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An equitable edge coloring of some classes of product of graphs

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Abstract
An equitable edge coloring for any graph G is an assignment of colors to all the edges of graph G such that adjacent edges receive the different color and for any two color classes different by at most one. In this paper, we prove theorem on equitable edge coloring for strong products of path and cycle.

Keywords
Equitable edge coloring, Strong product, Cycle graph.

AMS Subject Classification
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1. Introduction

Coloring problem is one among the most important research area in graph theory. As an extension of proper edge coloring [3, 9, 10] and conjectures on equitable edge coloring [1, 4, 6, 8] is established. It is tough to find a result using equitable edge chromatic number. In this paper, we consider a graph G as finite, simple and undirected. Let $G=(V(G),E(G))$ be an ordered pair of graph G with the vertices and the edges respectively. An equitable edge coloring of graph G is a mapping $f: E(G) \rightarrow N$, where N is a set of colors satisfying the following conditions.

1. $f(e) \neq f(e')$ for any two adjacent edges $e, e' \in E(G)$.
2. $||E_i|-|E_j|| \leq 1; i,j = 1,2,\ldots,k$.

The minimum number of colors are required for an equitable edge coloring of graph G is called the equitable edge chromatic number of G and is denoted by $\chi'_e(G)$. The edge chromatic number of graph G is related to the maximum degree $\Delta(G)$, the greatest number of edges incident to any single vertex of G. it is clear that $\chi'(G) \geq \Delta(G)$, for if Δ various number of edges join at a single vertex v, then all of these edges to be received different colors from each other and that can be possible if there are at least Δ colors available to be received. The edge chromatic number of graph G must be at least Δ, the greatest vertex degree of graph G given by Skiena [9]. However, Vizing[10] and Gupta[3] proved that any graph G can be edge colored with at most $\Delta+1$ colors. Vizing's theorem states that, the tight bound of edge coloring for any simple graph G, $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. If a graph G with edge chromatic number equal to $\Delta(G)$, then the graph G is called Type 1 and if edge chromatic number is equal to $\Delta(G)+1$ then it is called Type 2 graph. The number of colors for bipartite graph and high degree planar graphs is always Δ and for the multi graph may be as large as $\frac{3\Delta}{2}$. In 1964 Paul Erdős[1] conjectured that an equitable coloring is achievable with only one more color; for any graph G with greatest degree Δ has an equitable coloring with $\Delta+1$ colors. This conjecture was proved in 1970 by Hajnal and Szemerédi [4] with lengthy and difficulted proof is called as the HajnalSzemerédi Theorem. In the year 2008,Kierstead and Kostochka[6] was presented the same proof in a simple way.

Theorem 1.1. *[2] For cycle graph C_n with maximum degree $\Delta(G)$, then*

$$\chi'_e(C_n) = \begin{cases} \Delta(G)+1, & \text{if } n \text{ is odd} \\ \Delta(G), & \text{if } n \text{ is even.} \end{cases}$$

In this paper, we study the conjecture on equitable edge coloring of strong product of path and cycle.

2. Results and Discussion

Definition 2.1. *[5] Consider G and H be two graphs. The strong product* $G \boxtimes H$*, defined by* $V(G \boxtimes H) = \{(g,h) | g \in V(G), h \in V(H)\}$ *and* $E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$*.*

Theorem 2.2. *The equitable edge coloring of* $P_n \boxtimes P_m$ *and its edge chromatic number is* $\Delta(P_n \boxtimes P_m)$ *for all* $n,m \geq 3$ *and* $n,m \in \mathbb{Z}^+$

Proof. Here $\Delta(P_n \boxtimes P_m)$ is the maximum degree of $P_n \boxtimes P_m$ and having
$n(R_1, R_2, \ldots, R_n)$ rows and $m(C_1, C_2, \ldots, C_m)$ columns. We divide $(P_n \boxtimes P_m)$ into three parts, say X_1, X_2, X_3. X_1 with three colors and X_2 with three colors and X_3 with the remaining colors of $\Delta(P_n \boxtimes P_m)$.

Case (i) If n is odd and for any $m \geq 3$

Color all the edges in $R_2, R_3, \ldots, R_{n-1}$ using X_3 color. Then assign the colors to the remaining edges of $(P_n \boxtimes P_m)$ in the following way: at each maximum degree vertex in R_p using X_1 color to the edges which are incident from R_{p-1} and using X_2 color to the edges which are incident from R_{p+1}. Both the color will be given cyclically in order from p
adjustment to the repetition to attains the equitable conditions.

The balanced edges at maximum degree vertex in C_2 from C_1 and C_{m-1} from C_m be color with the missing color of $\Delta(P_n \boxtimes P_m)$. Finally, we color the remaining edges and boundary edges R_1, R_n, C_1 and C_m according to satisfying the equitable edge coloring conditions.

Case (ii) If n is even and m is odd:

Color all the edges in $C_2, C_3, \ldots, C_{m-1}$ using X_3 color. Then assign the colors to the remaining edges of $(P_n \boxtimes P_m)$ in the following way: at each maximum degree vertex in C_p using X_1 color to the edges which are incident from C_{p-1} and using X_2 color to the edges which are incident from C_{p+1}. Both the color will be given cyclically in order from $p = 2, 4, 6, \ldots, m-1$, using a repeated pattern with the minor adjustment to the repetition to attains the equitable conditions.

The balanced edges at maximum degree vertex in R_2 from R_1 and R_{n-1} from R_n be color with the missing color of $\Delta(P_n \boxtimes P_m)$. Finally, we color the remaining edges and boundary edges C_1, C_m, R_1, and R_n according to satisfying the equitable edge coloring conditions.

Case (iii) If both n and m are even:

Color all the edges in $R_2, R_3, \ldots, R_{n-1}$ using X_3 color. Then assign the colors to the remaining edges of $(P_n \boxtimes P_m)$ in the following way: at each maximum degree vertex in R_p from C_2 to $C_{\frac{m}{2}}$, using X_1 color to the edges which are incident from R_{p-1} and using X_2 color to the edges which are incident from R_{p+1}. Then at each maximum degree vertex in R_p from $C_{\frac{m}{2}+1}$ to C_m using X_2 color to the edges which are incident from R_{p-1} and using X_1 color to the edges which are incident from R_{p+1}. Both the color will be given cyclically in order from $p = 2, 4, 6, \ldots, n-2$, using a repeated pattern with the minor adjustment to the repetition to attains the equitable conditions.

At each vertex in R_n from C_2 to $C_{\frac{m}{2}}$, using X_1 color to the edges which are incident from R_{n-1} and from $C_{\frac{m}{2}+1}$ to C_{m-1} using X_2 color to the edges which are incident from R_{n-1}. The balanced edges between (C_1, C_2) and (C_{m-1}, C_m) be colors with the missing color of $\Delta(P_n \boxtimes P_m)$. Finally, we color the remaining edges and boundary edges R_1, R_n, C_1 and C_m according to satisfying the equitable edge coloring conditions.
Therefore, $\chi'_e(P_n \boxtimes P_m) = \Delta(P_n \boxtimes P_m)$. □

Theorem 2.3. *The equitable edge coloring of* $P_n \boxtimes C_m$ *and its edge chromatic number is* $\Delta(P_n \boxtimes C_m)$ *for all* $n,m \geq 3$ *and* $n,m \in \mathbb{Z}^+$

Proof. Here $\Delta(P_n \boxtimes C_m)$ is the maximum degree of $P_n \boxtimes C_n$ and having
$n(R_1, R_2, \ldots, R_n)$ rows and $m(C_1, C_2, \ldots, C_m)$ columns. We divide $\Delta(P_n \boxtimes C_m)$ into three parts, say X_1, X_2, X_3. X_1 with three colors and X_2 with three colors and X_3 with the remaining colors of $\Delta(P_n \boxtimes C_m)$.

Case (i) If n is odd and m is even:

Color all the edges in $R_2, R_3, \ldots, R_{n-1}$ using X_3 color. Then assign the colors to all the edges between R_1 and R_2 using X_1 color and edges between R_2 and R_3 with X_2 color. Then color the remaining edges between
$((R_3, R_4), (R_4, R_5), (R_5, R_6), (R_6, R_7), \ldots, (R_{n-1}, R_n))$ using the colors of X_1, X_2 cyclically. Finally, we color the remaining edges R_1, R_n using $\Delta(P_n \boxtimes C_m)$ colors according to satisfying the equitable edge coloring conditions.

Case (ii) If both n and m are odd:

Assign X_3 color to all the edges in $R_2, R_3, \ldots, R_{n-1}$ from C_1 to C_m. Color the edges of C_m using one of the color of X_3 and assign the remaining color of X_3 in C_1 edges. Then color the remaining edges between
$((R_1, R_2), (R_2, R_3), (R_3, R_4), (R_4, R_5), \ldots, (R_{n-1}, R_n))$ using the colors of X_1, X_2 cyclically. Now, we color the remaining edges between (C_m, C_1) using the missing color of $\Delta(P_n \boxtimes C_m)$. Finally, we color the remaining edges R_1 and R_n using $\Delta(P_n \boxtimes C_m)$ colors according to satisfying the equitable edge coloring conditions.

Case (iii) If both n and m are even:

Assign X_3 color to all the edges in $C_1, C_2, \ldots, C_m$ from R_1 to R_{n-1}. Color the edges of R_{n-1} using one of the color of X_3 and assign the remaining color of X_3 in R_n edges. Then color the remaining edges between
$((C_1, C_2), (C_2, C_3), (C_3, C_4), (C_4, C_5), \ldots, (C_{m-1}, C_m))$ using the colors of X_1, X_2 cyclically using a repeated pattern with the minor adjustment to the repetition to attains the equitable conditions except the boundary edges. Finally, we color the remaining boundary edges R_1 and R_n using $\Delta(P_n \boxtimes C_m)$ colors according to satisfying the equitable edge coloring conditions.

Case (iv) If n is even and m is odd:

Assign X_3 color to all the edges in $R_2, R_3, \dots, R_{n-1}$ from C_1 to C_m. Color the edges of C_m using one of the color of X_3 and assign the remaining color of X_3 in C_1 edges. Then assign the color to the remaining edges of $(P_n \boxtimes C_m)$ in the following way: at each maximum degree vertex in R_p from C_2 to $C_{\frac{m+1}{2}}$, using X_1 color to the edges which are incident from R_{p-1} and use X_2 color to the edges which are incident from R_{p+1}. Then at each maximum degree vertex in R_p from $C_{\frac{m+3}{2}}$ to C_m using X_2 color to the edges which are incident from R_{p-1} and use X_1 color to the edges which are incident from R_{p+1}. Both the color will be given cyclically in order from $p = 2,4,6,\dots,n-2$. At each vertex in R_n from C_2 to $C_{\frac{m+1}{2}}$, use X_1 color to the edges which are incident from R_{n-1} and from $C_{\frac{m+3}{2}}$ to C_m use X_2 color to the edges which are incident from R_{n-1}. Now, we color the remaining edges between(C_m, C_1) and (C_1, C_2) using the missing color of $\Delta(P_n \boxtimes C_m)$. Finally, we color the remaining edges R_1 and R_n using $\Delta(P_n \boxtimes C_m)$ colors according to satisfying the equitable edge coloring conditions.
Therefore, $\chi'_e(P_n \boxtimes C_m) = \Delta(P_n \boxtimes C_m)$. □

Theorem 2.4. *The equitable edge coloring of $C_n \boxtimes C_m$ for all $n,m \geq 3$ and $n,m \in \mathbb{Z}^+$*

$$\chi'_e(C_n \boxtimes C_m) = \begin{cases} \Delta(C_n \boxtimes C_m) + 1, & \text{if both n and m are odd} \\ \Delta(C_n \boxtimes C_m), & \text{otherwise.} \end{cases}$$

Proof. **Case (1) If both n and m are odd:**

Here $\Delta(C_n \boxtimes C_m)$ is the maximum degree of$\Delta C_n \boxtimes C_m$ and having
$n(R_1, R_2, \dots, R_n)$ rows and $m(C_1, C_2, \dots, C_m)$ columns. We divide $\Delta(C_n \boxtimes C_m) + 1$ into three equal parts, say X_1, X_2, X_3.

Subcase (1.1) If $n = 3k, k = 1,3,5,\dots$.

Color all the edges in R_1 using X_1, R_2 with X_2 and R_3 with X_3 color. Then assign the colors to all the edges in the remaining rows $R_4, R_5, \dots, R_n$ of $(C_n \boxtimes C_m)$ using X_1, X_2, X_3 colors cyclically using a repeated pattern with the minor adjustment to the repetition to attains the equitable conditions. Then assign the colors to all the edges between R_1 and R_2 using X_3 color and edges between R_2 and R_3 with X_1 color and edges between R_3 and R_4 with X_2 color. Finally, we color the remaining edges between $((R_4,R_5),(R_5,R_6),(R_6,R_7),\dots,(R_n,R_1))$ using the colors of X_3, X_1, X_2 cyclically which satisfies the conditions of equitable edge coloring.

Subcase (1.2) If $n = 3k+2, k = 1,3,5,\dots$.

Color all the edges in R_1 using X_1, R_2 with X_2 and R_3 with X_3 color. Then assign the colors to all the edges R_4, $R_5, \dots,$ R_{n-2} of $(C_n \boxtimes C_m)$ using X_1, X_2, X_3 colors cyclically using a repeated pattern with the minor adjustment to the repetition to attains the equitable conditions and for all the edges in the remaining rows R_{n-1} and R_n use X_1 color.Thencolor the remaining edges between
$((R_1,R_2),(R_2,R_3),(R_3,R_4),\dots,(R_{n-2},R_{n-1}))$ using the colors of X_3, X_1, X_2 cyclically. Finally, we color the edges between (R_{n-1}, R_n) using X_3 color and the missing color of X_1 and the edges between (R_n, R_1) using X_2 and the missing color of X_1 according to satisfying the equitable edge coloring conditions.

Subcase (1.3) If $n = 3k+4, k = 1,3,5,\dots$.

Color all the edges in R_1 using X_1, R_2 with X_2 and R_3 with X_3 color. Then assign the colors to all the edges $R_4, R_5, \dots, R_{n-2}$ of $(C_n \boxtimes C_m)$ using X_1, X_2, X_3 colors cyclically using a repeated pattern with the minor adjustment to the repetition to attains the equitable conditions. To color all the edges in the remaining rows R_{n-1} use X_2 and for R_n use X_1 colors. Then color the remaining edges between $((R_1,R_2),(R_2,R_3),(R_4,R_5),\dots,$ $(R_{n-3},R_{n-2}))$ using the colors of X_3, X_1, X_2 cyclically. Finally, we color the edges between (R_{n-2}, R_{n-1}) using X_1 color and the missing color of X_2 and the edges between (R_{n-1}, R_n) using X_3 and the missing colors of X_1 and X_2 and the edges between (R_n, R_1) using X_2 and the missing color of X_1 according to satisfying the equitable edge coloring conditions.
Therefore, $\chi'_e(C_n \boxtimes C_m) = \Delta(C_n \boxtimes C_m) + 1$.

Case(2) Both m and n are not odd:

Here, we divide $\Delta(C_n \boxtimes C_m)$ into three parts, say X_1, X_2, X_3. X_1 with three colors and X_2 with three colors and X_3 with the remaining colors of $\Delta(C_n \boxtimes C_m)$

Subcase (2.1) If both n and m are even:

Color all the edges in $R_1, R_2, R_3, \dots, R_n$ using X_3 color. Then assign the colors to all the edges between R_1 and R_2 using X_1 color and edges between R_2 and R_3 with X_2 color. Then color the remaining edges between
$((R_3,R_4),(R_4,R_5),(R_5,R_6),(R_6,R_7),\dots,(R_n,R_1))$ using the colors of X_1, X_2 cyclically which satisfies the conditions of equitable edge coloring.

Subcase (2.2) If n is even and m is odd:

Assign X_3 color to all the edges in $R_1, R_2, \dots, R_n$ from C_1 to C_m. Color the edges of C_m using one of the color of X_3 and assign the remaining color of X_3 in C_1 edges. Then color the remaining edges between
$((R_1,R_2),(R_2,R_3),(R_3,R_4),(R_4,R_5),\dots,(R_n,R_1))$ using the colors of X_1, X_2 cyclically. Now, we color the remaining edges between (C_m, C_1) using the missing colors of $\Delta(C_n \boxtimes C_m)$ which satisfies the conditions of equitable edge coloring.

Subcase (2.3) If n is odd and m is even:

Assign X_3 color to all the edges in $C_1, C_2, \dots, C_m$ from R_1 to R_n. Color the edges of R_n using one of the color of X_3 and assign the remaining color of X_3 in R_1 edges. Then color the remaining edges between
$((C_1,C_2),(C_2,C_3),(C_3,C_4),(C_4,C_5),\dots,(C_m,C_1))$ using the colors of X_1, X_2 cyclically. Now, we color the remaining edges between (R_n, R_1) using the missing colors of $\Delta(C_n \boxtimes C_m)$ which satisfies the conditions of equitable edge coloring.
Therefore, $\chi'_e(C_n \boxtimes C_m) = \Delta(C_n \boxtimes C_m)$. □

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