha*b)*\beta*c)(e_0)>\theta$.*

Proof. Let $a,b,c\in M$ and $\alpha,\beta\in\Gamma$ such that

$$\bigvee S(a,\alpha,b,\alpha',e_0)>\theta \tag{4.1}$$

Now

$$\begin{aligned}(c*\beta*(a*\alpha*b))(e_0)=\bigvee_{\alpha',\beta'\in\Gamma,d\in M}&(S(a,\alpha,b,\alpha',d)\\ &\wedge S(c,\beta,e_0,\beta',e_0))\\ &>(S(a,\alpha,b,\alpha',e_0)\\ &\wedge S(c,\beta,e_0,\beta',e_0))>\theta\end{aligned}$$

Similarly $((a*\alpha*b)*\beta*c)(e_0)>\theta$. □

Definition 4.11. *An element a of a fuzzy gamma ring (M,Γ,R,S) is called a nilpotent element if there exists some positive integer r such that*
$(a*\gamma_1*a*\gamma_2*a*\gamma_3*\ldots*\gamma_n*a)(e_0)>\theta$ implies $a=e_0$

Theorem 4.12. *If (M,Γ,R,S) is a left strongly regular ring then (M,Γ,R,S) is without nilpotent elements.*

Proof. Suppose $(a*\alpha*a)(e_0)>\theta$. Since M is a left strongly regular there exists a $x\in M$ such that

$$(x*\gamma*(a*\alpha*a))(a)>\theta \tag{4.2}$$

By Theorem 4.10, since $(a*\alpha*a)(e_0)>\theta$ implies

$$(x*(a*\alpha*a))(e_0)>\theta \tag{4.3}$$

(from (4.2) and (4.3)) $a=e_0$ □

Theorem 4.13. *Suppose (M,Γ,R,S) is a left strongly regular if $(a*\alpha*b)(e_0)>\theta$ for some $a,b\in M$ then $(b*\alpha*a)(e_0)>\theta$.*

Proof. Given $(a*\alpha*b)(e_0) > \theta$

Since (M,Γ,R,S) is a left strongly regular ring then there exists a $x \in M$ such that $(x*\gamma_1*(b*\alpha*a)*\gamma_2*(b*\alpha*a))(e_0) > \theta$. Therefore $(b*\alpha*a)(e_0) > \theta$. □

Definition 4.14. *A fuzzy gamma ring (M) is said to fulfill the insertion of factor property (IFP) provided that for any $a,b \in M$, $(a*\alpha*b)(e_0) > \theta$ implies $(a*\alpha*m*\beta*b)(e_0) > \theta$ for all $m \in M$.*

Theorem 4.15. *Let (M,Γ,R,S) i.e., a left strongly regular fuzzy gamma ring . Then for any $a,b \in M$ and $\alpha,\beta \in \Gamma$, $(a*\alpha*b)(e_0) > \theta$ implies $(a*\alpha*m*\beta*b)(e_0) > \theta$ for all $m \in M$.*

Proof. Given $(a*\alpha*b)(e_0) > \theta$ by Theorem 4.10 $(b*\alpha*a)(e_0) > \theta$.

Let $m \in M$ using theorem we have $(m*b*\alpha'*a')(e_0) > \theta$ and hence $(a*\alpha*m*\beta*b*\gamma_2*a*\alpha*m*\beta*b)(e_0) > \theta$. Then there exists $x \in M$ such that $(x*\gamma_1*(a*\alpha*m*\beta*b)*\gamma_2*(a*\alpha*m*\beta*b))(e_0) > \theta$. Since M is left strongly regular $(a*\alpha*m\beta*b)(e_0) > \theta$. Therefore$M$ has insertion factor property. □

Theorem 4.16. *Let (M,Γ,R,S) is a left strongly regular then for any $a \in M$, left annihilator of "a" is an ideal.*

Proof. By Theorem 3.2, $l(a)$ is a left ideal. Let $x \in l(a)$ and $m \in M, (x*\alpha*a)(e_0) > \theta$. Let us show that $(x*\alpha*m*\beta*a)(e_0) > \theta$. Since $(x*\alpha*a)(e_0) > \theta$, by Theorem 4.15, M has IFP, $(x*\alpha*m*\beta*a)(e_0) > \theta$. □

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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