Malaya Journal of Matematik, Vol. 8, No. 3, 1328-1330, 2020
https://doi.org/10.26637/MJM0803/0104

# On bounds of perfect number

Uma Dixit*

**Abstract**
In this paper we discuss about odd perfect numbers. We first prove an important inequality and use it to discuss about bounds of sum of reciprocal prime divisors of the perfect number. We then derive an important conclusion about improving the upper bound incase when $(15,n)=5$.

**Keywords**
Perfect number, divisors, upper-bounds.

**AMS Subject Classification**
11A41, 11A99.

*Department of Mathematics, University Post Graduate College, Secunderabad -500003, Telangana, India.
***Corresponding author**: umadixit@gmail.com
**Article History**: Received **19** April **2020**; Accepted **29** July **2020**


## Contents



## 1. Introduction

For any positive integer $n$, let $\sigma(n)$ denote the sum of the positive divisors of $n$. It is well known that a positive integer $n$ is said to be perfect if $\sigma(n)=2n$. According to Euler an even number $n$ is perfect if and only if it is of the form $n=2^{k-1}\left(2^k-1\right)$, where $2^k-1$ is a prime number. However the existence or otherwise of an odd perfect number is still an open question. Lot of research has been done on odd perfect numbers problem and various results in various directions are given. Review of these results can be seen in [1]. If we consider all the primes which actually divide n and then consider addition of all the inverse primes that is adding the reciprocals and denote it by R ( $n$ ), then much of the research was done on calculating upper and lower bounds for R ( $n$ ). The first study was done by [4] and later lot of improvement in these was done, which we can refer in [1],[3],[5],[2] and [6] and the best bounds given by researchers is in table below.

A result by Euler also gives that:

If n is product of prime powers where exactly one $p_k$, is such that $p_k \equiv \alpha_k \equiv 1(\bmod 4)$, then whenever $i$ and $k$ are different, then we get $\alpha_i$ is even and $p_i$ is an odd prime. (1.1)

| I | II | III |
|---|---|---|
| | Lower bound | Upper bound |
| $(15,n)=5$ | 0.64738 | 0.677637 |
| $(15,n)=1$ | 0.66745 | 0.69315 |
| $(15,n)=15$ | 0.59606 | 0.673634 |
| $(15,n)=3$ | 0.60383 | 0.65731 |

**Table 1.** Upper and Lower bound

## 2. Preliminaries

We require the following result already given by John A. Ewell [2].

**Theorem 2.1.** *([2], Theorem 3) let us consider n which is of the form* (1.1), *where* $p_i \neq 3$ *for* $i=1,2,\ldots,t; p_i=2\pi_i-1, \alpha_i=2e_i$ *then for distinct i&k and* $\alpha_k=4\varepsilon+1$, *the following congruences hold.*

*(i)* $\varepsilon=0$ *or* $-1(\bmod 3)$.

*(ii)* $\pi_i e_i \equiv$ *o or -1 (mod 3 ) and finally*

*(iii)* $p_k \equiv 1(\bmod 12)$.

We state the following lemma.

**Lemma 2.2.** *Whenever* $0<x<\frac{1}{7}$ *we get* $1+\mathrm{x}+x^2>e^{\left(x+c_1 x^2\right)}$ *where* $c_1=0.406$.

**Lemma 2.3.** *Whenever n is quasi-perfect and* $\alpha$ *is of the form* $2+\frac{1}{n}, m<n$ *m is divisor of n and where* $R_1(n)=R(n)$ *and* $R_2(n)=\sum_{i=1}^t \frac{1}{p_i^2}$, *then we get* $R_1(n)<R_1(m)+cR_2(m)+$

$\log\left(\frac{\alpha m}{\beta\sigma(m)}\right)-cR_2(n) given$

$$\beta=\begin{cases}1 & \text{if } (n,15)=1 \text{ or } (n,15)=(m,15)=3 \\ & \text{or } (n,15)=(m,15)=5 \\ & \text{or } (n,15)=(m,15)=15 \\ 1+\frac{1}{3}+\frac{1}{3^2} & \text{if } (n,15)=3 \text{ and } (m,15)=1 \\ & \text{or} (n,15)=15 \text{ and } (m,15)=5 \\ 1+\frac{1}{5}+\frac{1}{5^2} & \text{if } (n,15)=5 \text{ and } (m,15)=1 \\ & \text{or } (n,15)=15 \text{ and } (m,15)=3 \\ \left(1+\frac{1}{3}+\frac{1}{3^2}\right)\left(1+\frac{1}{5}+\frac{1}{5^2}\right) & \text{if } (n,15)=15 \text{ and } (m,15)=1.\end{cases}$$

**Lemma 2.4.** *If $m$ is divisor of $n'$ with $m<n'$ and $n$ is odd perfect number of the form (1.1) and $n'=\frac{n}{p_k\alpha_k}\cdot$ we get $R_1(n)<R_1(m)+cR_2(m)+\log\left(\frac{\alpha m}{\beta\sigma(m)}\right)-cR_2(n)$ where $c$ and $\beta$ have the same value as Lemma 2.2 and Lemma 2.3 respectively.*

*Proof.* Suppose $n=\prod_{i=1}^{t}p_i^{\alpha_i}$, where $p_i$ 's are odd primes, $p_k\equiv\alpha_k\equiv 1(\bmod 4)$ for exactly one $k$ and $\alpha_i\equiv 0 \pmod 2$ for $i\neq k$. Then

$$2=\frac{\sigma(n)}{n}=\prod_{i=1}^{t}\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}+\cdots+\frac{1}{p_i^{\alpha_i}}\right). \tag{1.2}$$

Suppose $m$ is a divisor of $n'$ with $m<n'$, then $m=\prod_{\substack{i=1\\ i\neq k}}^{t}p_i^{b_i}$ where $0\leq b_i\leq\alpha_i$ for each $i$ and $b_i<\alpha_i$ for at least one $i$. Therefore

$$\frac{\sigma(m)}{m}=\prod_{\substack{i=1\\ i\neq k}}^{t}\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}+\cdots+\frac{1}{p_i^{b_i}}\right). \tag{1.3}$$

Now from (1.2), (1.3) and Lemma 2.2 we get

$$\begin{aligned}2&=\prod_{\substack{i=1\\ p_i|m}}^{t}\left(1+\frac{1}{p_i}+\cdots+\frac{1}{p_i^{\alpha_i}}\right)\prod_{\substack{i=1\\ p_i\nmid m}}^{t}\left(1+\frac{1}{p_i}+\cdots+\frac{1}{p_i^{\alpha_i}}\right)\\ &>\prod_{\substack{i=1\\ p_i|m}}^{t}\left(1+\frac{1}{p_i}+\cdots+\frac{1}{p_i^{b_i}}\right)\prod_{\substack{i=1\\ p_i\nmid m}}^{t}\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}\right)\\ &=\frac{\sigma(m)}{m}\prod_{\substack{p_i|n\\ p_i\nmid m}}\left(1+\frac{1}{p_i}+\frac{1}{p_i^2}\right)\\ &=\frac{\sigma(m)}{m}\prod_{\substack{p_i|n\\ p_i\nmid m, p_i\geq 7}}\left(\frac{1}{p_i}+c\cdot\frac{1}{p_i^2}\right)\cdot\beta\\ &>\frac{\sigma(m)}{m}\prod_{\substack{p_i|n\\ p_i\nmid m, p_i\geq 7}}\left(\frac{1}{p_i}+c\cdot\frac{1}{p_i^2}\right)\cdot\beta\\ &=\beta\cdot\frac{\sigma(m)}{m}\cdot\exp\left\{\sum_{\substack{p|n\\ p\nmid m}}\left(\frac{1}{p}+\frac{c}{p^2}\right)\right\},\end{aligned}$$

which on taking logarithm gives

$$\begin{aligned}\log 2&>\log\left(\frac{\beta\sigma(m)}{m}\right)+\sum_{\substack{p|n\\ p\nmid m}}\left(\frac{1}{p}+\frac{c}{p^2}\right)\\ &=\log\left(\frac{\beta\sigma(m)}{m}\right)+\sum_{\substack{p|n\\ p\nmid m}}\frac{1}{p}+c\cdot\sum_{\substack{p|n\\ p\nmid m}}\frac{1}{p^2}\\ &=\log\left(\frac{\beta\sigma(m)}{m}\right)+R_1(n)-R_1(m)+c\left[R_2(n)-R_2(m)\right]\end{aligned}$$

Therefore,

$$R_1(n)<R_1(m)+cR_2(m)+\log\left(\frac{2m}{\beta\sigma(m)}\right)-cR_2(n).$$

□

## 3. Main Results

In this section we improve the upper-bound in the first case i.e. $(15,n)=5$ which means 5 is a divisor and 3 is not a divisor and also $5^2$ does not divide $n$ unitarily.

**Theorem 3.1.** *If $(15,n)=5$ then $p_1=5$ and $p_i\neq 3$ for $i=2,3,\ldots,t$. It follows from (i) and (ii) of Theorem 2.1, that $k\neq 1, p_k\geq 13$ and $\alpha_k=4\varepsilon+1$ where $\varepsilon\equiv 0$ or $-1 \pmod 3$. Therefore, $p_k\geq 13, k>1$ and $\alpha_k\equiv 1$ or $-3 (\bmod 12)$. Also since $\pi_1=3$, we get from Theorem 2.1 (iii), that $\pi_1 e_1\equiv 0(\bmod 3)$ so that $e_1\geq 1$ and hence $\alpha_1\geq 2$. Thus $5^2$ divides $n$ whenever $(15,n)=5$. By our assumption, $5^2$ does not unitarily divide $n$ and hence $5^4$ divides $n$.*

*Proof.* Case (i) I f $p_k=13$ and $\alpha_k=1$. Then since $\sigma(13)=14$ divides $2n$, we get 7 divides $n$. Therefore $p_2=7$, so that $\pi_2=4$. Again by Theorem ( 2.1 ) (iii), we have $e_2\equiv 0$ or -1 (mod 3 ) so that $e_2\geq 2$ and hence $\alpha_2\geq 4$. That is, $7^4$ divides $n$ in this case. Also since $5^4$ divides $n, \sigma\left(5^4\right)=781=11\times 71$ divides $n$. Taking $m=5^4\cdot 11^2\cdot 71^2$, where $m$ divides $n'$ and $m<n'$, we get by Lemma 2.4 that

$$\begin{aligned}R_1(n)<{}&\tfrac{1}{5}+\tfrac{1}{11}+\tfrac{1}{71}+(0.406)\left(\tfrac{1}{5^2}+\tfrac{1}{11^2}+\tfrac{1}{71^2}\right)\\ &+\log\left[\frac{2\cdot 5^4\cdot 11^2\cdot 71^2\cdot 4\cdot 10\cdot 70}{(5^5-1)(11^3-1)(71^3-1)}\right]\\ &-(0.406)\left[\tfrac{1}{5^2}+\tfrac{1}{7^2}+\tfrac{1}{11^2}+\tfrac{1}{13^2}+\tfrac{1}{71^2}\right]\end{aligned}$$

$$\begin{aligned}&=\frac{1}{5}+\frac{1}{11}+\frac{1}{71}+(0.406)\left(\frac{1}{5^2}+\frac{1}{11^2}+\frac{1}{71^2}\right)\\ &\quad+\log\left(5^4\cdot 11^2\cdot 71^2\cdot 80\cdot 70\right)\\ &\quad-\log\left[\left(5^5-1\right)\left(11^3-1\right)\left(71^3-1\right)\right]\\ &\quad-(0.406)\left(\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{71^2}\right)\\ &=0.2+0.09090909+0.014084507+0.019675911\\ &\quad+28.38942383-28.02784057-0.030363992\\ &=0.655888773.\end{aligned}\tag{1.4}$$

Subcase(i): If $p_k=13$ and $\alpha_k>1$.

Then again taking $m = 5^4 \cdot 11^2 \cdot 71^2$, where $m$ divides $n'$ and $m < n'$, by Lemma 2.4, we get

$$\begin{aligned}
&R_1(n) < \tfrac{1}{5} + \tfrac{1}{11} + \tfrac{1}{71} + (0.406)\left(\tfrac{1}{5^2} + \tfrac{1}{11^2} + \tfrac{1}{71^2}\right) \\
&+\log\left[\tfrac{2\cdot 5^4\cdot 11^2\cdot 71^2\cdot 4\cdot 10\cdot 70}{(5^5-1)\cdot(11^3-1)\cdot(71^3-1)}\right] - (0.406)\left(\tfrac{1}{5^2} + \tfrac{1}{11^2} + \tfrac{1}{71^2}\right) \\
&= \tfrac{1}{5} + \tfrac{1}{11} + \tfrac{1}{71} + \log\left(5^4 \cdot 11^2 \cdot 71^2 \cdot 80 \cdot 70\right) \\
&\quad -\log\left[\left(5^5-1\right)\left(11^3-1\right)\left(71^3-1\right)\right] \\
&= \quad 0.2 + 0.09090909 + 0.014084507 - 28.38942383 \\
&\quad -28.02784057 \\
&= 0.666576854. \qquad (1.5)
\end{aligned}$$

Case(ii): If $p_k \neq 13$, then $p_k > 13$.

Taking $m = 5^4 \cdot 11^2 \cdot 71^2$, where $m$ divides $n'$ and $m < n'$, we get by Lemma 2.4 that

$$\begin{aligned}
&R_1(n) < \tfrac{1}{5} + \tfrac{1}{11} + \tfrac{1}{71} + (0.406)\left(\tfrac{1}{5^2} + \tfrac{1}{11^2} + \tfrac{1}{71^2}\right) \\
&+\log\left[\left(\tfrac{2\cdot 5^4\cdot 11^2\cdot 71^2\cdot 4\cdot 10\cdot 70}{(5^5-1)\cdot(11^3-1)\cdot(71^3-1)}\right]\right. - (0.406)\left(\tfrac{1}{5^2} + \tfrac{1}{11^2} + \tfrac{1}{71^2}\right) \\
&= 0.2 + 0.09090909 + 0.014084507 - 28.38942383 \\
&\quad -28.02784057 \\
&= 0.666576854. \qquad (1.6)
\end{aligned}$$

By the results (1.4), (1.5) and (1.6), we get

$$R_1(n) < 0.666576854.$$

Hence the theorem. □

## References


[1] G.L. Cohen, On odd perfect numbers, *Fibonacci Quarterly*, 16(1978), 523–527.

[2] John A. Ewell, On the multiplicative structure of odd perfect numbers, *J. Number Theory*, 12(1980), 339–342.

[3] Kishore Masao, Odd integers $N$ with five distinct prime factors for which 2–$10^{-12} < \sigma(N)/N < 2 + 10^{-12}$, *Math. Comp.,* 32(1978), 303–309.

[4] M. Perisastry, A note on odd perfect numbers, *Math. Student*, 26(1958), 179–181.

[5] W. Sierpinski, Theoria liczb, czesc II, Warszawa, 1959.

[6] W. Sierpinski, *A Selection of Problems in the Theory of Numbers*, New York; 1964.



⋆⋆⋆⋆⋆⋆⋆⋆⋆

ISSN(P):2319 − 3786

Malaya Journal of Matematik

ISSN(O):2321 − 5666

⋆⋆⋆⋆⋆⋆⋆⋆⋆