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Distributions in CR-lightlike submanifolds of an indefinite Kaehler statistical manifold

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Abstract
In this paper, the distributions in CR-lightlike submanifolds of an indefinite Kaehler Statistical manifold have been characterized using second fundamental form and the necessary and sufficient conditions for integrability of the same have been obtained. Also, the conditions for the distributions to be totally geodesic with respect to the dual connections in the statistical manifold have been developed.

Keywords
CR-lightlike submanifolds, distribution, indefinite Kaehler Statistical manifold, totally geodesic submanifold, integrability.

AMS Subject Classification
53B05, 53B15, 53B30.

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Contents

1. Introduction

A statistical manifold is a contemporary and an interesting branch of manifolds which has developed from the investigation of geometric structures on sets of certain probability distributions. It is a differentiable manifold where each point represents a probability distribution.The set of all probability measures consists of an infinite dimensional statistical manifold which associates each location in parameter space to a probability density function. A parameter space and a manifold together enables us to generalize many concepts from the Euclidean space to the statistical manifold.These manifolds are also geometrically formulated as Riemannian manifolds with a certain affine connection.

Rao [14] was the one to relate geometry with statistics resulting in the formation of the statistical manifold. He used Fisher information matrix to introduce the concept of Riemannian metric. Although various researchers worked in this direction in the subsequent years, yet an appreciable amount of work was done by Amari [20] [1] and Simon [17] when they introduced statistical manifold on the basis of information geometry which is the study of probability and information from the view point of differential geometry having applications in the fields of statistics and applied mathematics. Then Vos [18] developed certain fundamental equations and structural formulae for the statistical manifold. Thereafter, Kurose [11] developed the concept of holomorphic statistical manifold which was further elaborated by Furuhata et.al [7][8][9][16]. Milijevic [22] [12] studied CR-submanifolds of holomorphic statistical manifold which was later on studied extensively by Boyom et.al [3].

The study of the concept of CR- submanifolds being earlier confined to manifolds with positive definite metric and its non applicability to other branches of mathematics motivated Duggal et.al [4][5][21][6][2] to introduce the theory of CR-lightlike submanifolds of Kaehler manifold where the underlying metric is indefinite. This study created widespread interest in the lightlike geometry among the researchers due to its applications in the theory of geneal relativity. In the recent years, a considerable amount of work has been done in almost Hermitian statistical manifolds.

Keeping in focus the above facts, in this paper, after giving certain important preliminary concepts, we have established same conditions on the integrability of distributions in CR-lightlike submanifolds of an indefinite Kaehler statistical manifold from their characterisations with the properties on the subbundles of the manifold. Also by using the projections on the distributions, we have developed certain results for the Kaehler statistical manifold with respect to the connection and dual connection.

2. CR-lightlike Submanifolds

Consider $(\bar{M},\bar{g})$ as an $(m+n)$-dimensional semi-Riemannian manifold with semi-Riemannian metric $\bar{g}$ and of constant index q such that $m,n \geq 1$, $1 \leq q \leq m+n-1$.
Let (M,g) be a m-dimensional lightlike submanifold of $\bar{M}$. In this case, there exists a smooth distribution $RadTM$ on M of rank $r>0$, known as Radical distribution on M such that $RadTM_p = TM_p \cap TM_p^{\perp}, \forall\, p \in M$ where TM_p and $TM_p^{\perp}$ are degenerate orthogonal spaces but not complementary. Then M is called an r-lightlike submanifold of $\bar{M}$.

Now, consider $S(TM)$, known as Screen distribution, as a complementary distribution of radical distribution in TM i.e.,

$$TM = RadTM \perp S(TM)$$

and $S(TM^{\perp})$, called screen transversal vector bundle, as a complementary vector subbundle to $Rad(TM)$ in $TM^{\perp}$ i.e.,

$$TM^{\perp} = RadTM \perp S(TM^{\perp})$$

As $S(TM)$ is non degenerate vector subbundle of $T\bar{M}|_M$, we have

$$T\bar{M}|_M = S(TM) \perp S(TM)^{\perp}$$

where $S(TM)^{\perp}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\bar{M}|_M$.

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}|_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\bar{M}|_M = TM \oplus tr(TM)$$

$$= (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}).$$

Theorem 2.1. *[21] Let $(M,g,S(TM),S(TM^{\perp}))$ be an r- lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ called a lightlike transversal bundle of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and basis of $\Gamma(ltr(TM)|_U)$ consisting of smooth sections $\{N_1,\cdots,N_r\}$ $S(TM^{\perp})^{\perp}|_U$ such that*

$$\bar{g}(N_i,\xi_j) = \delta_{ij}, \quad \bar{g}(N_i,N_j) = 0, \quad i,j = 0,1,\cdots,r$$

where $\{\xi_1,\cdots,\xi_r\}$ is a lightlike basis of $\Gamma(RadTM)|_U$.

Let $\hat{\nabla}$ be the Levi-Civitia connection on $\bar{M}$. We have, from the above mentioned theory, the Guass and Weingarten formulae as:

$$\hat{\nabla}_X Y = \nabla_X Y + h(X,Y), \quad \forall\, X,Y \in \Gamma(TM) \tag{2.1}$$

and

$$\hat{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \quad \forall\, X \in \Gamma(TM), V \in \Gamma(tr(TM))$$

Using the projections $L: tr(TM) \to ltr(TM)$ and $S: tr(TM) \to S(TM^{\perp})$, from [21], we have the following equations from the above formulae:

$$\hat{\nabla}_X Y = \nabla_X Y + h^l(X,Y) + h^s(X,Y)$$

$$\hat{\nabla}_X V = -A_V X + D^l_X V + D^s_X V$$

In particular,

$$\hat{\nabla}_X N = -A_N X + \nabla^l_X N + D^s(X,N)$$

$$\hat{\nabla}_X W = -A_W X + \nabla^s_X W + D^l(X,W)$$

for any $X,Y \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$ here $h^l(X,Y) = Lh(X,Y)$, $h^s(X,Y) = Sh(X,Y)$, $D^l_X V = L(\nabla^{\perp}_X V)$, $D^s_X V = S(\nabla^{\perp}_X V)$, $\nabla^l_X N$, $D^l(X,W) \in \Gamma(ltr(TM))$, $\nabla^s_X W$, $D^s(X,N) \in \Gamma(S(TM^{\perp}))$ and $\nabla_X Y, A_N X, A_W X \in \Gamma(TM)$.

Denoting by P, the projection morphism of tangent bundle TM to the screen distribution, we consider the following decomposition:

$$\nabla_X PY = \nabla'_X PY + h'(X,PY)$$

$$\nabla_X \xi = -A'_{\xi} X + \nabla''_X \xi$$

for any $X,Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$, where $\{\nabla'_X PY, A'_{\xi} X\}$ and $\{h'(X,PY), \nabla''_X \xi\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively; ∇' and ∇'' are linear connections on complementary distributions $S(TM)$ and $Rad(TM)$ respectively. Then we have the following equations:

$$\bar{g}(h^l(X,PY),\xi) = g(A'_{\xi} X, PY), \quad \bar{g}(h'(X,PY),N) = g(A_N X, PY)$$

$$g(A'_{\xi} PX, PY) = g(PX, A'_{\xi} PY), \quad A'_{\xi}\xi = 0$$

for any $X,Y \in \Gamma((TM))$, $\xi \in \Gamma(Rad(TM))$ and $N \in \Gamma(ltr(TM))$

Considering the above theory, the CR-lightlike submanifold is defined as:

Definition 2.2. *[21] A submanifold M of an indefinite almost Hermitian manifold $\bar{M}$ is said to be a CR-lightlike submanifold if the following two conditions are fullfilled:*
(i)$\bar{J}(Rad(TM))$ is a distribution on M such that

$$Rad(TM) \cap \bar{J}Rad(TM) = \{0\}$$

(ii) There exist vector bundles $S(TM)$, $S(TM^{\perp})$, $ltr(TM)$, $D_{\circ}$ and D' over M such that

$$S(TM)=\{\bar{J}(RadTM)\oplus D'\}\perp D_{\circ},\quad \bar{J}D_{\circ}=D_{\circ},\quad \bar{J}D'=L_1\perp L_2$$

where $D_{\circ}$ is a nondegenerate distribution on M and L_1 , L_2 are vector bundles of $ltr(TM)$ and $S(TM^{\perp})$, respectively.

Using the above definition , the tangent bundle TM of M is decomposed as:

$$TM=D\oplus D'$$

where

$$D=RadTM\perp\bar{J}RadTM\perp D_{\circ}$$

We denote by S and Q, the projections on D and D' respectively. Then we have

$$\bar{J}X=fX+wX \tag{2.2}$$

for any $X,Y\in\Gamma(TM)$. where $fX=\bar{J}SX$ and $wX=\bar{J}QX$. Also we set

$$\bar{J}V=BV+CV \tag{2.3}$$

for any $V\in\Gamma(tr(TM))$. where $BV\in\Gamma(TM)$ and $CV\in\Gamma(tr(TM))$.

Unless otherwise stated, M_1 and M_2 are supposed to be as $\bar{J}L_1$ and $\bar{J}L_2$ where $\bar{J}(L_1)=M_1\subset D'$ and $\bar{J}(L_2)=M_2\subset D'$ respectively.

3. The Lightlike approach to an indefinite Statistical Manifold

Following are certain basic known definitions related to the theory of lightlike submanifolds of an indefinite statistical manifold.

Let $\bar{M}$ be a C^{∞} manifold of dimension $\bar{m}\geq 2$, $\bar{\nabla}$ be an affine connection on $\bar{M}$ and $\bar{g}$ be a semi- Riemannian metric of constant index $q\geq 1$ on $\bar{M}$. Then
1. $(\bar{M},\bar{\nabla},\bar{g})$ is called an indefinite statistical manifold if
(i) $\bar{\nabla}$ is of torsion free and
(ii) $(\bar{\nabla}_X\bar{g})(Y,Z)=(\bar{\nabla}_Y\bar{g})(X,Z)$ for $X,Y,Z\in\Gamma(T\bar{M})$.
2. If $X\bar{g}(Y,Z)=\bar{g}(\bar{\nabla}_XY,Z)+\bar{g}(Y,\bar{\nabla}^*_XZ)$; for $X,Y,Z\in\Gamma(T\bar{M})$, then $\bar{\nabla}^*$ is called the dual connection of $\bar{\nabla}$ with respect to $\bar{g}$.

If $(\bar{M},\bar{\nabla},\bar{g})$ is an indefinite statistical manifold, then $(\bar{M},\bar{\nabla}^*,\bar{g})$ is also an indefinite statistical manifold. We therefore denote the indefinite statistical manifold by $(\bar{M},\bar{g},\bar{\nabla},\bar{\nabla}^*)$.

Let M be a submanifold of a statistical manifold $(\bar{M},\bar{\nabla},\bar{g})$ and g be the induced metric on M. An affine connection ∇ on M is defined by ([12], [9]) as:

$$\nabla_XY=(\bar{\nabla}_XY)^T$$

where $(\bar{\nabla}_XY)^T$ denotes the orthogonal projection of ∇_XY on the tangent space with respect to $\bar{g}$, that is $\langle\nabla_XY,Z\rangle=\langle\bar{\nabla}_XY,Z\rangle$ for $X,Y,Z\in\Gamma(TM)$.Then (M,∇,g) becomes a statistical manifold and (∇,g) is called the induced statistical structure on M.
(M,∇,g) is said to be a statistical submanifold in $(\bar{M},\bar{\nabla},\bar{g})$ if (∇,g) is induced statistical structure on M.
Now $T_x^{\perp}M$ denote the normal space of M i.e.
$T_x^{\perp}M:=\{v\in T_x\bar{M}\mid\bar{g}(v,w)=0,w\in T_xM\}$ and g, the induced metric on M. It follows that

$$\nabla,\nabla^*:\Gamma(TM)\times\Gamma(TM)\longrightarrow\Gamma(TM)$$
$$h,h^*:\Gamma(TM)\times\Gamma(TM)\longrightarrow\Gamma(T^{\perp}M)$$
$$A,A^*:\Gamma(T^{\perp}M)\times\Gamma(TM)\longrightarrow\Gamma(TM)$$
$$\nabla^{\perp},\nabla^{\perp*}:\Gamma(TM)\times\Gamma(T^{\perp}M)\longrightarrow\Gamma(T^{\perp}M)$$
$$\bar{\nabla}_XY=\nabla_XY+h(X,Y),\quad\bar{\nabla}_XV=-A_VX+\nabla^{\perp}_XV,$$

$$\bar{\nabla}^*_XY=\nabla^*_XY+h^*(X,Y),\quad\bar{\nabla}^*_XV=-A^*_VX+\nabla^{\perp*}_XV, \tag{3.1}$$

for X, $Y\in\Gamma(TM)$, $V\in\Gamma(T^{\perp}M)$.

Then the following hold for X, $Y\in\Gamma(TM)$, $V\in\Gamma(T^{\perp}M)$:

$$\bar{g}(h(X,Y),V)=g(A^*_VX,Y),\quad\bar{g}(h^*(X,Y),V)=g(A_VX,Y) \tag{3.2}$$

From the concept of structural equations in the lightlike theory available so far, the Gauss and Weingarten formulae for a lightlike submanifold (M,g) of an indefinite statistical manifold $(\bar{M},\bar{g},\bar{\nabla},\bar{\nabla}^*)$ are as follows:

$$\begin{aligned}\bar{\nabla}_XY&=\nabla_XY+h^l(X,Y)+h^s(X,Y),\\ \bar{\nabla}^*_XY&=\nabla^*_XY+h^{*l}(X,Y)+h^{*s}(X,Y)\end{aligned} \tag{3.3}$$

$$\begin{aligned}\bar{\nabla}_XV&=-A_VX+D^l_XV+D^s_XV,\\ \bar{\nabla}^*_XV&=-A^*_VX+D^{*l}_XV+D^{*s}_XV,\end{aligned}$$

$$\begin{aligned}\bar{\nabla}_XN&=-A_NX+\nabla^l_XN+D^s(X,N),\\ \bar{\nabla}^*_XN&=-A^*_NX+\nabla^{*l}_XN+D^{*s}(X,N)\end{aligned} \tag{3.4}$$

$$\begin{aligned}\bar{\nabla}_XW&=-A_WX+\nabla^s_XW+D^l(X,W),\\ \bar{\nabla}^*_XW&=-A^*_WX+\nabla^{*s}_XW+D^{*l}(X,W)\end{aligned} \tag{3.5}$$

for any $X,Y\in\Gamma(TM)$, $V\in\Gamma(tr(TM))$, $N\in\Gamma(ltr(TM))$ and $W\in\Gamma(STM^{\perp})$.

Considering the corresponding projection morphism P of tangent bundle TM to the screen distribution, we have the following decomposition w.r.t ∇ and ∇^*:

$$\nabla_X PY = \nabla'_X PY + h'(X,PY), \quad \nabla^*_X PY = \nabla^{*\prime}_X PY + h^{*\prime}(X,PY) \tag{3.6}$$

$$\nabla_X \xi = -A'_\xi X + \nabla'^t_X \xi, \quad \nabla^*_X \xi = -A^{*\prime}_\xi X + \nabla^{*\prime t}_X \xi \tag{3.7}$$

for any $X,Y \in \Gamma(TM)$, $\xi \in \Gamma(Rad(TM))$.
Then the following holds:

$$\bar{g}(h^l(X,PY),\xi) = g(A^{*\prime}_\xi X, PY), \quad \bar{g}(h^{*l}(X,PY),\xi) = g(A'_\xi X, PY) \tag{3.8}$$

$$\bar{g}(h'(X,PY),N) = g(A^*_N X, PY), \quad \bar{g}(h^{*\prime}(X,PY),N) = g(A_N X, PY) \tag{3.9}$$

3.1 Indefinite Kaehler Statistical Manifold

Let $\bar{\nabla}^\circ$ be the Levi-Civita connection w.r.t $\bar{g}$. Then, we have $\bar{\nabla}^\circ = \frac{1}{2}(\bar{\nabla} + \bar{\nabla}^*)$.
For a statistical manifold $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$, the difference $(1,2)$ tensor K of a torsion free affine connection $\bar{\nabla}$ and Levi-civita connection $\bar{\nabla}^\circ$ is defined as

$$K(X,Y) = K_X Y = \bar{\nabla}_X Y - \bar{\nabla}^\circ_X Y \tag{3.10}$$

Since $\bar{\nabla}$ and $\bar{\nabla}^\circ$ are torsion free, we have

$$K(X,Y) = K(Y,X) \quad , \quad \bar{g}(K_X Y, Z) = \bar{g}(Y, K_X Z)$$

for any $X,Y,Z \in \Gamma(TM)$.
Also we have

$$K(X,Y) = \bar{\nabla}^\circ_X Y - \bar{\nabla}^*_X Y.$$

From the above equations, we have

$$K(X,Y) = \frac{1}{2}(\bar{\nabla}_X Y - \bar{\nabla}^*_X Y).$$

Also, from (3.10), we have

$$\bar{g}(\bar{\nabla}_X Y, Z) = \bar{g}(K(X,Y),Z) + \bar{g}(\bar{\nabla}^\circ_X Y, Z)$$

We have the following result from [13]:

$$\bar{g}((\bar{\nabla}_X \bar{J})Y, Z) = -\bar{g}(Y, (\bar{\nabla}^*_X \bar{J})Z) \tag{3.11}$$

holds for any $X,Y,Z \in \Gamma(TM)$ for an almost Hermitian manifold $(\bar{M}, \bar{g}, \bar{J}, \bar{\nabla}, \bar{\nabla}^*)$. Now , from [19] , we have the following equations for the almost Hermitian manifold:

$$(\bar{\nabla}_X \bar{J})Y = (\bar{\nabla}^\circ_X \bar{J})Y + (K_X \bar{J})Y$$

$$(\bar{\nabla}^*_X \bar{J})Y = (\bar{\nabla}^\circ_X \bar{J})Y - (K_X \bar{J})Y$$

for any $X,Y,Z \in \Gamma(TM)$.
This implies

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}^*_X \bar{J})Y = 2(\bar{\nabla}^\circ_X \bar{J})Y$$

Let $(\bar{M}, \bar{J}, \bar{g})$ be an indefinite almost Hermitian manifold with an almost complex structure $\bar{J}$ and Hermitian metric $\bar{g}$ such that for all $X,Y \in \Gamma(T\bar{M})$,

$$\bar{J}^2 = -I, \quad \bar{g}(\bar{J}X, \bar{J}Y) = \bar{g}(X,Y). \tag{3.12}$$

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ with respect to metric $\bar{g}$, then the covariant derivative of $\bar{J}$ is defined by

$$(\bar{\nabla}_X \bar{J})Y = \bar{\nabla}_X \bar{J}Y - \bar{J}\bar{\nabla}_X Y$$

An indefinite almost Hermitian manifold $\bar{M}$ is called an indefinite Kaehler manifold if $\bar{J}$ is parallel with respect to $\bar{\nabla}$, i.e.,

$$(\bar{\nabla}_X \bar{J})Y = 0$$

Definition 3.1. *Let $(\bar{g}, \bar{J})$ be an indefinite Hermitian structure on $\bar{M}$. A triplet $(\bar{\nabla} = \bar{\nabla}^\circ + K, \bar{g}, \bar{J})$ is called an indefinite Hermitian Statistical structure on $\bar{M}$ if $(\bar{\nabla}, \bar{g})$ is a statistical structure on $\bar{M}$.*

Then $(\bar{M}, \bar{\nabla}, \bar{\nabla}^*, \bar{g}, \bar{J})$ is called an indefinite Hermitian Statistical manifold.

An indefinite Hermitian Statistical manifold is called indefinite Kaehler Statistical manifold if its almost complex structure is parallel with respect to Levi-Civita connection i.e. if,

$$(\bar{\nabla}^\circ_X \bar{J})Y = 0$$

Equivalently

$$(\bar{\nabla}_X \bar{J})Y + (\bar{\nabla}^*_X \bar{J})Y = 0$$

for all $X,Y \in \Gamma(T\bar{M})$.

The CR-lightlike submanifolds of the indefinite Kaehler Statistical manifolds are studied in the following section.

4. Chracterizations of Distributions

Definition 4.1. *A CR-lightlike submanifold of an indefinite Kaehler Statistical manifold is called D-totally geodesic with respect to $\bar{\nabla}$ (respectively $\bar{\nabla}^*$) if $h(X,Y) = 0$ (respectively $h^*(X,Y) = 0$) for all $X,Y \in D$.*

Definition 4.2. *A CR-lightlike submanifold of an indefinite Kaehler statistical manifold is called mixed totally geodesic with respect to $\bar{\nabla}$ (resp.$\bar{\nabla}^*$) if $h(X,Y) = 0$ (resp. $h^*(X,Y) = 0$) for $X \in D$ and $Y \in D'$.*

Theorem 4.3. *Let $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$ be an indefinite Kaehler statistical manifold and M be CR-lightlike submanifold of $\bar{M}$. If M is D-totally geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$, then $w\nabla_X \bar{J}Y + w\nabla^*_X \bar{J}Y = 0$.*

Proof: For the connections $\bar{\nabla}$ and $\bar{\nabla}^*$ in $\bar{M}$, for any $X,Y \in \Gamma(D)$

$$h(X,Y)+h^*(X,Y) = \bar{\nabla}_X Y - \nabla_X Y + \bar{\nabla}^*_X Y - \nabla^*_X Y$$

Using the relation of connections $\bar{\nabla}$ and $\bar{\nabla}^*$ with Levi Civita connection $\bar{\nabla}^\circ$ in statistical manifold $\bar{M}$, we get

$$h(X,Y)+h^*(X,Y) = 2(\bar{\nabla}^\circ_X Y - \nabla^\circ_X Y)$$
$$= 2(-\bar{\nabla}^\circ_X \bar{J}^2 Y - \nabla^\circ_X Y)$$
$$= -2((\bar{\nabla}^\circ_X \bar{J})\bar{J}Y + \overline{J\nabla}^\circ_X \bar{J}Y + \nabla^\circ_X Y)$$

From the fact that $\bar{M}$ is a Kaehler statistical manifold and from the equations (2.1), (2.2) (2.3), we have

$$h(X,Y)+h^*(X,Y) = -2\overline{J\nabla}^\circ_X \bar{J}Y - 2\nabla^\circ_X Y$$
$$= -2[f\nabla^\circ_X \bar{J}Y + w\nabla^\circ_X \bar{J}Y + \frac{1}{2}(Bh(X,\bar{J}Y)+Bh^*(X,\bar{J}Y)) + \frac{1}{2}(Ch(X,\bar{J}Y)+Ch^*(X,\bar{J}Y)) - \nabla^\circ_X Y]$$

On equating normal parts, we obtain

$$h(X,Y)+h^*(X,Y) = -w\nabla_X \bar{J}Y - w\nabla^*_X \bar{J}Y - Ch(X,\bar{J}Y) - Ch^*(X,\bar{J}Y)$$

Using the given hypothesis that M is D-totally geodesic with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$, we get the desired result.

Lemma 4.4. *Let $(\bar{M},\bar{g},\bar{\nabla},\bar{\nabla}^*)$ be an indefinite Kaehler statistical manifold and M be a CR-lightlike submanifold of $\bar{M}$. Then we have*

$$f[X,Y] = \nabla^\circ_X \bar{J}Y - \nabla^\circ_Y \bar{J}X$$

for any $X,Y \in \Gamma(D)$.

Proof: For any $X,Y \in \Gamma(D)$, we have

$$h(X,\bar{J}Y)+h^*(X,\bar{J}Y) = \bar{\nabla}_X \bar{J}Y - \nabla_X \bar{J}Y + \bar{\nabla}^*_X \bar{J}Y - \nabla^*_X \bar{J}Y$$

$$= 2(\bar{\nabla}^\circ_X \bar{J}Y - \nabla^\circ_X \bar{J}Y)$$

similarly

$$h(\bar{J}X,Y)+h^*(\bar{J}X,Y) = 2(\bar{\nabla}^\circ_Y \bar{J}X - \nabla^\circ_Y \bar{J}X)$$

From above equations, we have

$$h(X,\bar{J}Y)+h^*(X,\bar{J}Y) - h(\bar{J}X,Y) - h^*(\bar{J}X,Y) = 2(\bar{\nabla}^\circ_X \bar{J}Y - \nabla^\circ_X \bar{J}Y) - 2(\bar{\nabla}^\circ_Y \bar{J}X - \nabla^\circ_Y \bar{J}X)$$
$$= 2(\bar{J}\bar{\nabla}^\circ_X Y - \bar{J}\bar{\nabla}^\circ_Y X - \nabla^\circ_X \bar{J}Y + \nabla^\circ_Y \bar{J}X)$$
$$= 2(f[X,Y] + w[X,Y] - \nabla^\circ_X \bar{J}Y + \nabla^\circ_Y \bar{J}X)$$

Taking tangential part of this equation,we obtain

$$f[X,Y] = \nabla^\circ_X \bar{J}Y - \nabla^\circ_Y \bar{J}X$$

Hence the proof.

Theorem 4.5. *Let $(\bar{M},\bar{g},\bar{\nabla},\bar{\nabla}^*)$ be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold of $\bar{M}$. Then D' is integrable with respect to $\bar{\nabla}$ and $\bar{\nabla}^*$ if and only if*

$$-A_{wW}Z + A_{wZ}W = f[Z,W]$$

$$\nabla^{\circ\perp}_Z wW - \nabla^{\circ\perp}_W wZ = w[Z,W]$$

for any $Z,W \in \Gamma(D')$

Proof : The given hypothesis implies that

$$\bar{\nabla}^\circ_Z \bar{J}W = \bar{J}\bar{\nabla}^\circ_Z W$$

for any $Z,W \in \Gamma(D')$. Thus from the relation of Levi- civita connection with dual connections $\bar{\nabla}$,$\bar{\nabla}^*$, we obtain

$$2\bar{\nabla}^\circ_Z \bar{J}W = 2\bar{J}\nabla^\circ_Z W + \bar{J}(h(Z,W)+h^*(Z,W))$$

$$2\bar{\nabla}^\circ_Z(fW+wW) = 2\bar{J}\nabla^\circ_Z W + \bar{J}h(Z,W) + \bar{J}h^*(Z,W)$$

$$2(-A_{wW}Z + \nabla^{\circ\perp}_Z wW) = 2(f\nabla^\circ_Z W + w\nabla^\circ_Z W) + Bh(Z,W) + Ch(Z,W) + Bh^*(Z,W) + Ch^*(Z,W)$$

Similarly, we get

$$2(-A_{wZ}W + \nabla^{\circ\perp}_W wZ) = 2(f\nabla^\circ_W Z + w\nabla^\circ_W Z) + Bh(W,Z) + Ch(W,Z) + Bh^*(W,Z) + Ch^*(W,Z)$$

Thus we have

$$2(-A_{wW}Z + A_{wZ}W + \nabla^{\circ\perp}_Z wW - \nabla^{\circ\perp}_W wZ) = 2(f[Z,W] + w[Z,W])$$

By taking tangential and normal components of this equation,we get the desired result.

Theorem 4.6. *Let $(\bar{M},\bar{g},\bar{\nabla},\bar{\nabla}^*)$ be a CR-lightlike submanifold of an indefinite Kaehler statistical manifold of $\bar{M}$. Then $D_\circ$ integrable if and only if*

(i) $\bar{g}(h'(X,Y),N) + \bar{g}(h^{*\prime}(X,Y),N) = \bar{g}(h'(Y,X),N) + \bar{g}(h^{*\prime}(Y,X),N)$,
(ii) $\bar{g}(h'(X,\bar{J}Y),N) + \bar{g}(h^{*\prime}(X,\bar{J}Y),N) = \bar{g}(h'(Y,\bar{J}X),N) + \bar{g}(h^{*\prime}(Y,\bar{J}X),N)$
(iii) $\bar{g}(h^s(X,\bar{J}Y),W) + \bar{g}(h^{*s}(X,\bar{J}Y),W) = \bar{g}(h^s(Y,\bar{J}X),W) + \bar{g}(h^{*s}(Y,\bar{J}X),W)$
(iv) $\bar{g}(\bar{\nabla}'_X Y,\bar{J}\xi) + \bar{g}(\bar{\nabla}^{*\prime}_X Y,\bar{J}\xi) = \bar{g}(\bar{\nabla}'_Y X,\bar{J}\xi) + \bar{g}(\bar{\nabla}^{*\prime}_Y X,\bar{J}\xi)$

for any $X,Y \in \Gamma(D_\circ)$, $N \in \Gamma(ltr(TM))$, $\xi \in \Gamma(Rad(TM))$ and $W \in \Gamma(S(TM^\perp))$.

Proof : The definition of CR-lightlike submanifold M of $\bar{M}$ implies that $D_\circ$ is integrable if and only if

$$\bar{g}([X,Y],N) = \bar{g}([X,Y],\bar{J}N) = \bar{g}([X,Y],\bar{J}W) = \bar{g}([X,Y],\bar{J}\xi) = 0 \quad (4.1)$$

for any $X,Y\in\Gamma(D_\circ)$, $N\in\Gamma(ltr(TM))$, $\xi\in\Gamma(Rad(TM))$, $W\in\Gamma(S(TM^\perp))$.
For connections $\bar\nabla$ and $\bar\nabla^*$, we have

$$\bar g([X,Y],N)=\bar g(\bar\nabla^\circ_XY,N)-\bar g(\bar\nabla^\circ_YX,N)$$

$$=-\bar g(Y,\bar\nabla^\circ_XN)+\bar g(X,\bar\nabla^\circ_YN)$$

$$=\frac{1}{2}[-\bar g(Y,\bar\nabla_XN)-\bar g(Y,\bar\nabla^*_XN)+\bar g(X,\bar\nabla_YN)+\bar g(X,\bar\nabla^*_YN)]$$

$$=\frac{1}{2}[\bar g(Y,A_NX)+\bar g(Y,A^*_NX)-\bar g(X,A_NY)-\bar g(X,A^*_NY)]$$

Using the equation (3.9), we obtain

$$\bar g([X,Y],N)=\frac{1}{2}[\bar g(h^{*\prime}(X,Y),N)+\bar g(h^{\prime}(X,Y),N)\\ -\bar g(h^{*\prime}(Y,X),N)-\bar g(h^{\prime}(Y,X),N)]$$

and using the statistical character of the manifold and equations (3.4), (3.9) we have

$$\bar g([X,Y],\bar JN)=-\bar g(\bar J\bar\nabla^\circ_XY,N)+\bar g(\bar J\bar\nabla^\circ_YX,N).$$

$$=-\bar g(\bar\nabla^\circ_X\bar JY,N)+\bar g(\bar\nabla^\circ_Y\bar JX,N).$$

$$=\frac{1}{2}[\bar g(\bar JY,\nabla_XN)+\bar g(\bar JY,\nabla^*_XN)-\bar g(\bar JX,\nabla_YN)-\bar g(\bar JX,\nabla^*_YN)]$$

$$=\frac{1}{2}[-\bar g(h^{\prime}(X,\bar JY),N)-\bar g(h^{*\prime}(X,\bar JY),N)\\ +\bar g(h^{\prime}(Y,\bar JX),N)+\bar g(h^{*\prime}(Y,\bar JX),N)]$$

Also

$$\bar g([X,Y],\bar JW)=\bar g(\bar\nabla^\circ_XY,\bar JW)-\bar g(\bar\nabla^\circ_YX,\bar JW).$$

$$=-\bar g(\bar\nabla^\circ_X\bar JY,W)+\bar g(\bar\nabla^\circ_Y\bar JX,W).$$

Using the relation of connections $\bar\nabla$, $\bar\nabla^*$ with Levi-Civita Connection and the equation 3.3, we get

$$\bar g([X,Y],\bar JW)=\frac{1}{2}[-\bar g(h^s(X,\bar JY),W)-\bar g(h^{*s}(X,\bar JY),W)\\ +\bar g(h^s(Y,\bar JX),W)+\bar g(h^{*s}(Y,\bar JX),W)]$$

Now from equations (3.6),(3.7) we obtain

$$\bar g([X,Y],\bar J\xi)=\frac{1}{2}[\bar g(\bar\nabla^{\prime}_XY,\bar J\xi)+\bar g(\bar\nabla^{*\prime}_XY,\bar J\xi)\\ -\bar g(\bar\nabla^{\prime}_YX,\bar J\xi)-\bar g(\bar\nabla^{*\prime}_YX,\bar J\xi)].$$

Therefore using the given hypothesis, we have the desired result.

Theorem 4.7. *Let $(\bar M,\bar g,\bar\nabla,\bar\nabla^*)$ be an indefinite Kaehler statistical manifold and M be a CR-lightlike submanifold of $\bar M$. Then RadTM is integrable if and only if*
(i) $\bar g(h^l(\xi,\bar J\xi''),\xi')+\bar g(h^{*l}(\xi,\bar J\xi''),\xi')=\bar g(h^l(\xi',\bar J\xi''),\xi)+\bar g(h^{*l}(\xi',\bar J\xi''),\xi)$
(ii) $\bar g(h^l(\xi,X),\xi')+\bar g(h^{*l}(\xi,X),\xi')=\bar g(h^l(\xi',X),\xi)+\bar g(h^{*l}(\xi',X),\xi)$
(iii) $\bar g(h^s(\xi,\bar J\xi'),W)+\bar g(h^{*s}(\xi,\bar J\xi'),W)=\bar g(h^{*s}(\xi',\bar J\xi),W)+\bar g(h^{*s}(\xi',\bar J\xi),W)$
(iv) $\bar g(h^{\prime}(\xi,\bar J\xi'),N)+\bar g(h^{*\prime}(\xi,\bar J\xi'),N)=\bar g(h^{\prime}(\xi',\bar J\xi),N)+\bar g(h^{*\prime}(\xi',\bar J\xi),N)$
for any $X\in\Gamma(D_\circ),\xi,\xi',\xi''\in\Gamma(Rad(TM))$ and $N\in\Gamma(ltr(TM))$.

Proof : We know that for a CR-lightlike submanifold M of $\bar M$, $RadTM$ is integrable if and only if

$$\bar g([\xi,\xi'],\bar J\xi'')=\bar g([\xi,\xi'],X)=\bar g([\xi,\xi'],\bar JW)\\ =\bar g([\xi,\xi'],\bar JN)=0. \tag{4.2}$$

Thus for connections $\bar\nabla$ and $\bar\nabla^*$, we have

$$\bar g([\xi,\xi'],\bar J\xi'')=\bar g(\bar\nabla^\circ_\xi\xi',\bar J\xi'')-\bar g(\bar\nabla^\circ_{\xi'}\xi,\bar J\xi'')$$

$$=-\bar g(\xi',\bar\nabla^\circ_\xi\bar J\xi'')+\bar g(\xi,\bar\nabla^\circ_{\xi'}\bar J\xi'')$$

$$=\frac{1}{2}[-\bar g(h^l(\xi,\bar J\xi''),\xi')-\bar g(h^{*l}(\xi,\bar J\xi''),\xi')\\ +\bar g(h^l(\xi',\bar J\xi''),\xi)+\bar g(h^{*l}(\xi',\bar J\xi''),\xi)]$$

and using (3.7) and (3.8), we obtain

$$\bar g([\xi,\xi'],X)=\bar g(\bar\nabla^\circ_\xi\xi',X)-\bar g(\bar\nabla^\circ_{\xi'}\xi,X)$$

$$=\frac{1}{2}[-\bar g(A^{\prime}_\xi\xi',X)-\bar g(A^{*\prime}_\xi\xi',X)+\bar g(A^{\prime}_{\xi'}\xi,X)+\bar g(A^{*\prime}_{\xi'}\xi,X)]$$

$$=\frac{1}{2}[-\bar g(h^{*l}(\xi',X),\xi)-\bar g(h^l(\xi',X),\xi)+\bar g(h^{*l}(\xi,X),\xi')\\ +\bar g(h^l(\xi,X),\xi')]$$

On the other hand, we get

$$\bar g([\xi,\xi'],\bar JW)=\bar g(\bar\nabla^\circ_\xi\xi',\bar JW)-\bar g(\bar\nabla^\circ_{\xi'}\xi,\bar JW)$$

$$=-\bar g(W,\bar\nabla^\circ_\xi\bar J\xi')+\bar g(W,\bar\nabla^\circ_{\xi'}\bar J\xi)$$

$$=\frac{1}{2}[-\bar g(h^s(\xi,\bar J\xi'),W)-\bar g(h^{*s}(\xi,\bar J\xi'),W)+\bar g(h^{*s}(\xi',\bar J\xi),W)\\ +\bar g(h^{*s}(\xi',\bar J\xi),W)]$$

and

$$\bar g([\xi,\xi'],\bar JN)=-\bar g(N,\bar\nabla^\circ_\xi\bar J\xi')+\bar g(N,\bar\nabla^\circ_{\xi'}\bar J\xi)$$

$$=\frac{1}{2}[-\bar g(h^{\prime}(\xi,\bar J\xi'),N)-\bar g(h^{*\prime}(\xi,\bar J\xi'),N)+\bar g(h^{\prime}(\xi',\bar J\xi),N)\\ +\bar g(h^{*\prime}(\xi',\bar J\xi),N)]$$

Thus the proof is complete by using given hypothesis from (4.2).

Theorem 4.8. *Let $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$ be an indefinite Kaehler statistical manifold and M be a CR-lightlike submanifold of $\bar{M}$ .Then each maximal integrable manifold of the Radical distribution is totally geodesic in M if and only if*
(i) $\bar{g}(\bar{J}Y, A'_{\xi}X) + \bar{g}(\bar{J}Y, A^{*\prime}_{\xi}X) = 0$
(ii) $\bar{g}(h'(X, \bar{J}Y), N) + \bar{g}(h^{*\prime}(X, \bar{J}Y), N) = 0$
(iii) $\bar{g}(h^s(X, \bar{J}Y), W) + \bar{g}(h^{*s}(X, \bar{J}Y), W) = 0$
for any $X, Y, \xi \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$, $W \in \Gamma(S(TM^{\perp}))$ *where* $M_1 = \bar{J}(L_1)$ *and* $\bar{\nabla}^*$ *is the dual connection of* $\bar{\nabla}$ *w.r.t* $\bar{g}$.

Proof : The given hypothesis of M being a CR-lightlike submanifold of $\bar{M}$ implies that each maximal integrable manifold is totally geodesic if and only if

$$\bar{g}(\nabla^{\circ}_X Y, \bar{J}\xi) = \bar{g}(\nabla^{\circ}_X Y, \bar{J}N) = \bar{g}(\nabla^{\circ}_X Y, \bar{J}W) = \bar{g}(\nabla^{\circ}_X Y, Z_{\circ}) = 0 \tag{4.3}$$

where $X, Y \in \Gamma(Rad(TM))$

Since $\bar{M}$ is an indefinite Kaehler statistical manifold and (3.7) holds, we have

$$\begin{aligned}\bar{g}(\nabla^{\circ}_X Y, \bar{J}\xi) &= \bar{g}(\bar{\nabla}^{\circ}_X Y, \bar{J}\xi)\\ &= -\bar{g}(Y, \bar{\nabla}^{\circ}_X \bar{J}\xi) = -\bar{g}(Y, \bar{J}\bar{\nabla}^{\circ}_X \xi)\\ &= \bar{g}(\bar{J}Y, \bar{\nabla}^{\circ}_X \xi) = \frac{1}{2}[-\bar{g}(\bar{J}Y, A'_{\xi}X) - \bar{g}(\bar{J}Y, A^{*\prime}_{\xi}X)]\end{aligned}$$

and from (3.4) and (3.9), we obtain

$$\begin{aligned}\bar{g}(\nabla^{\circ}_X Y, \bar{J}N) &= \bar{g}(\bar{\nabla}^{\circ}_X Y, \bar{J}N)\\ &= -\bar{g}(Y, \bar{\nabla}^{\circ}_X \bar{J}N) = -\bar{g}(\bar{J}Y, \bar{\nabla}^{\circ}_X N)\\ &= \frac{1}{2}[-\bar{g}(\bar{J}Y, A_N X) - \bar{g}(\bar{J}Y, A^*_N X)] = \frac{1}{2}[-\bar{g}(h^{*\prime}(X, \bar{J}Y), N)\\ &\quad -\bar{g}(h'(X, \bar{J}Y), N)]\end{aligned}$$

Also

$$\bar{g}(\nabla^{\circ}_X Y, \bar{J}W) = \frac{1}{2}[-\bar{g}(h^s(X, \bar{J}Y), W) - \bar{g}(h^{*s}(X, \bar{J}Y), W)]$$

Hence the result holds using (4.3).

Theorem 4.9. *Let $(\bar{M}, \bar{g}, \bar{\nabla}, \bar{\nabla}^*)$ be an indefinite Kaehler statistical manifold and M be a CR-lightlike submanifold of $\bar{M}$. Then $\bar{J}Rad(TM)$ is integrable if and only if*
(i) $\bar{g}(\bar{J}\xi, A_N \bar{J}\bar{\xi}') + \bar{g}(\bar{J}\xi, A^*_N \bar{J}\bar{\xi}') = \bar{g}(\bar{J}\xi', A_N \bar{J}\xi)$
$+\bar{g}(\bar{J}\xi', A^*_N \bar{J}\xi)$
(ii) $\bar{g}(h^s(\bar{J}\bar{\xi}', \xi), W) + \bar{g}(h^{*s}(\bar{J}\bar{\xi}', \xi), W) = \bar{g}(h^s(\bar{J}\xi, \xi'), W)$
$+\bar{g}(h^{*s}(\bar{J}\xi, \xi'), W)$
(iii) $\bar{g}(h^l(\bar{J}\bar{\xi}', \xi), \xi'') + \bar{g}(h^{*l}(\bar{J}\bar{\xi}', \xi), \xi'') = \bar{g}(h^l(\bar{J}\xi, \xi'), \xi'')$
$+\bar{g}(h^{*l}(\bar{J}\xi, \xi'), \xi'')$
(iv) $\bar{g}(\bar{J}X, A'_{\xi}\bar{J}\xi') + \bar{g}(\bar{J}X, A'^{*}_{\xi}\bar{J}\xi') = \bar{g}(\bar{J}X, A'_{\xi'}\bar{J}\xi)$
$+\bar{g}(\bar{J}X, A'^{*}_{\xi'}\bar{J}\xi)$
for any $\xi, \xi', \xi'' \in \Gamma(Rad(TM))$, $N \in \Gamma(ltr(TM))$,
$W \in \Gamma(S(TM^{\perp}))$.

Proof : We have

$$\begin{aligned}\bar{g}([\bar{J}\xi', \bar{J}\xi], N) &= \bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\bar{\xi}'}\bar{J}\xi, N) - \bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\xi}\bar{J}\xi', N)\\ &= -\bar{g}(\bar{J}\xi, \bar{\nabla}^{\circ}_{\bar{J}\bar{\xi}'}N) + \bar{g}(\bar{J}\xi', \bar{\nabla}^{\circ}_{\bar{J}\xi}N)\\ &= \frac{1}{2}[-\bar{g}(\bar{J}\xi, A_N\bar{J}\bar{\xi}') - \bar{g}(\bar{J}\xi, A^*_N\bar{J}\bar{\xi}') + \bar{g}(\bar{J}\xi', A_N\bar{J}\xi)\\ &\quad +\bar{g}(\bar{J}\xi', A^*_N\bar{J}\xi)]\end{aligned}$$

and

$$\begin{aligned}\bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}W) &= -\bar{g}(\bar{J}\bar{\nabla}^{\circ}_{\bar{J}\bar{\xi}'}\bar{J}\xi, W) + \bar{g}(\bar{J}\bar{\nabla}^{\circ}_{\bar{J}\xi}\bar{J}\xi', W)\\ &= -\bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\bar{\xi}'}\bar{J}^2\xi, W) + \bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\xi}\bar{J}^2\xi', W)\\ &= \frac{1}{2}[\bar{g}(h^s(\bar{J}\bar{\xi}', \xi), W) + \bar{g}(h^{*s}(\bar{J}\bar{\xi}', \xi), W) - \bar{g}(h^s(\bar{J}\xi, \xi'), W)\\ &\quad -\bar{g}(h^{*s}(\bar{J}\xi, \xi'), W)]\end{aligned}$$

Also

$$\begin{aligned}\bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}\xi'') &= \bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\bar{\xi}'}\bar{J}\xi, \bar{J}\xi'') - \bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\xi}\bar{J}\xi', \bar{J}\xi'')\\ &= \bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\bar{\xi}'}\xi, \xi'') - \bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\xi}\xi', \xi'')\\ &= \frac{1}{2}[\bar{g}(h^l(\bar{J}\bar{\xi}', \xi), \xi'') + \bar{g}(h^{*l}(\bar{J}\bar{\xi}', \xi), \xi'') - \bar{g}(h^l(\bar{J}\xi, \xi'), \xi'')\\ &\quad -\bar{g}(h^{*l}(\bar{J}\xi, \xi'), \xi'')]\end{aligned}$$

and

$$\begin{aligned}\bar{g}([\bar{J}\xi', \bar{J}\xi], X) &= \bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\bar{\xi}'}\bar{J}\xi, X) - \bar{g}(\bar{\nabla}^{\circ}_{\bar{J}\xi}\bar{J}\xi', X)\\ &= \bar{g}(\bar{J}\bar{\nabla}^{\circ}_{\bar{J}\bar{\xi}'}\xi, X) - \bar{g}(\bar{J}\bar{\nabla}^{\circ}_{\bar{J}\xi}\xi', X)\\ &= \frac{1}{2}[\bar{g}(\bar{J}X, A'_{\xi}\bar{J}\xi') + \bar{g}(\bar{J}X, A^{*\prime}_{\xi}\bar{J}\xi') - \bar{g}(\bar{J}X, A'_{\xi'}\bar{J}\xi)\\ &\quad -\bar{g}(\bar{J}X, A^{*\prime}_{\xi'}\bar{J}\xi)]\end{aligned}$$

Now M being a CR-lightlike submanifold of the manifold $\bar{M}$ concludes that $\bar{J}Rad(TM)$ is integrable if and only if

$$\begin{aligned}\bar{g}([\bar{J}\xi', \bar{J}\xi], N) = \bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}W) &= \bar{g}([\bar{J}\xi', \bar{J}\xi], \bar{J}\xi'')\\ &= \bar{g}([\bar{J}\xi', \bar{J}\xi], X) = 0.\end{aligned}$$

Hence using the above assertion, we get the desired result.

5. Conclusion and Scope

In this paper, the concept of lightlike geometry in the indefinite Kaehler Statistical manifold has been discussed and the characterisation of distributions in the CR lightlike submanifolds of the same with respect to their geodescity and integrablity has been done.The statistical manifold is based on the information geometry having applications in the field of neural networks and when endowed with the lightlike geometry , becomes applicable in various brances of mathematics and physics. So this branch of study can imbibe great interest in the contemporary geometers to work upon the further properties of distributions of CR lightlike submanifolds in the indefinite Kaehler statistical manifold .

References

[1] Amari, S., *Differential geometry in statistical inference*, 10, 19-94 (1987).

[2] Bejancu, C. L. and Duggal, K. L., Global lightlike manifolds and Harmonicity, *Kodai Mathematical Journal*, 28(1)(2005), 131-145.

[3] Boyom, M. N., Siddiqui, A. N., Othman, W. A. M. and Shahid, M. H., Classification of totally umbilical CR-statistical submanifolds in holomorphic statistical manifolds with constant holomorphic curvature, *In International Conference on Geometric Science of Information*, Springer, (2017), 809-817.

[4] Duggal, K., Lorentzian geometry of CR submanifolds, *Acta Applicandae Mathematica*, 17(2)(1989), 171-193.

[5] Duggal, K., Lorentzian geometry of globally framed manifolds, *Acta Applicandae Mathematica*, 19(2)(1990), 131-148.

[6] Duggal, K. and Bejancu, A., Lightlike CR hypersurfaces of indefinite Kaehler manifolds, *Acta Applicandae Mathematica*, 31(2)(1993), 171-190.

[7] Furuhata, H., Hypersurfaces in statistical manifolds, *Differential Geometry and its Applications*, 27(3)(2009), 420-429.

[8] Furuhata, H., Statistical hypersurfaces in the space of Hessian curvature zero, *Differential Geometry and its Applications*, 29(S1)(2011), S86-S90.

[9] Furuhata, H. and Hasegawa, I., Submanifold theory in holomorphic statistical manifolds, *Geometry of Cauchy-Riemann Submanifolds*, Springer, Singapore, (2016), 179-215 .

[10] Furuhata, H., Hasegawa, I., Okuyama, Y., Sato, K., and Shahid, M. H., Sasakian statistical manifolds, *Journal of Geometry and Physics*, 117(2017), 179-186.

[11] Kurose, T., Geometry of statistical manifolds, *Mathematics in the 21st Century*, (2004), 34-43.

[12] Milijevic, M., Totally real statistical submanifolds, *Interdisciplinary Information Sciences*, 21(2)(2015), 87-96.

[13] Noda, T., Symplectic structures on statistical manifolds, *Journal of the Australian Mathematical Society*, 90(3)(2011), 371-384.

[14] Rao, C., Information and the accuracy attainable in the estimation of statistical parameters, *Bulletin of Calcutta Mathematical Society*, 37(1945), 81-91.

[15] Sahin, B. and Gunes, R., Integrability of distributions in CR-lightlike submanifolds, *Tamkang Journal of Mathematics*, 33(3)(2002), 209-222.

[16] Siddiqui, A. and Shahid, M., On totally real statistical submanifolds, Filomat, 32(13)(2018), 1-11.

[17] Simon, U., *Affine differential geometry*, In Handbook of differential geometry, Elsevier, 1(2000), 905-961.

[18] Vos, P. W., Fundamental equations for statistical submanifolds with applications to the bartlett correction, *Annals of the Institute of Statistical Mathematics*, 41(3)(1989), 429-450.

[19] Yazla, A., Erken, I. K., and Murathan, C., Almost cosymplectic statistical manifolds, *arXiv preprint arXiv:1801.09890*, (2018).

[20] Amari, S., *Differential geometrical methods in statistics*, Lecture Notes in Statistics, Springer, New York, 28 (1985).

[21] Duggal, K. and Bejancu, A., *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Mathematics and its applications, Kluwer Academic (1996).

[22] Milijevic, M., *CR submanifolds in holomorphic statistical manifold*, Ph.D Thesis in Science, Department of Mathematics Graduates School of Science, Hokkaido University, (2015).

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