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The lattice of pre-complements of a classic interval valued fuzzy graph

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Abstract

We prove that a complement Interval Valued Fuzzy Graph (IVFG), unlike the crisp and fuzzy cases, may have several non-isomorphic pre-complements. We introduce the notion of complement numbers, and show that, by assigning complement numbers to the edges of a complement IVFG, we can ensure uniqueness of pre-complement. We introduce the concepts *superior pre-complement* $\mathscr{G}^*$ and *inferior pre-complement* $\mathscr{G}_*$, for any given classic IVFG $\mathscr{G}$. A partial order $\underset{P}{\subseteq}$ is defined on $\mathscr{P} = C^{-1}(\mathscr{G})$, the collection of all pre-complements of $\mathscr{G}$. It is proved that ($\mathscr{P}$, $\underset{P}{\subseteq}$) is a lattice with $\mathscr{G}^*$ as the greatest element and $\mathscr{G}_*$ as the least element. We derive a necessary and sufficient condition for this lattice to become a chain.

Keywords

Interval Valued Fuzzy Graph, Complement, Complement Number, Lattice.

AMS Subject Classification

05C72, 06B99.

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1. Introduction

In 1736, Euler who solved the *Königsberg bridge problem* laid the foundation of Graph theory. It has grown into a significant area of mathematical research, with applications in chemistry, operations research, social sciences, computer science, etc. In 1965, L. A. Zadeh [15] introduced the notion of *fuzzy set* and in 1975, Rosenfeld [3] used the idea of fuzzy set to develop the concept *fuzzy graph*. Mordeson and Peng [14] defined the concept of complement of fuzzy graph and described some operations on fuzzy graphs. In [22], the definition of complement of a fuzzy graph was modified so that the complement of the complement is the original fuzzy graph, which agrees with the crisp graph case. In [16], Zadeh extended the concept of fuzzy sets to *interval valued fuzzy sets* , in which the values of the membership degree are intervals of numbers instead of fixed numbers between 0 and 1. Ju and Wang gave the definition of interval-valued fuzzy graph (IVFG) in [9]. Akram et al. ([17] - [21]) introduced many new concepts including bipolar fuzzy graphs, interval-valued line fuzzy graphs, and strong intuitionistic fuzzy graphs. [2], [12], [24] are some recent works in this area. In [1] *complement of interval valued fuzzy graphs* was defined as an extension of complement fuzzy graph [22].

We observed that the definition of complement of IVFG in [1] does not work in all cases and so redefined the concept in [5]. Some properties of the new definition was studied in [4] and [6]. It was noted that several non-isomorphic IVFGs provide the same complement. To overcome this limitation we introduced the notion of *complement number* of an edge in [6] and proved that by assigning complement numbers to the edges of a complement IVFG, we can ensure uniqueness of pre-complement. In this paper, we give a necessary and sufficient condition for a complement IVFG to have a unique pre-complement, without using complement numbers. We observe that a given classic IVFG $\mathscr{G}$ may have a unique or infinitely many pre-complements. We construct two special pre-complements $\mathscr{G}^*$ and $\mathscr{G}_*$ for a classic IVFG $\mathscr{G}$. The notation $\mathscr{P} = C^{-1}(\mathscr{G})$ is used for the collection of all pre-complements of $\mathscr{G}$. We define a partial order $\underset{P}{\subseteq}$ on $\mathscr{P}$, and prove that ($\mathscr{P}$, $\underset{P}{\subseteq}$) is a lattice with $\mathscr{G}^*$ as greatest element and $\mathscr{G}_*$ as least element. We have proved that the lattice ($\mathscr{P}$, $\underset{P}{\subseteq}$) becomes a chain if, and only if, $\mathscr{G}$ has atmost one edge uv with real number membership r, where $r \neq min\{\mathscr{V}_u^-, \mathscr{V}_v^-\}$ and $\mathscr{V}_u^-$ represents the lower end point of the interval membership of the vertex u.

2. Preliminaries

In this section, we present the basic concepts and results used in this work. Most of them are well-known, and yet we include them for the sake of completeness.

Definition 2.1. *[7] A relation $\mathscr{R}$ on a non-empty set A is called a **partial ordering** (or **partial order**) if it satisfies the following conditions:*

1. *Reflexivity ($x \mathscr{R} x$, for all $x \in A$).*
2. *Anti-symmetry ($x\mathscr{R}y$ and $y\mathscr{R}x \Rightarrow x = y$, for all $x, y \in A$).*
3. *Transitivity ($x\mathscr{R}y$ and $y\mathscr{R}z \Rightarrow x\mathscr{R}z$, for all $x, y, z \in A$).*

*A set A together with a partial ordering $\mathscr{R}$ is called a **partially ordered set** (or **poset**) and is denoted as $(A, \mathscr{R})$.*

*Let x and y be the elements of the poset $(A, \mathscr{R})$. Then x and y are said to **comparable** if either $x\mathscr{R}y$ or $y\mathscr{R}x$. Otherwise, x and y are said to be **incomparable**. If every two element of a poset $(A, \mathscr{R})$ are comparable, we call it a **totally ordered set** or a **chain**.*

*An element x in the poset $(A, \mathscr{R})$ is said to be the **greatest element** if $y\mathscr{R}x$, for all $y \in A$; and the **least element** if $x\mathscr{R}y$, for all $y \in A$.*

*Let S be any subset of A. Then $x \in A$ is called an **upper bound** of S if $y\mathscr{R}x$ for all $y \in S$; and a **lower bound** of S if $x\mathscr{R}y$ for all $y \in S$. Let U and L denote the set of all upper bounds and lower bounds of S, respectively. Then, if there exist $u \in U$ such that $u\mathscr{R}x$ for all $x \in U$, we say that, u is the **least upper bound** (l.u.b) of S and if there exist $l \in L$ such that $x\mathscr{R}l$ for all $x \in L$, we say that, l is the **greatest lower bound** (g.l.b) of S.*

We use the notations $x \vee y$ and $x \wedge y$, respectively, for the l.u.b and g.l.b of $\{x, y\}$.

Definition 2.2. *[7] A poset $(A, \mathscr{R})$ is said to be a **lattice** if both $x \vee y$ and $x \wedge y$ exist in A for all $x, y \in A$.*

Definition 2.3. *[13] A **graph** (or **crisp graph**) is defined as a pair $G = (V, E)$ consisting of a non-empty finite set V of elements called **vertices** and a finite set E of elements called **edges** such that each edge e in E is assigned unordered pair of vertices (u, v) called the **end vertices** of e. We can represent e by uv or vu.*

*An edge $e = uv$ is called a **loop** if $u = v$ and two edges e and f are said to be **parallel** if they have the same end vertices. A graph with no loops and parallel edges is called a **simple graph**.*

*The **complement** $\overline{G}$ of a simple graph G is defined as the simple graph with the same vertex set as G and there is an edge connecting vertices u and v if, and only if, there is no such edge in G.*

From the above definition it is clear that for any graph G, $\overline{\overline{G}} = G$.

Definition 2.4. *[8] A **fuzzy set** A on a set X is characterized by a mapping $\mathscr{M} : X \to [0, 1]$, which is called the membership function and the fuzzy set A on X is denoted by $A = \{(x, \mathscr{M}(x)) | x \in X\}$. Here $\mathscr{M}(x)$ is called the membership level of x in A.*

Definition 2.5. *[22] A **fuzzy graph** $G_F = (V, E, \mathscr{V}, \mathscr{E})$ consists of a non-empty vertex set V together with an edge set E of all unordered pairs of vertices and a pair of functions $\mathscr{V} : V \to [0, 1]$, $\mathscr{E} : E \to [0, 1]$ such that $\mathscr{V}(u) \neq 0$ for at least one $u \in V$ and $\mathscr{E}(uv) \leq min\{\mathscr{V}(u), \mathscr{V}(v)\}$, for all $uv \in E$.*

*The **complement of fuzzy graph** $G_F = (V, E, \mathscr{V}, \mathscr{E})$ is a fuzzy graph $\overline{G_F} = (V, E, \mathscr{V}, \overline{\mathscr{E}})$ where $\overline{\mathscr{E}}(uv) = min\{\mathscr{V}(u), \mathscr{V}(v)\} - \mathscr{E}(uv)$.*

Similar to any crisp graph, all fuzzy graphs will satisfy the property $\overline{\overline{G_F}} = G_F$.

Definition 2.6. *[16] An **interval valued fuzzy set (IVFS)** A on X is defined by $A = \{(x, i(x)) | x \in X\}$ where i is an interval–valued function from X to $P[0, 1]$, set of all subsets of $[0, 1]$, such that $i(x) = [A_x^-, A_x^+]$ where $0 \leq A_x^- \leq A_x^+ \leq 1$. Here $i(x)$ is called the membership interval of x*

Definition 2.7. *[9] An **interval valued fuzzy graph (IVFG)** $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ consists of a non–empty vertex set V together with an edge set E of all unordered pairs of vertices and a pair of interval valued functions $\mathscr{V}$ and $\mathscr{E}$ which satisfy the following conditions:*

1. *$\mathscr{V} : V \to P[0, 1]$ such that $\mathscr{V}(u) = [\mathscr{V}_u^-, \mathscr{V}_u^+], 0 \leq \mathscr{V}_u^- \leq \mathscr{V}_u^+ \leq 1$.*
2. *$\mathscr{V}(u) \neq 0$ for atleast one $u \in V$.*
3. *$\mathscr{E} : E \to P[0, 1]$ such that $\mathscr{E}(uv) = [\mathscr{E}_{uv}^-, \mathscr{E}_{uv}^+], 0 \leq \mathscr{E}_{uv}^- \leq \mathscr{E}_{uv}^+ \leq 1$; and*
4. *$\mathscr{E}_{uv}^- \leq min\{\mathscr{V}_u^-, \mathscr{V}_v^-\}$ and $\mathscr{E}_{uv}^+ \leq min\{\mathscr{V}_u^+, \mathscr{V}_v^+\}$, for all $u, v \in V$.*

Definition 2.8. *[11] Let $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ and $\mathscr{H} = (W, F, \sigma, \mu)$ be two IVFGs. Then $\mathscr{G}$ and $\mathscr{H}$ are said to be **isomorphic** if there exist a bijection $h : V \to W$ such that*

1. *$\mathscr{V}_u^- = \sigma_{h(u)}^-, \mathscr{V}_u^+ = \sigma_{h(u)}^+$ for every $u \in V$; and*
2. *$\mathscr{E}_{uv}^- = \mu_{h(u)h(v)}^-, \mathscr{E}_{uv}^+ = \mu_{h(u)h(v)}^+$ for any $uv \in E$.*

We write $\mathscr{G} \cong \mathscr{H}$ for the statement "$\mathscr{G}$ is isomorphic to $\mathscr{H}$".

3. Complement of an Interval Valued Fuzzy Graph

The complement of an IVFG $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ is defined in [1] as the IVFG $\overline{\mathscr{G}} = (V, E, \mathscr{V}, \overline{\mathscr{E}})$ where

$$\overline{\mathscr{E}}(uv) = [min\{\mathscr{V}_u^-, \mathscr{V}_v^-\} - \mathscr{E}_{uv}^-, min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \mathscr{E}_{uv}^+]$$

for every edge $uv \in E$. This is a direct extension of the notion of *complement* in crisp graph theory and *fuzzy complement* in fuzzy graph theory. But it has a serious defect. It does not apply to all IVFGs.

Example 3.1. *Consider the IVFG G given in figure(1).*

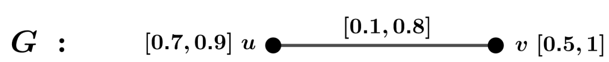

Figure 1. An IVFG for which complement cannot be formed using the definition in [1]

Using the above definition we cannot construct its complement since

$$min\{0.7, 0.5\} - 0.1 = 0.4 > min\{0.9, 1\} - 0.8 = 0.1$$

and so the membership-interval of the edge uv has to be $[0.4, 0.1]$ *which is absurd.*

So in [5] we have redefined complement of an IVFG as follows.

Definition 3.2. *[5] The* ***complement of IVFG*** $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ *is an IVFG* $\overline{\mathscr{G}} = (V, E, \mathscr{V}, \overline{\mathscr{E}})$ *where* $\overline{\mathscr{E}}(uv) = [\overline{\mathscr{E}}_{uv}^-, \overline{\mathscr{E}}_{uv}^+]$

$$= \begin{cases} [min\{\mathscr{V}_u^-, \mathscr{V}_v^-\} - \mathscr{E}_{uv}^-, min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \mathscr{E}_{uv}^+]; \\ \quad \text{if } min\{\mathscr{V}_u^-, \mathscr{V}_v^-\} - \mathscr{E}_{uv}^- \leq min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \mathscr{E}_{uv}^+ \\ .[min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \mathscr{E}_{uv}^+, min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \mathscr{E}_{uv}^+]; \\ \quad \text{otherwise} \end{cases}$$

Using the above definition, we can form the complement of every IVFG. For example, the complement of the IVFG G given in figure(1), whose complement could not be formed using the definition in [1], is obtained as the IVFG $\overline{G}$ given in figure(2)

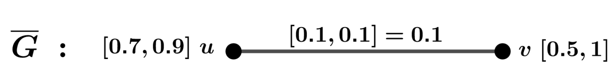

Figure 2. Complement of the IVFG G in figure(1) obtained using definition(3.2).

Moreover, definition(3.2) generalises the notions of complement and fuzzy complement.

These observations motivate the following definitions.

Definition 3.3. *[5] An IVFG* $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ *is called a* ***classic IVFG*** *if it satisfies the condition*

$$min\{\mathscr{V}_u^-, \mathscr{V}_v^-\} - \mathscr{E}_{uv}^- \leq min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \mathscr{E}_{uv}^+ \text{, for all edges } uv \in E.$$

The above condition is called the classic condition. Otherwise we call it a ***non-classic IVFG****.*

Definition 3.4. *[5] Let* $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ *be an IVFG. Then an edge* $uv \in E$ *is called a* ***perfect edge*** *if* $min\{\mathscr{V}_u^-, \mathscr{V}_v^-\} - \mathscr{E}_{uv}^- \leq min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \mathscr{E}_{uv}^+$ *. Otherwise, edge* $uv \in E$ *is called an* ***imperfect edge****.*

It may be noted that all edges of a classic IVFG are perfect. An IVFG is non-classic, if and only if, it has atleast one imperfect edge.

Throughout this paper, we use the convention that, if the membership-interval of a vertex is not given, it will be assumed as $1 = [1, 1]$*. Also, if the membership-interval of an edge is not given, or if an edge is not drawn, its membership-interval will be assumed as* $0 = [0, 0]$*. Intervals of the form* $[r, r]$ *represent the real number r and is refered to as a* ***degenerate interval****.*

Example 3.5. *Consider the IVFG given in figure(3).*

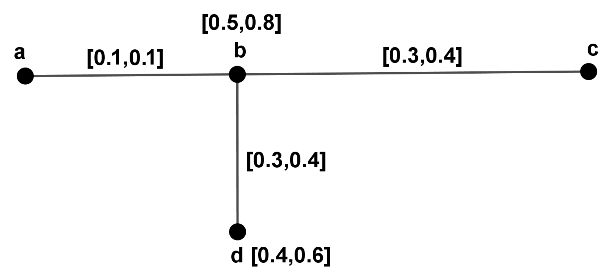

Figure 3. Example for classic IVFG.

We will show that it is a classic IVFG. The condition for being classic IVFG is

$$min\{\mathscr{V}_u^-, \mathscr{V}_v^-\} - \mathscr{E}_{uv}^- \leq min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \mathscr{E}_{uv}^+$$

for each edge uv. Consider the edges one by one.

1. *ab (Memership-interval is [0.1,0.1]) :* $min\{\mathscr{V}_a^-, \mathscr{V}_b^-\} - \mathscr{E}_{ab}^- = min\{1, 0.5\} - 0.1 = 0.4 < min\{\mathscr{V}_a^+, \mathscr{V}_b^+\} - \mathscr{E}_{ab}^+ = min\{1, 0.8\} - 0.1 = 0.7$.

2. *ac (Not drawn. So, memership-interval is* $0 = [0, 0]$*) :* $min\{\mathscr{V}_a^-, \mathscr{V}_c^-\} - \mathscr{E}_{ac}^- = min\{1, 1\} - 0 = 1 = min\{\mathscr{V}_a^+, \mathscr{V}_c^+\} - \mathscr{E}_{ac}^+ = min\{1, 1\} - 0 = 1$.

In a similar way, we can verify the property for edges ad, bc, bd and cd. Hence every edge in the IVFG in figure(3) is perfect, and so it is a classic IVFG.

Next, we will give an example of a non-classic IVFG.

Example 3.6. *Consider the edge bc in the IVFG given in figure(4).*

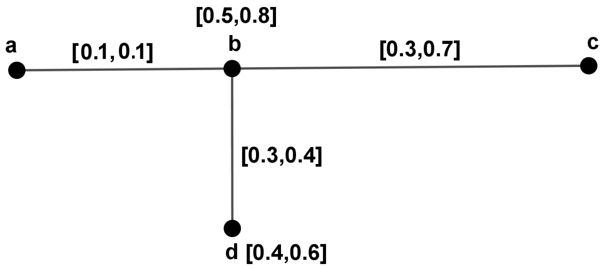

Figure 4. Example for non-classic IVFG.

Here, $min\left\{\mathscr{V}_b^-,\mathscr{V}_c^-\right\}-\mathscr{E}_{bc}^- = min\left\{0.5,1\right\}-0.3 = 0.2 > min\left\{\mathscr{V}_b^+,\mathscr{V}_c^+\right\}-\mathscr{E}_{bc}^+ = min\left\{0.8,1\right\}-0.7 = 0.1.$ *Therefore it is an imperfect edge. Hence the IVFG in figure(4) is non-classic.*

Remark 3.7. *In the case of crisp and fuzzy graphs, for a given complement graph [fuzzy graph], we can uniquely find the original graph [fuzzy graph] for which the complement was formed. But in the case of IVFGs, the situation is different.*

Let $\mathscr{G}$ be an IVFG and $\overline{\mathscr{G}}$ be its complement. Then we shall refer to $\mathscr{G}$ as the *pre-complement* of $\overline{\mathscr{G}}$. For a given IVFG there is a unique complement. But for a given complement IVFG there may be several non-isomorphic pre-complements.

Example 3.8. *Consider the IVFGs* $\mathscr{G}$ *and* $\mathscr{H}$ *given in figure(5).*

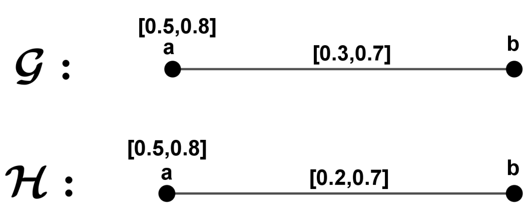

Figure 5. Two non-isomorphic IVFGs having the same complement.

They are obviuosly non-isomorphic. But they have the same complement IVFG given in figure(6).

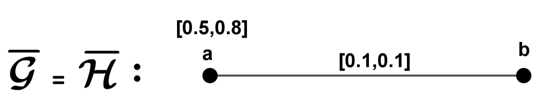

Figure 6. Complement of the IVFGs given in figure(5)

Hence $\mathscr{G}$ *and* $\mathscr{H}$ *are non-isomorphic pre-complements of the IVFG in figure(6).*

It may be noted that the IVFGs in figure(5) are non-classic. But the complement IVFG in figure(6) is classic. We have proved the following results in [5].

Theorem 3.9. *[5] For any IVFG* $\mathscr{G}$, $\overline{\mathscr{G}}$ *is a classic IVFG.*

Theorem 3.10. *[5] An IVFG* $\mathscr{G}$ *is classic if and only if* $\overline{\overline{\mathscr{G}}} = \mathscr{G}$.

In view of theorem(3.9), we can say that, only classic IVFGs can have pre-complement. Moreover, for every classic IVFG, we can make at least one pre-complement. This observation gives us the following result.

Proposition 3.11. *An IVFG has pre-complement if and only if it is classic.*

Proof. Straight forward □

4. Complement numbers and pre-complements

In the previous section, we have seen that, a given complement IVFG may have several pre-complements. In the present section we introduce an extra feature which can ensure uniqueness of pre-complement.

Definition 4.1. *[6] Let* $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ *be an IVFG and* $\overline{\mathscr{G}} = (V,E,\mathscr{V},\overline{\mathscr{E}})$ *be its complement. Then the* ***complement number of an edge*** uv ***in*** $\overline{\mathscr{G}}$ ***w.r.t.*** $\mathscr{G}$, *denoted as* $c_{uv}^{\mathscr{G}}$ *or simply* c_{uv}, *is defined as*

$$c_{uv} = \begin{cases} min\left\{\mathscr{V}_u^-,\mathscr{V}_v^-\right\}-\mathscr{E}_{uv}^-, & \text{if } uv \text{ is an imperfect edge of } \mathscr{G} \\ 0, & \text{otherwise} \end{cases}$$

In this section, if c_{uv} is not given for an edge uv of a complement IVFG, we will assume that its complement number is 0. When a complement IVFG is given, along with complement numbers of edges, we will call it a *numbered complement.*

Example 4.2. *Consider the non-isomorphic IVFGs* $\mathscr{G}$ *and* $\mathscr{H}$ *in figure(5) having the same complement given in figure(6). Their numbered complements are given in figure(7)*

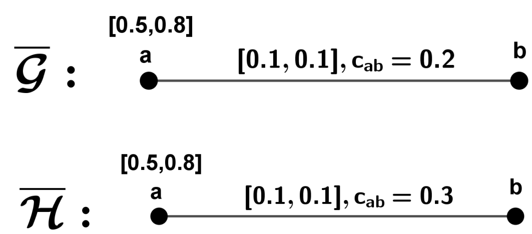

Figure 7. Numbered complements of the IVFGs in figure(5).

We observe that they differ only in complement numbers.

A numbered complement has a unique pre-complement. In [6], we have given a method to obtain the unique pre-complement when a numbered complement is given. This method is summarized in the following theorem.

Theorem 4.3. *[6] Let any complement graph* $\overline{\mathscr{G}} = (V,E,\mathscr{V},\overline{\mathscr{E}})$ *along with the complement numbers of each edge be given.*

Then the unique IVFG from which $\overline{\mathscr{G}}$ is made is $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ where

$$\mathscr{E}_{uv} = \begin{cases} [min\{\mathscr{V}_u^-, \mathscr{V}_v^-\} - c_{uv}, min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \overline{\mathscr{E}}_{uv}^+], \\ \qquad \qquad if\, c_{uv} \neq 0 \\ [min\{\mathscr{V}_u^-, \mathscr{V}_v^-\} - \overline{\mathscr{E}}_{uv}^-, min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \overline{\mathscr{E}}_{uv}^+], \\ \qquad \qquad if\, c_{uv} = 0 \end{cases}$$

Example 4.4. *Consider the complement IVFG given in figure(8). By using theorem(4.3), we can determine the IVFG $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ from which this complement is made.*

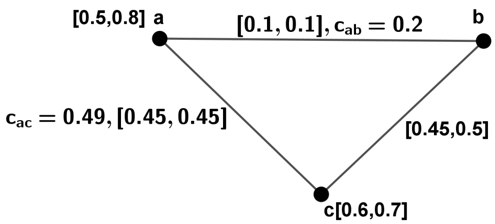

Figure 8. A numbered complement IVFG.

Here, $\overline{\mathscr{E}}(ab) = [0.1, 0.1]$ with $c_{ab} = 0.2$. So,

$$\begin{aligned} \mathscr{E}(ab) &= [min\{0.5, 1\} - 0.2, min\{0.8, 1\} - 0.1] \\ &= [0.5 - 0.2, 0.8 - 0.1] = [0.3, 0.7]. \end{aligned}$$

Similarly, $\mathscr{E}(ac) = [0.5 - 0.49, 0.7 - 0.45] = [0.01, 0.25]$. Since c_{bc} is not given we consider it as zero and hence $\mathscr{E}(bc) = [0.6 - 0.3, 0.7 - 0.5] = [0.15, 0.2]$.

Hence we get its pre-complement as the IVFG in figure(9).

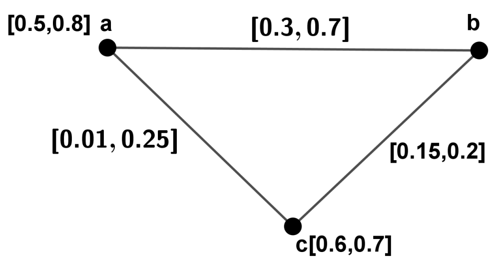

Figure 9. The unique pre-complement of the numbered complement IVFG in figure(8)

It can be easily verified that the complement of IVFG in figure(9) is the numbered complement IVFG given in figure(8).

Remark 4.5. *In theorem(3.9), we have seen that the complement of any IVFG is classic. Conversely, any classic IVFG is the complement of some IVFG. For example, consider any classic IVFG $\mathscr{G}$. We can assign complement numbers 0 to its edges and form a particular pre-complement for it, using the technique given in theorem(4.3), with $\overline{\mathscr{E}}$ in the formula replaced by $\mathscr{E}$.*

Example 4.6. *Consider the classic IVFG G given in figure(10).*

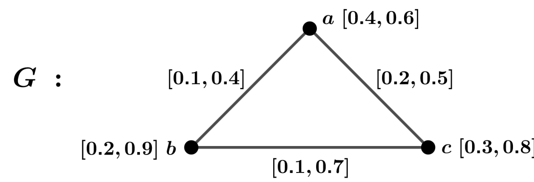

Figure 10. A classic IVFG for which pre-complement is formed in figure(11)

Assuming the complement numbers of all its edges as 0, we can form the pre-complement given in figure(11) for it.

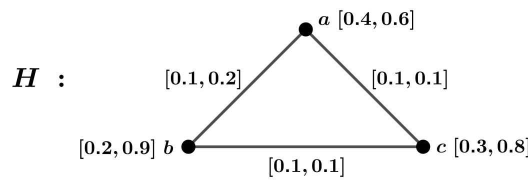

Figure 11. A pre-complement of the classic IVFG in figure(10)

One can easily verify that $\overline{H} = G$; and hence H is a pre-complement of G.

We have the following results regarding complement numbers.

Proposition 4.7. *[6] Let $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ be any IVFG. Then in $\overline{\mathscr{G}} = (V, E, \mathscr{V}, \overline{\mathscr{E}})$*

$$c_{uv} = 0 \Leftrightarrow uv \text{ is a perfect edge of } \mathscr{G}.$$

Proposition 4.8. *[6] Let $\overline{\mathscr{E}}(uv) = [\overline{\mathscr{E}}_{uv}^-, \overline{\mathscr{E}}_{uv}^+]$ for an edge uv in $\overline{\mathscr{G}} = (V, E, \mathscr{V}, \overline{\mathscr{E}})$. Then either $c_{uv} = 0$ or $\overline{\mathscr{E}}_{uv}^+ < c_{uv} \leq min\{\mathscr{V}_u^-, \mathscr{V}_v^-\}$.*

Now we give a necessary and sufficient condition for a complement IVFG to have a unique pre-complement.

Theorem 4.9. *Let $\overline{\mathscr{G}} = (V, E, \mathscr{V}, \overline{\mathscr{E}})$ be any complement IVFG. Then $\overline{\mathscr{G}}$ has a unique pre-complement if, and only if, for any $uv \in E$ with $\overline{\mathscr{E}}(uv) = r$, a real number, then $r = min\{\mathscr{V}_u^-, \mathscr{V}_v^-\}$.*

Proof. Let $uv \in E$ such that $\overline{\mathscr{E}}(uv) = [\overline{\mathscr{E}}_{uv}^-, \overline{\mathscr{E}}_{uv}^+] = r \in [0, 1]$. Then $r = min\{\mathscr{V}_u^-, \mathscr{V}_v^-\}$. Hence by proposition(4.8), $c_{uv} = 0$.

Let $uv \in E$ such that $\overline{\mathscr{E}}(uv) = [\overline{\mathscr{E}}_{uv}^-, \overline{\mathscr{E}}_{uv}^+] \neq r \in [0, 1]$.

$\Rightarrow min\{\mathscr{V}_u^-, \mathscr{V}_v^-\} - \mathscr{E}_{uv}^- \leq min\{\mathscr{V}_u^+, \mathscr{V}_v^+\} - \mathscr{E}_{uv}^+$, by definition(3.2).

$\Rightarrow uv$ is a perfect edge, by definition(3.4).

$\Rightarrow c_{uv} = 0$, by proposition(4.7).

Hence $c_{uv} = 0$ for all $uv \in E$ and so by theorem(4.3), we can uniquely determine an IVFG whose complement is $\overline{\mathscr{G}}$.

To prove the converse part, it is enough to show that if there exist $uv \in E$ such that $\overline{\mathscr{E}}(uv) = [\overline{\mathscr{E}}^-_{uv}, \overline{\mathscr{E}}^+_{uv}] = r \in [0,1]$ but $r \neq min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$, then there exist infinitely many IVFGs whose complement is $\overline{\mathscr{G}}$.

Let $uv \in E$ such that $\overline{\mathscr{E}}(uv) = [\overline{\mathscr{E}}^-_{uv}, \overline{\mathscr{E}}^+_{uv}] = r \in [0,1]$ but $r \neq min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$. Then by proposition(4.8), $c_{uv} = 0$ or $r < c_{uv} \leq min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$. Since $r \neq min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$, c_{uv} has infinitely many choices and hence, by theorem(4.3), corresponding to each collection of complement numbers, we will get an IVFG whose complement is $\overline{\mathscr{G}}$. □

Corollary 4.10. *Let $\overline{\mathscr{G}} = (V, E, \mathscr{V}, \overline{\mathscr{E}})$ be any complement IVFG and $\overline{\mathscr{E}}(uv) = [\overline{\mathscr{E}}^-_{uv}, \overline{\mathscr{E}}^+_{uv}] \neq r \in [0,1)$ for all $uv \in E$. Then $\overline{\mathscr{G}}$ has a unique pre-complement, and it will be a classic IVFG.*

Corollary 4.11. *Any classic IVFG $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ has a unique pre-complement if, and only if, for any $uv \in E$ with $\mathscr{E}(uv) = r$, a real number, then $r = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$.*

Corollary 4.12. *A classic IVFG $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ with $\mathscr{E}(uv) \neq r \in [0,1)$, for every edge $uv \in E$, has a unique pre-complement, which also is a classic IVFG.*

5. The lattice of pre-complements

We have observed that a given complement IVFG may have a unique or several pre-complements. The same is true for a classic IVFG also. Let $C^{-1}(\mathscr{G})$ denote the collection of pre-complements of a classic IVFG $\mathscr{G}$. In this section, we denote $C^{-1}(\mathscr{G})$ by $\mathscr{P}$.

Definition 5.1. *Let $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ be any classic IVFG. Then the pre-complement of $\mathscr{G}$ obtained by assigning $c_{uv} = 0$ to all edges $uv \in E$ is called the **superior pre-complement of** $\mathscr{G}$ and is denoted as $\mathscr{G}^*$.*

The pre-complement of $\mathscr{G}$ obtained by assigning the complement numbers

$$c_{uv} = \begin{cases} 0, & \text{if } \mathscr{E}(uv) \neq r, \text{ a real number} \\ min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}, & \text{otherwise} \end{cases}$$

*to each edge $uv \in E$ is called the **inferior pre-complement of** $\mathscr{G}$. It is denoted as $\mathscr{G}_*$.*

Example 5.2. *The IVFG H in figure(11) is the pre-complement obtained by assigning the complement number 0 to each edge of the IVFG G in figure(10). Therefore, $H = G^*$.*

Now, to make G_*, since all edges have non-degenerate membership intervals, we have to assign complement number 0 to each edge, and form the pre-complement. Hence $G_* = G^* = H$.

Remark 5.3. *If $\mathscr{G}$ is a classic IVFG whose each edge has a membership interval, which is not degenerate, then $G_* = G^*$.*

Example 5.4. *Consider the classic IVFG in figure(12).*

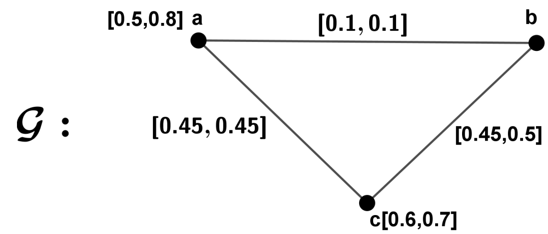

Figure 12. A classic IVFG with non-degenerate membership interval for an edge.

Assigning complement number 0 to each edge and forming its pre-complement, we get the IVFG in figure(13)

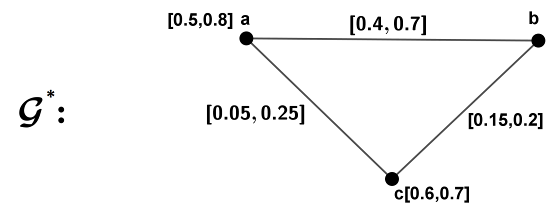

Figure 13. Superior pre-complement for the IVFG in figure(12).

Now, we proceed to make $\mathscr{G}_*$. First we assign complement numbers $c_{ab} = min\{0.5, 1\} = 0.5$, $c_{bc} = min\{0.5, 0.6\} = 0.5$, $c_{ac} = 0$ and then form the pre-complement. Then we get the IVFG in figure(14) as $\mathscr{G}_*$.

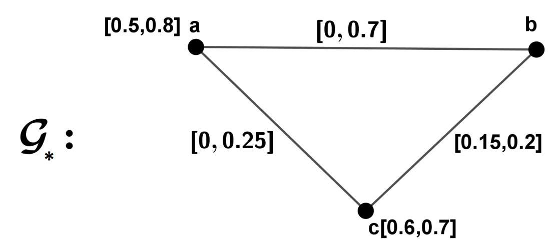

Figure 14. Inferior pre-complement of the IVFG in figure(12).

Here in $\mathscr{G}$, the membership interval is non-degenerate for one edge; and so $\mathscr{G}^* \neq \mathscr{G}_*$

Remark 5.5. .

1. *If $\mathscr{G}$ has at least one edge whose membership interval is not degenerate, then $\mathscr{G}^* \neq \mathscr{G}_*$.*

2. *For every classic IVFG $\mathscr{G}$, $\mathscr{G}^*$ and $\mathscr{G}_*$ are uniquely formed.*

3. *Using theorem(4.3), we can rewrite the definitions of $\mathscr{G}^*$ and $\mathscr{G}_*$ directly without involving complement numbers, as follows.*

   Let $\mathscr{G} = (V, E, \mathscr{V}, \mathscr{E})$ be any classic IVFG. Then

(a) $\mathscr{G}^* = (V,E,\mathscr{V},\mathscr{E}^*)$ *where for every edge* $uv \in E$
$\mathscr{E}^*(uv) =$
$[min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - \mathscr{E}_{uv}^-, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - \mathscr{E}_{uv}^+]$

(b) $\mathscr{G}_* = (V,E,\mathscr{V},\mathscr{E}_*)$ *where for each edge* $uv \in E$

$\mathscr{E}_*(uv) =$

$$\begin{cases} [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - \mathscr{E}_{uv}^-, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - \mathscr{E}_{uv}^+], \\ \quad \text{if } \mathscr{E}(uv) \neq \text{a real number} \\ [0, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r], \\ \quad \text{if } \mathscr{E}(uv) = r, \text{ a real number} \end{cases}$$

Theorem 5.6. *Let* $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ *be any classic IVFG. Then* $\mathscr{G}^* = \mathscr{G}_*$ *if, and only if, for any* $uv \in E$ *with* $\mathscr{E}(uv) = r$*, a real number, then* $r = min\{\mathscr{V}_u^-,\mathscr{V}_v^-\}$.

Proof. Let $\mathscr{G}^* = (V,E,\mathscr{V},\mathscr{E}^*)$ and $\mathscr{G}_* = (V,E,\mathscr{V},\mathscr{E}_*)$. Assume, for any $uv \in E$ such that $\mathscr{E}(uv) = [\mathscr{E}_{uv}^-,\mathscr{E}_{uv}^+] = r \in [0,1] \Rightarrow r = min\{\mathscr{V}_u^-,\mathscr{V}_v^-\}$. To show that $\mathscr{G}^* = \mathscr{G}_*$, it is enough to prove $\mathscr{E}^*(uv) = \mathscr{E}_*(uv)$ for all $uv \in E$.

Let $uv \in E$ such that $\mathscr{E}(uv) \neq r \in [0,1]$. Then by definition(5.1), $c_{uv}^{\mathscr{G}^*} = 0$ and $c_{uv}^{\mathscr{G}_*} = 0$. $\Rightarrow \mathscr{E}^*(uv) = [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - \mathscr{E}_{uv}^-, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - \mathscr{E}_{uv}^+] = \mathscr{E}_*(uv)$, by theorem(4.3).

Now, let $uv \in E$ such that $\mathscr{E}(uv) = r \in [0,1]$
$\Rightarrow r = min\{\mathscr{V}_u^-,\mathscr{V}_v^-\}$ (given). Then by definition(5.1), $c_{uv}^{\mathscr{G}^*} = 0$ and hence, by theorem(4.3)

$$\begin{aligned} &\mathscr{E}^*(uv) \\ &= [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - \mathscr{E}_{uv}^-, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - \mathscr{E}_{uv}^+] \\ &= [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - r, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r] \\ &= [r - r, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r] \\ &= [0, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r] \end{aligned}$$

Case 1. $\quad min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} = 0$
By definition(5.1), $c_{uv}^{\mathscr{G}_*} = min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} = 0$
$\Rightarrow \mathscr{E}_*(uv) = [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - \mathscr{E}_{uv}^-, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - \mathscr{E}_{uv}^+]$, by theorem(4.3).

$$\begin{aligned} &= [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - r, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r] \\ &= [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - min\{\mathscr{V}_u^-,\mathscr{V}_v^-\}, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r] \\ &= [0, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r] \end{aligned}$$

Case 2. $\quad min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} \neq 0$
By definition(5.1), $c_{uv}^{\mathscr{G}_*} = min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} \neq 0$
$\Rightarrow \mathscr{E}_*(uv) = [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - c_{uv}^{\mathscr{G}_*}, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - \mathscr{E}_{uv}^+]$, by theorem(4.3).

$$= [0, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r]$$

Hence $\mathscr{E}^*(uv) = \mathscr{E}_*(uv)$ for all $uv \in E$.

Conversely, suppose there exist $uv \in E$ such that $\mathscr{E}(uv) = [\mathscr{E}_{uv}^-,\mathscr{E}_{uv}^+] = r \in [0,1]$ but $r \neq min\{\mathscr{V}_u^-,\mathscr{V}_v^-\}$. To show that $\mathscr{G}^* \neq \mathscr{G}_*$, it is enough to prove $\mathscr{E}^*(uv) \neq \mathscr{E}_*(uv)$

Case 1. $\quad c_{uv}^{\mathscr{G}_*} = 0$
$\Rightarrow min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} = 0$, by definition(5.1)
$\Rightarrow \mathscr{E}(uv) = [0,\mathscr{E}_{uv}^+]$ (since $\mathscr{G}$ is an IVFG, $0 \leq \mathscr{E}_{uv}^-$ and $\mathscr{E}_{uv}^- \leq min\{\mathscr{V}_u^-,\mathscr{V}_v^-\}$)
$\Rightarrow \mathscr{E}(uv) = 0$ (since $\mathscr{E}(uv) = [\mathscr{E}_{uv}^-,\mathscr{E}_{uv}^+] = r$)
$\Rightarrow r = 0$, which is a contradiction to our assumption $r \neq min\{\mathscr{V}_u^-,\mathscr{V}_v^-\}$.

Case 2. $\quad c_{uv}^{\mathscr{G}_*} \neq 0$
$\Rightarrow \mathscr{E}_*(uv) = [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - c_{uv}^{\mathscr{G}_*}, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - \mathscr{E}_{uv}^+]$, by theorem(4.3).

$$= [0, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r], \text{by definition(5.1)}$$

Now, by definition(5.1) $c_{uv}^{\mathscr{G}^*} = 0$.
$\Rightarrow \mathscr{E}^*(uv) = [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - \mathscr{E}_{uv}^-, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - \mathscr{E}_{uv}^+]$, by theorem(4.3).

$$\begin{aligned} &= [min\{\mathscr{V}_u^-,\mathscr{V}_v^-\} - r, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r] \\ &\neq [0, min\{\mathscr{V}_u^+,\mathscr{V}_v^+\} - r] \text{ (since } r \neq min\{\mathscr{V}_u^-,\mathscr{V}_v^-\}) \end{aligned}$$

$\Rightarrow \mathscr{E}^*(uv) \neq \mathscr{E}_*(uv)$. Hence proved. □

Corollary 5.7. *Let* $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ *be any classic IVFG. Then* $\mathscr{G}$ *has unique pre-complement if, and only if,* $\mathscr{G}^* = \mathscr{G}_*$.

We need some more ideas to develop the lattice structure of $\mathscr{P} = C^{-1}(\mathscr{G})$, where $\mathscr{G}$ is some classic IVFG.

Definition 5.8. *[10] Let* $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ *and* $\mathscr{H} = (V,E,\mathscr{T},\mathscr{U})$ *be two fuzzy graphs. Then* $\mathscr{H}$ *is said to be a* ***fuzzy partial-subgraph*** *of* $\mathscr{G}$ *if* $\mathscr{T}(u) \leq \mathscr{V}(u)$ *for all* $u \in V$ *and* $\mathscr{U}(uv) \leq \mathscr{E}(uv)$ *for all* $uv \in E$.

Definition 5.9. *Let* $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ *and* $\mathscr{H} = (V,E,\mathscr{T},\mathscr{U})$ *be two IVFGs. Then* $\mathscr{H}$ *is said to be a* ***interval valued fuzzy partial-subgraph (IVFPSG)*** *of* $\mathscr{G}$*, denoted as* $\mathscr{H} \underset{P}{\subseteq} \mathscr{G}$*, if it satisfies the following conditions:*

1. $\mathscr{T}_u^- \leq \mathscr{V}_u^-$ *and* $\mathscr{T}_u^+ \leq \mathscr{V}_u^+$*, for all* $u \in V$*; and*
2. $\mathscr{U}_{uv}^- \leq \mathscr{E}_{uv}^-$ *and* $\mathscr{U}_{uv}^+ \leq \mathscr{E}_{uv}^+$*, for all* $uv \in E$.

The following proposition is a direct consequence of definition(5.9)

Proposition 5.10. *For any classic IVFG* $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$

1. $\mathscr{G}_* \underset{P}{\subseteq} \mathscr{G}^*$*; and*
2. $(\mathscr{P}, \underset{P}{\subseteq})$ *is a poset.*

Proof. Straight forward □

Proposition 5.11. *Let* $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ *be any classic IVFG and* $\mathscr{P} = C^{-1}(\mathscr{G})$*. Then* $\mathscr{G}^*$ *is the greatest element in the poset* $(\mathscr{P}, \underset{P}{\subseteq})$.

Proof. Let $\mathscr{G}^* = (V,E,\mathscr{V},\mathscr{E}^*)$ and let $G_1 = (V,E,\mathscr{V},\mathscr{E}_1)$ be any IVFG in $\mathscr{P}$. To prove $G_1 \underset{P}{\subseteq} \mathscr{G}^*$, it is enough to show that ${\mathscr{E}_1}^-_{uv} \le {\mathscr{E}^*}^-_{uv}$ for all $uv \in E$.

By definition(5.1) $c^{\mathscr{G}^*}_{uv} = 0$, for all $uv \in E$
$\Rightarrow {\mathscr{E}^*}^-_{uv} = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\} - \mathscr{E}^-_{uv}$ for all $uv \in E$, by theorem(4.3).
Similarly, $c^{G_1}_{uv} = 0 \Rightarrow {\mathscr{E}_1}^-_{uv} = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\} - \mathscr{E}^-_{uv}$
$\Rightarrow {\mathscr{E}_1}^-_{uv} = {\mathscr{E}^*}^-_{uv}$

Now, let $uv \in E$ such that $c^{G_1}_{uv} \neq 0$.
$\Rightarrow uv$ is an imperfect edge of G_1 and ${\mathscr{E}_1}^-_{uv} = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\} - c^{G_1}_{uv}$, by proposition(4.7) and by theorem(4.3) respectively.
$\Rightarrow \overline{\mathscr{E}_1}^-_{uv} = \overline{\mathscr{E}_1}^+_{uv}$ (from definitions(3.2 and 3.4)) and ${\mathscr{E}_1}^-_{uv} = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\} - c^{G_1}_{uv}$.
$\Rightarrow \overline{\mathscr{E}_1}^-_{uv} < c^{G_1}_{uv}$ (since, by proposition(4.8) $\overline{\mathscr{E}_1}^+_{uv} < c^{G_1}_{uv}$) and ${\mathscr{E}_1}^-_{uv} = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\} - c^{G_1}_{uv}$.
$\Rightarrow \mathscr{E}^-_{uv} < c^{G_1}_{uv}$ (since $G_1 \in \mathscr{P}$, $\overline{G_1} = \mathscr{G}$) and ${\mathscr{E}_1}^-_{uv} = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\} - c^{G_1}_{uv}$.
$\Rightarrow min\{\mathscr{V}^-_u, \mathscr{V}^-_v\} - c^{G_1}_{uv} < min\{\mathscr{V}^-_u, \mathscr{V}^-_v\} - \mathscr{E}^-_{uv}$ and ${\mathscr{E}_1}^-_{uv} = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\} - c^{G_1}_{uv}$.
$\Rightarrow {\mathscr{E}_1}^-_{uv} < {\mathscr{E}^*}^-_{uv}$
Hence ${\mathscr{E}_1}^-_{uv} \le {\mathscr{E}^*}^-_{uv}$ for all $uv \in E$, which completes our proof □

Using similar arguments we can prove the following result. We state it without proof.

Proposition 5.12. *Let $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ be any classic IVFG and $\mathscr{P} = C^{-1}(\mathscr{G})$. Then $\mathscr{G}_*$ is the least element in the poset $(\mathscr{P}, \underset{P}{\subseteq})$.*

Theorem 5.13. *Let $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ be any classic IVFG and $\mathscr{P} = C^{-1}(\mathscr{G})$. Then the poset $(\mathscr{P}, \underset{P}{\subseteq})$ is a lattice.*

Proof. Consider $R = \{uv \in E : \mathscr{E}(uv) = r \in [0,1]\}$. Let $G_1 = (V,E,\mathscr{V},\mathscr{E}_1)$ and $G_2 = (V,E,\mathscr{V},\mathscr{E}_2)$ be any two IVFGs in $\mathscr{P}$. Define two IVFGs, $G_{g.l.b} = (V,E,\mathscr{V},\mathscr{E}_L)$ and $G_{l.u.b} = (V,E,\mathscr{V},\mathscr{E}_U)$ where

$$\mathscr{E}_L(uv) = \begin{cases} \mathscr{E}_1(uv), & \text{if } uv \notin R \\ [min\{{\mathscr{E}_1}^-_{uv}, {\mathscr{E}_2}^-_{uv}\}, {\mathscr{E}_1}^+_{uv}], & \text{if } uv \in R \end{cases}$$

and

$$\mathscr{E}_U(uv) = \begin{cases} \mathscr{E}_1(uv), & \text{if } uv \notin R \\ [max\{{\mathscr{E}_1}^-_{uv}, {\mathscr{E}_2}^-_{uv}\}, {\mathscr{E}_1}^+_{uv}], & \text{if } uv \in R \end{cases}$$

We can easily verify that $G_{g.l.b}, G_{l.u.b} \in \mathscr{P}$; $G_1 \vee G_2 = G_{l.u.b}$ and $G_1 \wedge G_2 = G_{g.l.b}$. This completes the proof □

Theorem 5.14. *Let $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ be any classic IVFG and $\mathscr{P} = C^{-1}(\mathscr{G})$. Then the poset $(\mathscr{P}, \underset{P}{\subseteq})$ is a chain if, and only if, there exist atmost one $uv \in E$ such that $\mathscr{E}(uv) = [\mathscr{E}^-_{uv}, \mathscr{E}^+_{uv}] = r \in [0,1]$ but $r \neq min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$.*

Proof. Let $\mathscr{G}_1, \mathscr{G}_2 \in \mathscr{P}$. Suppose there exist atmost one edge $uv \in E$ such that $\mathscr{E}(uv) = [\mathscr{E}^-_{uv}, \mathscr{E}^+_{uv}] = r \in [0,1]$ but $r \neq min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$ (say) ab. Then for all other uv, either $\mathscr{E}(uv) \neq r$ or $\mathscr{E}(uv) = r$ and $r = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$. We want to show that $\mathscr{G}_1 \underset{P}{\subseteq} \mathscr{G}_2$ or $\mathscr{G}_2 \underset{P}{\subseteq} \mathscr{G}_1$.

If $\mathscr{E}(uv) \neq r$, then $c_{uv} = 0$ (by definitions(3.2, 3.4) and proposition(4.7)).

If $\mathscr{E}(uv) = r$ and $r = min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$, then either $c_{uv} = 0$ or $\mathscr{E}^-_{uv} < c_{uv} \le min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$ (by proposition(4.8)).
$\Rightarrow c_{uv} = 0$.

Now, for edge ab, $\mathscr{E}(ab) = r$ but $r \neq min\{\mathscr{V}^-_a, \mathscr{V}^-_b\}$.
$\Rightarrow c_{ab} = 0$ or $r < c_{ab} \le min\{\mathscr{V}^-_a, \mathscr{V}^-_b\}$ (by proposition(4.8)).

Hence we can conclude that :

1. for all edges $uv \in E - \{ab\}$, $c^{\mathscr{G}_1}_{uv} = 0$ and $c^{\mathscr{G}_2}_{uv} = 0$; and

2. $c^{\mathscr{G}_1}_{ab} \le c^{\mathscr{G}_2}_{ab}$ or $c^{\mathscr{G}_2}_{ab} \le c^{\mathscr{G}_1}_{ab}$

$\Rightarrow \mathscr{G}_2 \underset{P}{\subseteq} \mathscr{G}_1$ or $\mathscr{G}_1 \underset{P}{\subseteq} \mathscr{G}_2$.
$\Rightarrow (\mathscr{P}, \underset{P}{\subseteq})$ is a chain.

Conversely, suppose there exist more than one edge $uv \in E$ such that $\mathscr{E}(uv) = r$ but $r \neq min\{\mathscr{V}^-_u, \mathscr{V}^-_v\}$ (say) ab and ef. Then as in proof of sufficient part we can conclude that :

1. either $c_{ab} = 0$ or $r_1 < c_{ab} \le min\{\mathscr{V}^-_a, \mathscr{V}^-_b\}$, where $r_1 = \mathscr{E}(ab)$; and

2. either $c_{ef} = 0$ or $r_2 < c_{ef} \le min\left\{\mathscr{V}^-_e, \mathscr{V}^-_f\right\}$, where $r_2 = \mathscr{E}(ef)$.

Choose $\mathscr{G}_1, \mathscr{G}_2 \in \mathscr{P}$ such that $c^{\mathscr{G}_1}_{ab} < c^{\mathscr{G}_2}_{ab}$ and $c^{\mathscr{G}_2}_{ef} < c^{\mathscr{G}_1}_{ef}$. Then $\mathscr{G}_2 \underset{P}{\not\subseteq} \mathscr{G}_1$ and $\mathscr{G}_1 \underset{P}{\not\subseteq} \mathscr{G}_2$. Hence $(\mathscr{P}, \underset{P}{\subseteq})$ is not a chain □

Example 5.15. *Consider the classic IVFG $\mathscr{G} = (V,E,\mathscr{V},\mathscr{E})$ in figure(15).*

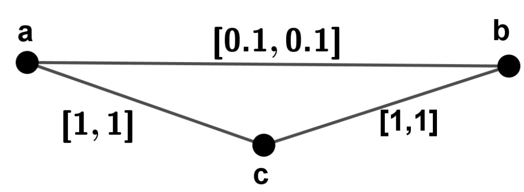

Figure 15. Classic IVFG satisfying the condition in theorem(5.14)

Here ab is the only edge with $\mathscr{E}(ab) = 0.1 = r \in [0,1]$ but $r \neq min\{\mathscr{V}^-_a, \mathscr{V}^-_b\}$ since $\mathscr{V}^-_a = \mathscr{V}^-_b = 1$. We can make pre-complements for $\mathscr{G}$, by assigning infinitely many complement numbers to edge ab. Some of the pre-complements are displayed in figure(16).

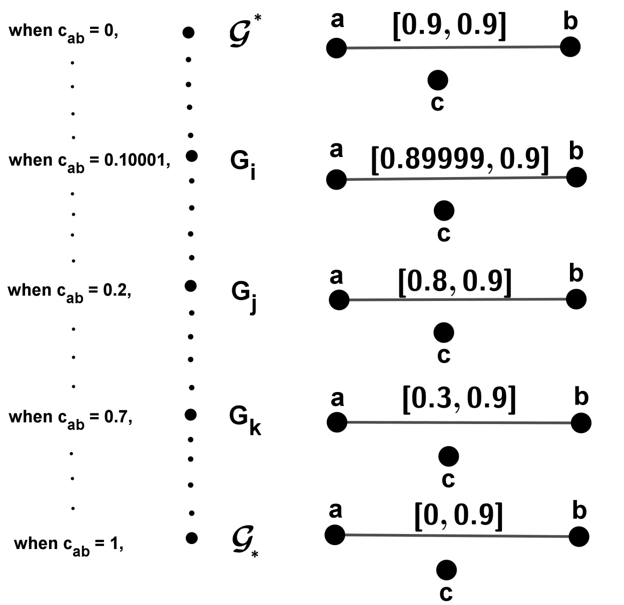

Figure 16. A section of the infinite chain of pre-complements of the IVFG in figure(15) constructed by allotting $c_{bc} = 0, c_{ac} = 0$ and varying values of c_{ab} as : $c_{ab} = 0$ and $0.1 < c_{ab} \leq 1$.

Obviously they form an infinite chain, with uncountably many elements, with $\mathscr{G}^*$ as greatest element and $\mathscr{G}_*$ as least element.

Remark 5.16. *We can conclude, from remark(5.3), that if $\mathscr{G}$ is a classic IVFG whose all edges have non-degenerate intervals as membership intervals, then $\mathscr{P}$ is the singleton lattice $\{\mathscr{G}^*\}$. If $\mathscr{G}$ has exactly one edge uv satisfying the condition $\mathscr{E}(uv) = r \in [0,1]$, where $r \neq min\{\mathscr{V}_u^-, \mathscr{V}_v^-\}$, then $\mathscr{P}$ is an infinite chain, with uncountable many members, and having $\mathscr{G}^*$ as the greatest element and $\mathscr{G}_*$ as the least element. Further, if $\mathscr{G}$ has more than one edge satisfying the above condition, then $\mathscr{P}$ is just an infinite lattice. It does not become a chain.*

6. Conclusion

This paper is a continuation of our work reported in [4], [5] and [6]. We begin with a summary of relevant ideas and results from those papers which are required for a proper understanding of the concepts discussed in this paper. Thus we discuss our new definition of complement of an interval valued fuzzy graph(IVFG), the concepts of classic and non-classic IVFGs and the complement numbers which we have introduced and developed. We observe that, unlike the crisp and fuzzy cases, a *complement IVFG* may have several non-isomorphic pre-complements. But by assigning complement numbers to its edges we can ensure uniqueness of pre-complement. It is proved that an IVFG has a pre-complement if, and only if, it is classic. In theorem(4.9), we give a necessary and sufficient condition for a general complement IVFG, without complement numbers, to have a unique pre-complement; and in corollary(4.11) we extend it to classic IVFGs.

For any given *classic IVFG* $\mathscr{G}$, we describe a method to construct its *superior pre-complement* $\mathscr{G}^*$ and *inferior pre-complement* $\mathscr{G}_*$. In theorem(5.6), we obtain a necessary and sufficient condition for the coincidence of $\mathscr{G}^*$ and $\mathscr{G}_*$. We have defined a partial order $\underset{P}{\subseteq}$ on $\mathscr{P} = C^{-1}(\mathscr{G})$, the collection of all pre-complements of a classic IVFG $\mathscr{G}$, and proved that $(\mathscr{P}, \underset{P}{\subseteq})$ is a lattice with $\mathscr{G}^*$ as the greatest element and $\mathscr{G}_*$ as the least element. Further, we have proved that $(\mathscr{P}, \underset{P}{\subseteq})$ becomes a chain if, and only if, $\mathscr{G}$ has atmost one edge uv such that $\mathscr{E}(uv) = r \in [0,1]$, where $r \neq min\{\mathscr{V}_u^-, \mathscr{V}_v^-\}$. We observe that this is the trivial singleton chain $\{\mathscr{G}^*\}$ when there is no edge in $\mathscr{G}$ satisfying the condition, and an infinite chain, with uncountably many members, if there is exactly one such edge. Moreover, we have included several structure revealing examples and constructions.

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