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Fuzzy optimization in gH- symmetrically differentiable fuzzy function of several variables

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Abstract

This paper defines a new concept called Levelwise generalized hukuhara symmetric (LgHs) derivative of functions which are of fuzzy valued and with several variables. We can see that LgHs derivative of functions which are of fuzzy valued and with several variables is very much general compared to existing ones. Here we introduced a concept of midpoint and radius function. This allows us for a connection between generalized hukuhara symmetric (gHs) differentiability and classic definitions for the cases of differentiability in functions which are real valued. The novel directional LgHs derivative and partial gHs derivative unifies and extends in latest papers.

Keywords

gHs derivative, Fuzzy optimization, Fuzzy functions of several variables, Fuzzy directional LgHs differentiability.

AMS Subject Classification

90C70.

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Contents

1. Introduction

In this paper, optimization problems considered have non linear constraints with fuzzy valued objective functions of several variables. In optimization, there are two components namely, Theory and mathematical programming. Practically objective functions rarely holds real number as coefficients. Most of the time, they have uncertainty. These values may not be accurate also. The disadvantages of uncertainty or inaccuracy can be tackled by the use of fuzzy programming approach.

Works of Rommelfanger [1] and Delgado et al. [2] viewed this from 90s onwards. Lodwick [3] gives a detailed literature review on this topic. Paper of Slowinski and Teghem [4] compares optimization problems with multiple objectives. Inuiguchi [5] has done a similar comparison but for problems to solve portfolio selection.

HD(hukuhara differentiability) of functions which are set valued will give HD of fuzzy valued functions where the differentiability is based upon Hukuhara difference. Hukuhara [6] developed the subtraction of two sets. Hukuhara derivatives introduced in [6] is widely used by researchers in the field of set and fuzzy valued functions due to its importance in fuzzy differential equations as well as optimization problems.

It is found from the works of [7], [8] and [9], compared to H differentiable functions gH differentiable fuzzy functions are relatively general.

Here we propose a new idea known as generalized Hukuhara symmetrically(gHs) differentiable fuzzy functions of several variables. We can see that gHs derivative of fuzzy functions of several variables is more general than the existing definitions.

Section 2 contains preliminaries. we define our main definition, LgHs differentiable fuzzy function and some theorems related to it in section 3. Section 4 deals with directional LgHs derivative of several variable functions which are of fuzzy valued. In last section, we obtain necessary and sufficient optimality conditions of non-dominated solution applying LgHs derivative to fuzzy functions of several variables.

2. Preliminaries

Assume $\mathscr{I}_{\mathscr{C}}$ represents the family of all intervals belongs to $\mathbb{R}$ which are bounded .
ie $\mathscr{I}_{\mathscr{C}}$={$[\underline{k},\overline{k}] | \underline{k},\overline{k} \in \mathbb{R}$ and $\underline{k} <= \overline{k}$ }.
suppose $A = [\underline{a},\overline{a}]$ and $B = [\underline{b},\overline{b}]$ denote the two fuzzy intervals. Now we explain the Hausdorff–Pompeiu distance(H_p) from A to B as

$$H_p(A,B) = \max\{|\underline{a}-\underline{b}| \,,\, |\overline{a}-\overline{b}\,|\}$$

Clearly $(\mathscr{I}_{\mathscr{C}}, H)$ denotes a complete metric space .
Consider $\mathbb{R}^n$ denotes a mapping $l : \mathbb{R}^n \to [0,1]$. We represent the α level set, $[l]^\alpha = \{t \in \mathbb{R}^n | l(t) >= \alpha\}$ for any $\alpha \in (0,1]$. Now we recall the definition of support as: supp(l) = $\{t \in \mathbb{R}^n | l(t) > 0\}$.

Definition 2.1. *Suppose l denotes fuzzy set on $\mathbb{R}$and l become a fuzzy interval only when the following conditions are hold:*
1. l is normal and upper semi continuous.
2. The value of $l(\lambda x + (1-\lambda)y)$ should be $\geq min\{l(x), l(y)\}$, where $x, y \in \mathbb{R}, \lambda \in [0,1]$
3. The value of $[l]^0$ should be compact.

Assume F_C stand for collection of every fuzzy intervals. The α levels of fuzzy intervals are defined as, $[l]^\alpha = [\underline{l}_\alpha, \overline{l}_\alpha]$, where $\underline{l}_\alpha, \overline{l}_\alpha \in \mathbb{R}$, $\forall\ \alpha \in [0,1]$ and $[l]^\alpha \in \mathscr{I}_{\mathscr{C}}$, $\forall\ \alpha \in [0,1]$. Now its time to define the arithmetic operations such as addition and scalar multiplication of fuzzy intervals $l, m \in F_C$ as follows:

$$(l+m)(t) = \sup_{y+z=t} min\{l(y), m(z)\}$$

$$(\lambda l)(t) = \begin{cases} l(\frac{t}{\lambda}) & if \lambda \neq 0 \\ 0 & if \lambda = 0 \end{cases}$$

Clearly $\forall\ \alpha \in$[0,1],

$$[l+m]^\alpha = [(\underline{l+m})_\alpha, (\overline{l+m})_\alpha] = [\underline{l}_\alpha + \underline{m}_\alpha, \overline{l}_\alpha + \overline{m}_\alpha] \quad (2.1)$$

and

$$[(\lambda l)]^\alpha = [(\underline{\lambda l})_\alpha, (\overline{\lambda l})_\alpha] = [min\{\lambda \underline{l}_\alpha, \lambda \overline{l}_\alpha\}, max\{\lambda \underline{l}_\alpha, \lambda \overline{l}_\alpha\}] \quad (2.2)$$

Definition 2.2. *(Stefanini [8])*

$$l \ominus_{gH} m = p \Leftrightarrow \begin{cases} (1) l = m + p, \\ or (2) m = l + (-1)p. \end{cases}$$

$[l \ominus_{gH} m]^\alpha = [l]^\alpha \ominus_{gH} [m]^\alpha = [min\{\underline{l}_\alpha - \underline{m}_\alpha, \overline{l}_\alpha - \overline{m}_\alpha\}, max\{\underline{l}_\alpha - \underline{m}_\alpha, \overline{l}_\alpha - \overline{m}_\alpha\}]$, $\forall\ \alpha \in$[0,1], where $[l]^\alpha \ominus_{gH} [m]^\alpha$ represents generalized hukuhara difference between l and m. Let $l, m \in F_C$, we can define distance between l and m by

$$D(l,m) = \sup_{\alpha \in [0,1]} H([l]^\alpha, [m]^\alpha) = \sup_{\alpha \in [0,1]} max\{|\underline{l}_\alpha - \underline{m}_\alpha|, |\overline{l}_\alpha - \overline{m}_\alpha|\}$$

so (F_C, D) denotes complete metric space.

2.1 gH derivative of fuzzy functions

Let E be an open subset of $\mathbb{R}^n$ and define the fuzzy function $\mathscr{F} : E \to F_C$, $\forall\ \alpha \in [0,1]$. Collection of all functions which are of interval valued can be represented by, $\mathscr{F}_\alpha : E \to \mathscr{I}_{\mathscr{C}}$ and $\mathscr{F}_\alpha(t) = [\mathscr{F}(t)]^\alpha$. $\forall\ \alpha \in [0,1]$, with lower function $\underline{\mathscr{F}}_\alpha(t)$ and upper function $\overline{\mathscr{F}}_\alpha(t)$ we denote $\mathscr{F}_\alpha(t) = [\underline{\mathscr{F}}_\alpha(t), \overline{\mathscr{F}}_\alpha(t)] = (\hat{\mathscr{F}}_\alpha; \tilde{\mathscr{F}}_\alpha)$ where $\hat{\mathscr{F}}_\alpha$ denotes the midpoint function of $\mathscr{F}$ and $\tilde{\mathscr{F}}_\alpha$ denotes the radius function of $\mathscr{F}$.

Definition 2.3. *Let $E \subset \mathbb{R}$ with the fuzzy function $\mathscr{F} : E \to \mathscr{F}_{\mathscr{C}}$, $t_o \in E$ and $\overline{h} \in \mathbb{R}$ such that $t_o + \overline{h} \in E$, the gH-derivative of $\mathscr{F}$ at t_o is as follows*

$$\mathscr{F}'(t_o) = \lim_{\overline{h} \to 0} \frac{\mathscr{F}(t_o + \overline{h}) \ominus_{gH} \mathscr{F}(t_o)}{\overline{h}} \quad (2.3)$$

If $\mathscr{F}'(t_o) \in \mathscr{F}_{\mathscr{C}}$ satisfying (2.3) exists, then $\mathscr{F}$ can be called as generalized hukuhara differentiable (gH differentiable) at the point t_o.

3. Main Results

Definition 3.1. *Let $E \subset \mathbb{R}$ with the fuzzy function $\mathscr{F} : E \to \mathscr{F}_{\mathscr{C}}$ and $t_o \in E$ and $t_o + \overline{h}, t_o - \overline{h} \in E$, the generalized hukuhara symmetric (gHs)-derivative of $\mathscr{F}$ at the point t_o is as follows*

$$\mathscr{F}^s(t_o) = \lim_{\overline{h} \to 0} \frac{\mathscr{F}(t_o + \overline{h}) \ominus_{gH} \mathscr{F}(t_o - \overline{h})}{2\overline{h}} \quad (3.1)$$

If there exists a $\mathscr{F}^s(t_o) \in F_C$ satisfying (3.1) then $\mathscr{F}$ is gHs differentiable at the point t_o.

Definition 3.2. *Let $E \subset \mathbb{R}$ and fuzzy function $\mathscr{F} : E \to \mathscr{F}_{\mathscr{C}}$ with $t_o \in E$, $t_o + \overline{h}, t_o - \overline{h} \in E$. Let $\alpha \in [0,1]$, the level wise gHs (LgHs) derivative of the corresponding interval valued function $\mathscr{F}_\alpha : E \to \mathscr{I}_{\mathscr{C}}$ at to is defined as*

$$\mathscr{F}^s_{(LgHs,\alpha)}(t_o) = \lim_{\overline{h} \to 0} \frac{\mathscr{F}_\alpha(t_o + \overline{h}) \ominus_{gH} \mathscr{F}_\alpha(t_o - \overline{h})}{2\overline{h}} \quad (3.2)$$

if it exists. If $\mathscr{F}^s_{(LgHs,\alpha)}(t_o) \in \mathscr{I}_{\mathscr{C}}$ $\forall\ \alpha \in [0,1]$, then (i) $\mathscr{F}$ is LgHs differentiable at the point t_o. (ii) The family of intervals $\mathscr{F}^s_{(LgHs,\alpha)}(t_o), \alpha \in [0,1]$ is level wise gH derivative of $\mathscr{F}$ at t_o. This can be denoted as $\mathscr{F}^s_{(LgHs)}(t_o)$

Theorem 3.3. *If $\mathscr{F} : E \to \mathscr{F}_{\mathscr{C}}$ is gHs differentiable then the fuzzy function is LgHs differentiable, $\forall\ \alpha \in [0,1]$. Furthermore $[\mathscr{F}^s_{LgHs}(t_o)]^\alpha = [\mathscr{F}^s(t_o)]^\alpha$.*

Example 3.4. *Let fuzzy function $\mathscr{F} : \mathbb{R} \to \mathscr{F}_{\mathscr{C}}$ and $\mathscr{F}(x)$ = p.x, where p denotes fuzzy interval with $[p]^\alpha = [\underline{p}_\alpha, \overline{p}_\alpha]$ such that $\underline{p}_\alpha < \overline{p}_\alpha$. Thus*

$$\mathscr{F}_\alpha(x) = \begin{cases} [\underline{p}_\alpha(x), \overline{p}_\alpha(x)], & x \geq 0 \\ [\overline{p}_\alpha(x), \underline{p}_\alpha(x)], & x < 0 \end{cases}$$

Here we notice that $\overline{\mathscr{F}}_\alpha$ and $\underline{\mathscr{F}}_\alpha$ are not symmetrically differentiable at t=0. $\mathscr{F}$ is gHs differentiable on $\mathbb{R}$. If $\mathscr{F}(x) =$

$p.c(x)$ where $c:\mathbb{R}\to\mathbb{R}$ is a symmetrically differentiable function and is known as the gHs-differential(total gHs-derivative) of $\mathscr{F}$ at $t^{(o)}$ and $D_{gHs}\ \mathscr{F}(t^{(o)})(\bar{h})$ is the interval valued differential of $\mathscr{F}$ at $t^{(o)}$ with respect to h.

The following definition explain LgHs-differentiability of an interval valued fuzzy function.

Definition 3.5. *Suppose* $\mathscr{F}:E\to\mathscr{F}_{\mathscr{C}},\mathscr{F}_\alpha(t)=(\hat{\mathscr{F}}_\alpha(t);\tilde{\mathscr{F}}_\alpha(t))$ $=[\hat{\mathscr{F}}_\alpha(t)-\tilde{\mathscr{F}}_\alpha(t),\hat{\mathscr{F}}_\alpha(t)+\tilde{\mathscr{F}}_\alpha(t)]\ \forall\ \alpha\in[0,1]$ *and assume that* $t^{(o)}\in E$ *such that* $t^{(o)}+\bar{h}\in E$, $\forall\ \bar{h}\in\mathbb{R}^n$ *with* $\|\bar{h}\|<\delta$.

(1) $\mathscr{F}$ is LgHs differentiable at $t^{(o)}$ iff for each $\alpha\in[0,1]$ $\exists$ two vectors $\hat{g}_\alpha,\tilde{g}_\alpha\in\mathbb{R}^n$, $\hat{g}_\alpha=(\hat{g}_{\alpha,1},...,\hat{g}_{\alpha,n}),\tilde{g}_\alpha=(\tilde{g}_{\alpha,1},...,\tilde{g}_{\alpha,n})$ and two functions $\hat{w}_\alpha(\bar{h}),\tilde{w}_\alpha(\bar{h})$ with

$$\lim_{\bar{h}\to 0}\hat{w}_\alpha(\bar{h})=\lim_{\bar{h}\to 0}\tilde{w}_\alpha(\bar{h})=0$$

such that

$$\hat{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\hat{\mathscr{F}}_\alpha(t^{(o)}-\bar{h})=\sum_{j=1}^{n}2h_j\hat{g}_{\alpha,j}+2\parallel\bar{h}\parallel\hat{w}_\alpha(\bar{h}) \tag{3.3}$$

$$|\tilde{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\tilde{\mathscr{F}}_\alpha(t^{(o)}-\bar{h})=\left|\sum_{j=1}^{n}2\bar{h}_j\tilde{g}_{\alpha,j}+2\parallel\bar{h}\parallel\tilde{w}_\alpha(\bar{h})\right| \tag{3.4}$$

family of interval valued functions $D_{gHs}\mathscr{F}(t^{(o)})\ \forall\alpha\in[0,1]$ is called the LgHs total derivative of fuzzy function $\mathscr{F}$ at $t^{(o)}$.
(2) $\mathscr{F}$ is fuzzy gHs differentiable at $t^{(o)}\iff$ (a) $\mathscr{F}$ is LgHs differentiable at $t^{(o)}$ as stated in (1) and (b) $\forall\bar{h}\in\mathbb{R}^n$ the LgHs differentials $D_{gHs}\mathscr{F}_\alpha(t^{(o)})(\bar{h})$ for $\alpha\in[0,1]$ form the α cuts of a fuzzy interval.
Equation (3.3) is similar to the differentiability of $\hat{\mathscr{F}}_\alpha$ at $t^{(o)}$ with partial derivatives $\frac{\partial\hat{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_j}=\hat{g}_{\alpha,j}$, where j=1,...,n. Also Equation (3.4) is similar to the differentiability of $\tilde{\mathscr{F}}_\alpha$ at $t^{(o)}$ with partial derivatives $\frac{\partial\tilde{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_j}=\tilde{g}_{\alpha,j}$, where j=1,...,n.

Proposition 3.6. *Suppose the fuzzy funtion* $\mathscr{F}:E\to\mathscr{F}_{\mathscr{C}}$ *with* $\mathscr{F}_\alpha(t)=(\hat{\mathscr{F}}_\alpha(t);\tilde{\mathscr{F}}_\alpha(t))\forall\alpha\in[0,1]$. *Assume* $t^{(o)}\in E$ *such that* $t^{(o)}+\bar{h}\in E$, $\forall\bar{h}\in\mathbb{R}^n$ *with* $|\bar{h}|<\delta$. *Then* $\mathscr{F}$ *is LgHs differnetiable at* $t^{(o)}\iff$ *for each* $\alpha\in[0,1]\ \exists\ g_\alpha^{(1)},\ g_\alpha^{(2)}\in\mathbb{R}$ *such that* $\exists$

$$\lim_{\bar{h}\to 0}\frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h})\ominus_{gH}\mathscr{F}_\alpha(t^{(o)}-\bar{h})}{2\bar{h}}=(g_\alpha^{(1)},\left|g_\alpha^{(2)}\right|) \tag{3.5}$$

where $\mathscr{F}^s_{(gHs,\alpha)}(t^{(o)})=(\hat{\mathscr{F}}^s_{(gHs,\alpha)}(t^{(o)});\tilde{\mathscr{F}}^s_{(gHs,\alpha)}(t^{(o)}))$ $=(g_\alpha^{(1)},\left|g_\alpha^{(2)}\right|)$ *denotes the LgHs-derivative of* $\mathscr{F}_\alpha$ *at* $t^{(o)}$.

Proof. Consider $\frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h})\ominus_{gH}\mathscr{F}_\alpha(t^{(o)}-\bar{h})}{2\bar{h}}=$

$$\Big(\frac{\hat{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\hat{\mathscr{F}}_\alpha(t^{(o)}-\bar{h})}{2\bar{h}},\frac{|\tilde{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\tilde{\mathscr{F}}_\alpha(t^{(0)}-\bar{h})|}{2|\bar{h}|}\Big) \tag{3.6}$$

for any $\bar{h}\in\mathbb{R}$. We can see that $\exists\ g_\alpha^{(1)},g_\alpha^{(2)}\in\mathbb{R}$ such that (3.5) is satisfied. Considering (3.6) and $(\mathscr{F}_{\mathscr{C}},\mathsf{H}_p)$ we say that $\exists$

$$\lim_{\bar{h}\to 0}\frac{\hat{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\hat{\mathscr{F}}_\alpha(t^{(o)}-\bar{h})}{2\bar{h}}=g_\alpha^{(1)}$$

$$\lim_{\bar{h}\to 0}\frac{|\tilde{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\tilde{\mathscr{F}}_\alpha(t^{(o)}-\bar{h})|}{2|\bar{h}|}=g_\alpha^{(2)}$$

Also $\exists w_\alpha^{(1)}(\bar{h}),w_\alpha^{(2)}(\bar{h})$ with $\lim_{\bar{h}\to 0}w^{(i)}\bar{h}=0,i=1,2$ such that

$$\hat{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\hat{\mathscr{F}}_\alpha(t^{(o)}-\bar{h})=2\bar{h}g_\alpha^{(1)}+2|\bar{h}|w_\alpha^{(1)}(\bar{h}) \tag{3.7}$$

$$|\tilde{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\tilde{\mathscr{F}}_\alpha(t^{(o)}-\bar{h})|=2|\bar{h}|g_\alpha^{(2)}+2|\bar{h}|\eta(\bar{h})\geq 0 \tag{3.8}$$

Now define $w_\alpha^{(2)}(\bar{h})=\eta(\bar{h})$ when $\bar{h}\geq 0$ and $w_\alpha^{(2)}(\bar{h})=-\eta(\bar{h})$ when $\bar{h}<0$, then

$$|\tilde{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\tilde{\mathscr{F}}_\alpha(t^{(o)}-\bar{h})|=|2\bar{h}g_\alpha^{(2)}+2|\bar{h}|w_\alpha^{(2)}(\bar{h})| \tag{3.9}$$

and $\lim_{\bar{h}\to 0}w_\alpha^{(2)}(\bar{h})=0$

$$|\tilde{\mathscr{F}}_\alpha(t^{(o)}+\bar{h})-\tilde{\mathscr{F}}_\alpha(t^{(o)}-\bar{h})|=|2\bar{h}g_\alpha^{(2)}+2|\bar{h}|w_\alpha^{(2)}(\bar{h})| \tag{3.10}$$

By equation (3.7),(3.9) and definition (3.1), $\mathscr{F}$ is LgHs differentiable at the point $t^{(o)}$. □

If $\frac{\partial_{gHs}\mathsf{F}_\alpha(t^{(o)})}{\partial t_i}=(\hat{g}_{\alpha,i},|\tilde{g}_{\alpha,i}|)$, i = 1,2,...,n (partial LgHs derivative) defines level cuts of a fuzzy interval then $\mathscr{F}$ is gHs differentiable at $t^{(o)}$. Partial derivative with respect to t_i at $t^{(o)}$ is represented as $\frac{\partial_{gHs}\mathscr{F}(t^{(o)})}{\partial t_i}$.

4. Directional LgHs derivative

Definition 4.1. *Let fuzzy function* $\mathscr{F}:E\to\mathscr{F}_{\mathscr{C}}$ *where* $E\subset\mathbb{R}^n$ *and* $t^{(o)}\in E$. *Let* $d\in\mathbb{R}^n$ *be any admissible direction at* $t^{(o)}$. *For given* $\alpha\in[0,1]$, *the directional LgHs derivative corresponding to the interval valued function* $\mathscr{F}_\alpha:E\to\mathscr{I}_{\mathscr{C}}$ *at* $t^{(o)}$ *in the direction d can be defined as*

$$\mathscr{F}^s_{(LgHs,\alpha)}(t^{(o)};d)=\lim_{\bar{h}\to 0^+}\frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h}.d)\ominus_{gH}\mathscr{F}_\alpha(t^{(o)}-\bar{h}.d)}{2\bar{h}} \tag{4.1}$$

if it exists.

1 *If $\forall\alpha \in [0,1], \mathscr{F}^s_{LgHs,\alpha}(t^{(o)};d)$ exists then $\mathscr{F}$ has directional LgHs derivative at the point $t^{(o)}$ in the direction d.*

2 *If $\forall\alpha \in [0,1], \mathscr{F}$ admits directional LgHs derivatives at the point $t^{(o)}$ and in direction $d \in \mathbb{R}^n$ then $\mathscr{F}$ is directionally LgHs differentiable at the point $t^{(o)}$.*

3 *Let $\mathscr{F}$ be directionally LgHs differentiable at the point $t^{(o)}$ in direction d then $\mathscr{F}$ is directionally gHs differentiable at the point $t^{(o)}$. Also directional LgHs derivative $\mathscr{F}^s_{LgHs,\alpha}(t^{(o)};d)$ defines a fuzzy interval.*

4 *Let $\mathscr{F}$ be directionally LgHs differentiable at each point $t^{(o)} \in E$ then $\mathscr{F}$ is directionally LgHs differentiable on E. Also, if $\mathscr{F}$ is directionally gHs differentiable at each point $t^{(o)} \in E$ then function $\mathscr{F}$ is said to be directionally gHs differentiable on E.*

Theorem 4.2. *Suppose $X \subset \mathbb{R}^n$ be an open set and assume the fuzzy function $\mathscr{F} : X \to \mathscr{F}_{\mathscr{C}}$ with interval valued function $\mathscr{F}_\alpha = (\hat{\mathscr{F}}_\alpha;\tilde{\mathscr{F}}_\alpha)$ for each $\alpha \in [0,1]$. If function $\mathscr{F}$ is LgHs differentiable at the point $t^{(o)}$ then all interval valued partial gHs derivatives of fuzzy function $\mathscr{F}$ exists.*

$$\frac{\partial_{gHs}\mathscr{F}_\alpha(t^{(o)})}{\partial t_i} = \mathscr{F}^s_{gHs,\alpha}(t^{(o)};e_i) = (\hat{g}_{\alpha,i};|\tilde{g}_{\alpha,i}|) = [\hat{g}_{\alpha,i} - |\tilde{g}_{\alpha,i}|, \hat{g}_{\alpha,i} + |\tilde{g}_{\alpha,i}|],$$

$\forall\alpha \in [0,1]$, for $i = 1,...,n$. Moreover all directional gHs derivatives of $\mathscr{F}_\alpha$, $\forall\alpha \in [0,1]$ exists and

$$\mathscr{F}^s_{gHs,\alpha}(t^{(o)};d) = \Big(\sum_{i=1}^{n} d_i\hat{g}_{\alpha,i};\Big|\sum_{i=1}^{n} d_i\tilde{g}_{\alpha,i}\Big|\Big)$$

Proof. Proved by definition (3.3) □

From the above theorem we can define the LgHs gradient of $\mathscr{F}$ at $t^{(o)}$.
For each α function, gHs gradient of LgHs differentiable function is

$$\tilde{\nabla}_{gHs}\mathscr{F}_\alpha(t^{(o)}) = \Big(\frac{\partial_{gHs}\mathscr{F}_\alpha(t^{(o)})}{\partial t_1},...,\frac{\partial_{gHs}\mathscr{F}_\alpha(t^{(o)})}{\partial t_n}\Big) = \Big((g^{(1)}_{\alpha,1};|g^{(2)}_{\alpha,1}|),...,(g^{(1)}_{\alpha,n};|g^{(2)}_{\alpha,n}|)\Big)$$

Let $(\hat{\mathscr{F}}_\alpha;\tilde{\mathscr{F}}_\alpha)$ are differentiable. This will give us

$$\tilde{\nabla}_{gHs}\mathscr{F}_\alpha(t^{(o)}) = \Big(\Big(\frac{\partial\hat{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_1};\Big|\frac{\partial\tilde{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_1}\Big|\Big),...,\Big(\frac{\partial\hat{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_n};\Big|\frac{\partial\tilde{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_n}\Big|\Big)\Big)$$

$$\frac{\partial_{gHs}\mathscr{F}_\alpha(t^{(o)})}{\partial t_i} = \Big[\frac{\partial\hat{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_i} - \Big|\frac{\partial\tilde{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_i}\Big|, \frac{\partial\hat{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_i} + \Big|\frac{\partial\tilde{\mathscr{F}}_\alpha(t^{(o)})}{\partial t_i}\Big|\Big]$$

$$\frac{\partial_{LgHs}\mathscr{F}(t^{(o)})}{\partial t_i} = \{\frac{\partial_{LgHs}\mathsf{F}_\alpha(t^{(o)})}{\partial t_i} : \alpha \in [0,1]\},$$

denotes the family of LgHs partial derivative if it exists. Thus LgHs gradient of $\mathscr{F}$ at the point $t^{(o)}$ is as follows

$$\tilde{\nabla}_{LgHs}\mathscr{F}(t^{(o)}) = \Big(\frac{\partial_{LgHs}\mathscr{F}(t^{(o)})}{\partial t_1},...,\frac{\partial_{LgHs}\mathscr{F}(t^{(o)})}{\partial t_n}\Big) \quad (4.2)$$

Theorem 4.3. *Assume $\mathscr{F} : E \to \mathscr{F}_{\mathscr{C}}$ where $E \subset \mathbb{R}$, is LgHs differentiable then $\mathscr{F}$ is directionally LgHs differentiable on E*

$$\mathscr{F}^s_{LgHs,\alpha}(t^{(o)};d_0) = \mathscr{F}^s_{LgHs,\alpha}(t^{(o)}).d_0$$

$\forall t^{(o)} \in E, d_0 \in \mathbb{R}$ and $\alpha \in [0,1]$

Proof. By definition (4.1) we will show that

$$\mathscr{F}^s_{(LgHs,\alpha)}(t^{(o)};d_0) = \lim_{\bar{h}\to 0^+} \frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h}.d_0) \ominus_{gH} \mathscr{F}_\alpha(t^{(o)}-\bar{h}.d_0)}{2\bar{h}} \quad (4.3)$$

exists.
Given $\mathscr{F}$ is LgHs differentiable. By definition (3.1) we get

$$\mathscr{F}^s_{(LgHs,\alpha)}(t^{(o)}) = \lim_{\bar{h}\to 0^+} \frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h}) \ominus_{gH} \mathscr{F}_\alpha(t^{(o)}-\bar{h})}{2\bar{h}} \quad (4.4)$$

where $\mathscr{F}^s(t^{(o)}) \in \mathscr{F}_{\mathscr{C}}$

$$\lim_{\bar{h}\to 0^+} \frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h}) \ominus_{gH} \mathscr{F}_\alpha(t^{(o)}-\bar{h})}{2\bar{h}} = \mathscr{F}^s_{LgHs,\alpha}(t^{(o)})$$

$$= \lim_{\bar{h}\to 0^-} \frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h}) \ominus_{gH} \mathscr{F}_\alpha(t^{(o)}-\bar{h})}{2\bar{h}} \quad (4.5)$$

From (4.3) and (4.5) we have

$$\mathscr{F}^s_{(LgHs,\alpha)}(t^{(o)};d_0) = \lim_{\bar{h}\to 0^+} \frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h}.d_0) \ominus_{gH} \mathscr{F}_\alpha(t^{(o)}-\bar{h}.d_0)}{2\bar{h}.d_0}.d_0$$

$$= \lim_{\bar{h}\to 0^+} \frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h}') \ominus_{gH} \mathscr{F}_\alpha(t^{(o)}-\bar{h}')}{2\bar{h}'}.d_0, if d_0 > 0$$

$$\lim_{\bar{h}\to 0^-} \frac{\mathscr{F}_\alpha(t^{(o)}+\bar{h}') \ominus_{gH} \mathscr{F}_\alpha(t^{(o)}-\bar{h}')}{2\bar{h}'}.d_0, if d_0 < 0$$

where $\bar{h}' = \bar{h}.d_0$

$$= \mathscr{F}^s_{LgHs,\alpha}(t^{(o)}).d_0$$

□

From the following example it is observed that converse of above theorem is false.

Example 4.4. *Let fuzzy function* $\mathscr{F} : [-5,5] \to \mathscr{F}_{\mathscr{C}}$ *be* $\mathscr{F}(t) =< 1,2,3 > |t-2|$. *Now the* α *levels can be expressed by functions*

$$\mathscr{F}_\alpha(t) = [< 1,2,3 > |t-2|]^\alpha$$

$$= [(1+\alpha)|t-2|, (3-\alpha)|t-2|]$$

Now midpoint function, $\hat{\mathscr{F}}_\alpha = 2|t-2|$ *and radius function,* $\tilde{\mathscr{F}}_\alpha = (1-\alpha)|t-2|$
$\forall\ \alpha \in [0,1]$.*Also* $\hat{\mathscr{F}}$ *cannot be differentiable at the point* $t_o = 2$. *Thus* $\mathscr{F}$ *is not LgHs differentiable at the point* t_o.
Now

$$\lim_{\bar{h}\to 0^+} \frac{1}{2\bar{h}} \mathscr{F}_\alpha(2+\bar{h}.d) \ominus_{gH} \mathscr{F}_\alpha(2-\bar{h}.d)$$

$$= \lim_{\bar{h}\to 0^+} \frac{1}{2\bar{h}} \{[1+\alpha, 3-\alpha]|\bar{h}.d| \ominus_{gH} [1+\alpha, 3-\alpha]|\bar{h}.d|\}$$

Thus at the point t_o, $\mathscr{F}$ *is directionally LgHs differentiable.*

5. Necessary and Sufficient α Optimality Condition

Now we discuss the partial orders over fuzzy numbers.

Definition 5.1. *Given* $l, m \in \mathscr{F}_{\mathscr{C}}$, $\alpha \in [0,1]$, *then*

i. $l \leqq_{\alpha-LU} m \iff l_\alpha \leqq_{LU} m_\alpha$, *ie* $\underline{l}_\alpha \leq \overline{m}_\alpha$,

ii. $l \leq_{\alpha-LU} m \iff l_\alpha \leq_{LU} m_\alpha$,

iii. $l <_{\alpha-LU} m \iff l_\alpha <_{LU} m_\alpha$,

Similarly we obtained the LU fuzzy orders as follows:

a. $l \leqq_{LU} m \iff l \leqq_{\alpha-LU} m, \forall \alpha \in [0,1]$

b. $l \leq_{LU} m \iff l \leq_{\alpha-LU} m, \forall \alpha \in [0,1]$

c. $l <_{LU} m \iff l <_{\alpha-LU} m, \forall \alpha \in [0,1]$

Definition 5.2. *Luciano et.al defines [10] "Given* $\bar{t} \in \mathbb{R}^n$ *and* $\alpha \in [0,1]$, *then* $\bar{t}$ *is a weak* $\alpha - LU$ *minimum point of* $\mathscr{F}$ *if there exists no* $t \in \mathbb{R}^n$ *such that* $\mathscr{F}(t) <_{\alpha-LU} \mathscr{F}(\bar{t})$. *Similarly we say that* $\bar{t}$ *is a weak LU minimum point of* $\mathscr{F}$ *if there exists no* $t \in \mathbb{R}^n$ *such that* $\mathscr{F}(t) <_{LU} \mathscr{F}(\bar{t})$. *Also* $\bar{t}$ *is a weak global LU-minimum of* $\mathscr{F}$ *if* $\bar{t}$ *is a weak* $\alpha - LU$ *minimum of* $\mathscr{F}$, $\forall \alpha$*".*

Definition 5.3. *Luciano et.al defines [10] "Given* $\bar{t} \in \mathbb{R}^n$ *and* $\alpha \in [0,1]$, *then* $\bar{t}$ *is a* $\alpha - LU$ *minimum point of* $\mathscr{F}$ *if there exists no* $t \in \mathbb{R}^n$ *such that* $\mathscr{F}(t) \leq_{\alpha-LU} \mathscr{F}(\bar{t})$. *Similarly we say that* $\bar{t}$ *is a weak LU minimum point of* $\mathscr{F}$ *if there exists no* $t \in \mathbb{R}^n$ *such that* $\mathscr{F}(t) \leq_{LU} \mathscr{F}(\bar{t})$. *Also* $\bar{t}$ *is a global LU-minimum of* $\mathscr{F}$ *if* $\bar{t}$ *is a* $\alpha - LU$ *minimum of* $\mathscr{F}$, $\forall \alpha$*".*

Theorem 5.4. *Assume that* $\mathscr{F}$ *be LgHs directional differentiable function,*$\bar{t} \in \mathbb{R}^n$ *and* $\alpha \in [0,1]$. *If* $\bar{t}$ *is a weak* $\alpha - LU$ *solution of* $\mathscr{F}$, *then* $\exists$ *no directional* $d \in \mathbb{R}^n$ *such that* $\mathscr{F}^s_{LgHs,\alpha}(\bar{t}; d) <_{\alpha-LU} [0,0]$

Proof. Given $\mathscr{F}$ is LgHs directional differentiable function. Thus we have

$$\mathscr{F}^s_{(LgHs,\alpha)}(\bar{t}; d) = \lim_{\bar{h}\to 0^+} \frac{\mathscr{F}_\alpha(\bar{t}+\bar{h}.d) \ominus_{gH} \mathscr{F}_\alpha(\bar{t}-\bar{h}.d)}{2\bar{h}} \quad (5.1)$$

exists.
Now assume that $\exists\ \bar{d} \in \mathbb{R}^n$ such that $\mathscr{F}^s_{LgHs,\alpha}(\bar{t}; \bar{d}) <_{\alpha-LU} [0,0]$. For any $\varepsilon > 0, \exists\ \delta > 0$ such that $||\bar{t}+\bar{h}.\bar{d} - (\bar{t}-\bar{h}.\bar{d})|| = ||2\bar{h}.\bar{d}|| < \delta$, then $H(F_\alpha(\bar{t}+\bar{h}.\bar{d}), F_\alpha(\bar{t}-\bar{h}.\bar{d})) < \varepsilon$. Thus we can see that $\exists\ k_0 > 0$ such that $\mathscr{F}(\bar{t}+\bar{h}.\bar{d}) \ominus_{gH} \mathscr{F}(\bar{t}-\bar{h}.\bar{d}) <_{LU} [0,0]$, $\forall \bar{k} \in]0, k_0]$.
$\mathscr{F}_\alpha(\bar{t}+\bar{h}.\bar{d}) \ominus_{gH} \mathscr{F}_\alpha(\bar{t}-\bar{h}.\bar{d}) <_{\alpha-LU} [0,0]$, where $\bar{k} \in]0, k_0]$.
ie

$$\Big[Min\{\underline{\mathscr{F}}_\alpha(\bar{t}+\bar{\bar{h}}.\bar{d}) - \underline{\mathscr{F}}_\alpha(\bar{t}-\bar{\bar{h}}.\bar{d}), \mathsf{F}_\alpha(\bar{t}+\bar{\bar{h}}.\bar{d}) - \overline{\mathscr{F}}_\alpha(\bar{t}-\bar{\bar{h}}.\bar{d})\}, Max\{\underline{\mathscr{F}}_\alpha(\bar{t}+\bar{\bar{h}}.\bar{d}) - \underline{\mathscr{F}}_\alpha(\bar{t}-\bar{\bar{h}}.\bar{d}), \overline{\mathscr{F}}_\alpha(\bar{t}+\bar{\bar{h}}.\bar{d}) - \overline{\mathscr{F}}_\alpha(\bar{t}-\bar{\bar{h}}.\bar{d})\} \Big] <_{\alpha-LU} [0,0]$$

$$\implies \begin{cases} \underline{\mathscr{F}}_\alpha(\bar{t}+\bar{\bar{h}}.\bar{d}) - \underline{\mathscr{F}}_\alpha(\bar{t}-\bar{\bar{h}}.\bar{d}) < 0 \\ \overline{\mathscr{F}}_\alpha(\bar{t}+\bar{\bar{h}}.\bar{d}) - \overline{\mathscr{F}}_\alpha(\bar{t}-\bar{\bar{h}}.\bar{d}) < 0 \end{cases}$$

$\implies F_\alpha(\bar{t}+\bar{\bar{h}}.\bar{d}) <_{\alpha-LU} \mathscr{F}_\alpha(\bar{t}-\bar{\bar{h}}.\bar{d})$, which contradicts with the assumption. Hence the proof. ☐

Definition 5.5. *Assume that* $\mathscr{F}$ *is LgHs directional differentiable. If*

$$\mathscr{F}^s_{LgHs,\alpha}(\bar{t}; t-\bar{t}) \leqq_{\alpha-LU} \mathscr{F}_\alpha(t) \ominus_{gH} \mathscr{F}_\alpha(\bar{t}) \quad (5.2)$$

then fuzzy function $\mathscr{F}$ *is said to be* α *convex at* $\bar{t} \in \mathbb{R}^n$ *on* $\mathscr{P} \subseteq \mathbb{R}^n$, $\alpha \in [0,1]$, $\forall t \in \mathscr{P}$. *Also fuzzy function* $\mathscr{F}$ *is said to be* α *convex on* $\mathscr{P}$ *if it is* α *convex at every* $\bar{t} \in \mathbb{R}^n$ *on* P. *If* $\mathscr{F}$ *is* α *convex on* $\mathbb{R}^n$ *then* $\mathscr{F}$ *is* α *convex at* $\bar{t} \in \mathbb{R}^n$. *If* $\mathscr{F}$ *is* α *convex at every* $\bar{t} \in \mathbb{R}^n$ *then* $\mathscr{F}$ *is* α *convex on* $\mathbb{R}^n$. *If* $\mathscr{F}$ *is* α *convex* $\forall\ \alpha \in [0,1]$ *then* $\mathscr{F}$ *is LU convex.*

Now we are going to state the sufficient optimality condition using α convex.

Theorem 5.6. *Assume* $\mathscr{F}$ *be the directional LgHs differentiable* α *convex fuzzy function ,* $\bar{t} \in \mathbb{R}^n$, $\alpha \in [0,1]$ *and if* $\exists$ *no* $d \in \mathbb{R}^n$ *such that*

$$\mathscr{F}^s_{LgHs,\alpha}(\bar{t}; d) <_{\alpha-LU} [0,0] \quad (5.3)$$

then $\bar{t}$ *is a weak* α*-LU solution of* $\mathscr{F}$.

Proof. We assume the converse that $\exists\ t \in \mathbb{R}^n$ such that $\mathscr{F}(t) <_{\alpha-LU} \mathscr{F}(\bar{t})$.ie $\mathscr{F}_\alpha(t) <_{LU} \mathscr{F}_\alpha(\bar{t}) \implies \mathscr{F}_\alpha(t) \ominus_{gH}$

$\mathscr{F}_\alpha(\bar{t}) <_{LU} [0,0]$. Given $\mathscr{F}$ is α convex at $\bar{t}$ so we can say that

$$\mathscr{F}^s_{LgHs,\alpha}(\bar{t};t-\bar{t}) \leqq_{\alpha-LU} \mathscr{F}_\alpha(t) \ominus_{gH} \mathscr{F}_\alpha(\bar{t}) <_{LU} [0,0]$$

$$\Longrightarrow$$

$$\mathscr{F}^s_{LgHs,\alpha}(\bar{t};d) <_{LU} [0,0]$$

with $d = t - \bar{t}$, which is a contradiction. Hence the proof. □

6. Conclusion

This paper defines a new concept called LgHs derivative of functions which are of fuzzy valued and with several variables. We can see that LgHs derivative of functions which are of fuzzy valued and with several variables is more general than the existing definitions. Here we define the LgHs differentiability of fuzzy function which are of interval valued. Also we propose directional LgHs derivative of an interval valued fuzzy function. Finally we proved the necessary and sufficient α optimality condition of directional LgHs differentiable fuzzy function.The novel directional LgHs and partial gHs derivative unifies and extends in latest papers.

References

[1] Rommelfanger. H, Slowinski.R. , Fuzzy linear programming with single or multiple objective functions, *Fuzzy Sets in Decision Analysis, Operations Research and Statistics: Handbook Fuzzy Sets Series*, 1 (1998), 179–213.

[2] Delgado. M, Kacprzyk.J, Verdegay.J. L, Vila. M. A., *Fuzzy optimization: Recent Advances*, New York: Physica-Verlag, 1994.

[3] Lodwick. W. A., Kacprzyk. J. , *Fuzzy optimization: Recent Advances and Applications*, Berlin: Springer, 2010.

[4] Slowinski.R., Teghem. J., *A comparison study of "STRANGE" and "FLIP": Stochastic Versus Fuzzy Approaches to Multiobjective Mathematical Programming Under Uncertainty*, Dordrecht: Kluwer Academic Publisher, 1990.

[5] Inuiguchi.M., Ramik. J., Possibilistic linear programming: A brief review of fuzzy mathematical programming and a comparison with stochastic programming in portfolio selection problem , *Fuzzy Sets and Systems,* 111(2000), 3–28.

[6] Hukuhara M. , Integration des applications mesurables dont la valeur est un compact con- vexe, *Funkc. Ekvac.*, 10 (1967), 205–223.

[7] Bede.B., Gal.S. G., Generalizations of the differentiability of fuzzy number valued functions with applications to fuzzy differential equations, *J. Fuzzy Sets and Systems,* 151(2005), 581–599.

[8] Bede.B., Stefanini. L. , Generalized differentiability of fuzzy-valued functions , *Fuzzy Sets and Systems* , 230(2013), 119–141.

[9] Chalco-Cano.Y., Roman-Flores, H., Jiménez-Gamero, M. D. , Generalized derivative and π derivative for set-valued functions , *Information Sciences,* 181(2011), 2177–2188.

[10] Luciano Stefanini, Manuel Arana Jimenez, Karush-Kuhn-Tucker conditions for interval and fuzzy optimization in several variables under total and directional generalized differentiability, *Fuzzy Sets and Systems*, (2018), 1–34.

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