



A note on almost $C(\alpha)$ – manifolds

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Abstract

In this paper, we concentrate on almost $C(\alpha)$ – manifolds. Firstly, we study almost $C(\alpha)$ – manifolds satisfying the curvature conditions $R.Q = 0$, $R(\xi, X).\mathcal{L} = 0$ and $R(\xi, X).R = 0$, where \mathcal{L} is the curvature tensor of type $(0, 2)$, R is the Riemannian curvature tensor and Q is the Ricci operator. Then, we deal with \mathcal{L} – recurrent and generalized Ricci recurrent $C(\alpha)$ – manifolds and give an important relationship between such manifolds.

Keywords

Almost $C(\alpha)$ –manifold, Einstein Manifold, Generalized Ricci recurrent manifold, \mathcal{L} –curvature tensor.

AMS Subject Classification

53C15, 53C25, 53D15.

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Article History: Received 15 February 2020; Accepted 09 August 2020

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1. Introduction

Considering the recent stage of the developments in contact geometry, there is an impression that the geometers are focused on problems in almost contact metric geometry. Many different classes of almost contact structures are defined in the literature such as Sasakian [13], Kenmotsu [8], almost cosymplectic [6], trans-Sasakian [12] and others.

In 1981, Jenessens and Vanhecke introduced a new class of almost contact metric manifolds which is called almost $C(\alpha)$ – manifolds [7]. They proved that cosymplectic, Sasakian and Kenmotsu manifolds are $C(0)$ –, $C(1)$ – and $C(-1)$ – manifolds, respectively. After this work, such manifolds have been studied extensively by many mathematicians in contact geometry. For instance, Kharitonova obtained necessary and sufficient conditions for an almost contact metric manifold to be an almost $C(\alpha)$ – manifold in [9]. Ashoka et al. studied the

flatness of the pseudo-Projective, quasi-conformal curvature tensor on such a manifold in [2]. Furthermore, Akbar and Sarkar characterized certain curvature conditions on conharmonic and concircular curvature tensors on almost $C(\alpha)$ – manifolds in [1]. In 2016, Blair and Yıldırım showed that if an almost $C(\alpha)$ – manifold of $\dim \geq 5$ is conformally flat, it is of constant sectional curvature α and obtained some important results for such a manifold in [4].

Motivated by the above studies, we study generalized Ricci recurrent, \mathcal{L} – recurrent almost $C(\alpha)$ – manifolds. Also, in Riemannian geometry, one of the basic interest is curvature properties and to what extent these determine the manifold itself. So, we investigate the geometric effects of the such manifolds when satisfies the curvature conditions $R.Q = 0$, $R(\xi, X).\mathcal{L} = 0$ and $R(\xi, X).R = 0$.

The paper is organized in the following way:

Section 1 is devoted to the introduction. In section 2, we give some fundamental notions which are going to be needed. In section 3, we study almost $C(\alpha)$ – manifolds satisfying certain curvature conditions. In section 4, we deal with generalized Ricci recurrent almost $C(\alpha)$ – manifolds. In last section, we discuss \mathcal{L} – recurrent almost $C(\alpha)$ – manifolds and obtain some important characterizations which classifies the manifold.

2. Preliminaries

In this section, we shall give a brief review of some fundamental definitions and formulas. We refer to [3], [7] and [14].

A $(2n + 1)$ -dimensional smooth manifold M is an almost contact metric manifold with an almost contact metric structure (φ, ξ, η, g) such that φ is a tensor field of type $(1, 1)$, ξ is a vector field (called the characteristic vector field) of type $(0, 1)$, $1 - \text{form } \eta$ is a tensor field of type $(1, 0)$ on M and the Riemannian metric g satisfies the following relations:

$$\varphi^2 X = -X + \eta(X)\xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0 \quad (2.1)$$

$$\eta(\xi) = 1, \quad \eta(X) = g(X, \xi) \quad (2.2)$$

and

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$$

$$g(\varphi X, Y) = -g(X, \varphi Y)$$

for any $X, Y \in \Gamma(TM)$.

On the other hand, in [3], D.E. Blair defined the fundamental 2-form Φ of M as follows:

$$\Phi(X, Y) = g(X, \varphi Y)$$

for any $X, Y \in \Gamma(TM)$. Furthermore, if the relation

$$\Phi(X, Y) = d\eta(X, Y)$$

holds for all $X, Y \in \Gamma(TM)$, an almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ is said to be contact metric manifold such that

$$d\eta(X, Y) = \frac{1}{2} \{ X\eta(Y) - Y\eta(X) - \eta([X, Y]) \}.$$

The Nijenhuis tensor field of φ is defined by

$$N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[X, \varphi Y] - \varphi[\varphi X, Y]$$

for all $X, Y \in \Gamma(TM)$. If M is an almost contact metric manifold and the Nijenhuis tensor of φ satisfies

$$N_\varphi + 2d\eta \otimes \xi = 0$$

then, M is called a normal contact metric manifold. A normal contact metric manifold (respectively, almost cosymplectic manifold, almost Kenmotsu manifold) is called Sasakian manifold (respectively, cosymplectic manifold, Kenmotsu manifold).

An almost contact metric manifold is called an almost $C(\alpha)$ - manifold if the following condition is satisfied

$$R(X, Y)Z = R(\varphi X, \varphi Y)Z + \alpha \{ g(Y, Z)X - g(X, Z)Y + g(Z, \varphi X)\varphi Y - g(Z, \varphi Y)\varphi X \},$$

where ∇ is the Levi-Civita connection on M and α is a real number [7]. For an almost $C(\alpha)$ - manifold we also have

$$R(\xi, Y)Z = \alpha(g(Y, Z)\xi - \eta(Z)Y), \quad (2.3)$$

$$R(\xi, Y)\xi = \alpha(\eta(Y)\xi - Y), \quad (2.4)$$

$$R(Y, \xi)Z = \alpha(\eta(Z)Y - g(Y, Z)\xi), \quad (2.5)$$

$$R(Y, \xi)\xi = \alpha(Y - \eta(Y)\xi), \quad (2.6)$$

$$S(X, \xi) = 2n\alpha\eta(X), \quad (2.7)$$

$$S(\xi, \xi) = 2n\alpha, \quad (2.8)$$

$$Q\xi = 2n\alpha\xi, \quad (2.9)$$

where S and R are the Ricci tensor and Riemann curvature tensor of M , respectively and Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$.

Furthermore, it is well known that Sasakian (respectively, cosymplectic, Kenmotsu, α -Sasakian and α -Kenmotsu) manifolds correspond to $C(1)$ - (respectively, $C(0)$ -, $C(-1)$ -, $C(\alpha^2)$ - and $C(-\alpha^2)$ -) manifolds.

On the other hand, a non-flat semi-Riemannian manifold (M, g) is called generalized Ricci recurrent manifold if its Ricci tensor field S satisfies

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z) \quad (2.10)$$

where A and B are non-zero 1-forms [5]. If B vanishes identically in (2.10), then M is called a Ricci recurrent manifold.

In 2012, Mantica and Molinari [10] introduced a new symmetric tensor of type $(0, 2)$ which is called \mathcal{Z} tensor as

$$\mathcal{Z}(X, Y) = S(X, Y) + fg(X, Y), \quad (2.11)$$

where f is a smooth function. From the equalities (2.7), (2.8) and (2.11), we get

$$\mathcal{Z}(X, \xi) = (f + 2n\alpha)\eta(X), \quad (2.12)$$

$$\mathcal{Z}(\xi, \xi) = (f + 2n\alpha). \quad (2.13)$$

Also, a non-flat semi-Riemannian manifold (M, g) is called \mathcal{Z} - recurrent manifold if the non-zero \mathcal{Z} - tensor satisfies

$$(\nabla_X \mathcal{Z})(Y, Z) = \eta(X)\mathcal{Z}(Y, Z) \quad (2.14)$$

such that

$$(\nabla_X \mathcal{Z})(Y, Z) = \nabla_X \mathcal{Z}(Y, Z) - \mathcal{Z}(\nabla_X Y, Z) - \mathcal{Z}(Y, \nabla_X Z), \quad (2.15)$$

where η is a non-zero 1-form, for any $X, Y, Z \in \Gamma(TM)$ [11]. Here we assume that 1-form η is the g -dual of ξ of type $(1, 0)$.

3. On Almost $C(\alpha)$ -Manifolds Satisfying Some Curvature Conditions

In this section, we study almost $C(\alpha)$ - manifolds which satisfy certain curvature conditions such as $R.Q = 0$, $R(\xi, X).\mathcal{Z} = 0$ and $R(\xi, X).R = 0$.

Theorem 3.1. *Let M be an almost $C(\alpha)$ - manifold such that the condition $R(\xi, X).\mathcal{Z} = 0$ is satisfied. Then, M is either a cosymplectic manifold or M is an Einstein manifold.*

Proof. Let us suppose that an almost $C(\alpha)$ - manifold M satisfies the condition $R(\xi, X).\mathcal{Z} = 0$, namely

$$\mathcal{Z}(R(\xi, X)Y, Z) + \mathcal{Z}(Y, R(\xi, X)Z) = 0 \quad (3.1)$$

for any $X, Y, Z \in \Gamma(TM)$. By putting $Z = \xi$ in (3.1), we have

$$\mathcal{Z}(R(\xi, X)Y, \xi) + \mathcal{Z}(Y, R(\xi, X)\xi) = 0. \quad (3.2)$$



From (2.4), (2.12) and (3.2) one has

$$(f + 2n\alpha)\eta(R(\xi, X)Y) + \alpha\eta(X)\mathcal{L}(Y, \xi) - \alpha\mathcal{L}(Y, X) = 0. \tag{3.3}$$

Also, making use of (2.3) in (3.3) and from (2.11), (2.12) we get

$$2n\alpha^2g(X, Y) - \alpha S(Y, X) = 0.$$

Therefore, we have

$$\alpha(2n\alpha g(X, Y) - S(X, Y)) = 0.$$

which implies that either

$$\alpha = 0$$

or

$$S(X, Y) = 2n\alpha g(X, Y).$$

Thus, we get the requested result. □

Theorem 3.2. *Let M be an almost $C(\alpha)$ - manifold such that the condition $R.Q = 0$ is satisfied. Then, M is either a cosymplectic manifold or M is an Einstein manifold.*

Proof. Let us suppose that an almost $C(\alpha)$ - manifold M satisfies the condition $(R(X, Y).Q)Z = 0$, that is,

$$R(X, Y)QZ - Q(R(X, Y)Z) = 0 \tag{3.4}$$

for any $X, Y, Z \in \Gamma(TM)$, where Q stands for the Ricci operator defined by $S(X, Y) = g(QX, Y)$ and R is the Riemannian curvature tensor. Substituting $Y = \xi$ in (3.4) gives

$$R(X, \xi)QZ - Q(R(X, \xi)Z) = 0. \tag{3.5}$$

Furthermore, in view of (2.5) and (3.5) we infer

$$\alpha(\eta(QZ)X - g(X, QZ)\xi) - Q(\alpha(\eta(Z)X - g(X, Z)\xi)) = 0. \tag{3.6}$$

Taking the inner product of (3.6) with vector field W and from (2.7), (2.9) we have

$$2n\alpha^2\eta(Z)g(X, W) - \alpha S(X, Z)\eta(W) - \alpha\eta(Z)S(X, W) + 2n\alpha^2\eta(W)g(X, Z) = 0. \tag{3.7}$$

Replacing W by ξ in (3.7) and using (2.7), (2.8) we find that

$$-\alpha S(X, Z) + 2n\alpha^2g(X, Z) = 0.$$

That is,

$$\alpha(S(X, Z) - 2n\alpha g(X, Z)) = 0.$$

Hence, we find that either $\alpha = 0$ or $S(X, Z) = 2n\alpha g(X, Z)$ which completes the proof of the theorem. □

The following theorem provides an important characterization for an almost $C(\alpha)$ - manifold.

Theorem 3.3. *Let M be an almost $C(\alpha)$ - manifold such that the condition $R(\xi, X).R = 0$ is satisfied. Then, M is either a cosymplectic manifold or M is a manifold of constant curvature α .*

Proof. Let us suppose that an almost $C(\alpha)$ - manifold M satisfies the condition $(R(\xi, X).R)(Y, Z)W = 0$. Then, we write

$$R(\xi, X)R(Y, Z)W - R(R(\xi, X)Y, Z)W - R(Y, R(\xi, X)Z)W - R(Y, Z)R(\xi, X)W = 0 \tag{3.8}$$

for any $X, Y, Z, W \in \Gamma(TM)$. Taking ξ instead of Y in (3.8), we have

$$R(\xi, X)R(\xi, Z)W - R(R(\xi, X)\xi, Z)W - R(\xi, R(\xi, X)Z)W - R(\xi, Z)R(\xi, X)W = 0. \tag{3.9}$$

For the first and second term of (3.9), using (2.3) and (2.4) we deduce that

$$R(\xi, X)R(\xi, Z)W = \alpha^2g(Z, W)\eta(X)\xi - \alpha^2g(Z, X)\eta(W)\xi - \alpha^2g(Z, W)X + \alpha^2\eta(Z)\eta(W)X \tag{3.10}$$

and

$$R(R(\xi, X)\xi, Z)W = \alpha^2g(Z, W)\eta(X)\xi - \alpha^2\eta(X)\eta(W)Z - \alpha R(X, Z)W. \tag{3.11}$$

For the third and fourth term of (3.9), again using (2.3) and after some calculations, we derive

$$R(\xi, R(\xi, X)Z)W = -\alpha^2g(X, W)\eta(Z)\xi + \alpha^2\eta(Z)\eta(W)X. \tag{3.12}$$

and

$$R(\xi, Z)R(\xi, X)W = \alpha^2g(X, W)\eta(Z)\xi - \alpha^2g(Z, X)\eta(W)\xi - \alpha^2g(X, W)Z + \alpha^2\eta(X)\eta(W)Z \tag{3.13}$$

If we replace the equalities (3.10)-(3.13) in (3.9), we find that

$$-\alpha^2g(Z, W)X + \alpha R(X, Z)W + \alpha^2g(X, W)Z = 0$$

which is equivalent to

$$\alpha\{R(X, Z)W + \alpha(g(X, W)Z - g(Z, W)X)\} = 0.$$

This implies that either

$$\alpha = 0$$

or

$$R(X, Z)W = \alpha(g(Z, W)X - g(X, W)Z).$$

This result ends the proof of the theorem. □



4. Generalized Ricci Recurrent Almost $C(\alpha)$ - Manifolds

In this section, we examine generalized Ricci recurrent almost $C(\alpha)$ - manifolds.

The first result of this section is the following:

Theorem 4.1. *If an almost $C(\alpha)$ - manifold M is a generalized Ricci recurrent, then the 1-forms A and B in (2.10) are linearly dependent.*

Proof. Suppose that an almost $C(\alpha)$ - manifold M is a generalized Ricci recurrent. Then, taking $Z = \xi$ in (2.10) and using (2.7) we have

$$(\nabla_X S)(Y, \xi) = 2n\alpha\eta(Y)A(X) + \eta(Y)B(X) \tag{4.1}$$

such that

$$(\nabla_X S)(Y, \xi) = \nabla_X S(Y, \xi) - S(\nabla_X Y, \xi) - S(Y, \nabla_X \xi) \tag{4.2}$$

for any $X, Y \in \Gamma(TM)$. If we use (2.7) in (4.2), the equation (4.2) takes the form

$$(\nabla_X S)(Y, \xi) = 2n\alpha g(Y, \nabla_X \xi) - S(Y, \nabla_X \xi). \tag{4.3}$$

Therefore, by combining (4.1) and (4.3) yields

$$2n\alpha g(Y, \nabla_X \xi) - S(Y, \nabla_X \xi) = 2n\alpha\eta(Y)A(X) + \eta(Y)B(X). \tag{4.4}$$

Putting $Y = \xi$ in (4.4) and with the help of (2.2), (2.7) we obtain

$$2n\alpha A(X) + B(X) = 0$$

which implies that the 1-forms A and B in (2.10) are linearly dependent. Thus, the proof is completed. \square

The following result follows easily from Proposition 4.1.

Corollary 4.2. *If an almost $C(\alpha)$ - manifold M is a Ricci recurrent, then M is a cosymplectic manifold.*

5. \mathcal{L} -Recurrent Almost $C(\alpha)$ -Manifolds

In this section, we deal with \mathcal{L} - recurrent almost $C(\alpha)$ - manifolds.

Now, we begin to this section with the following:

Theorem 5.1. *If an almost $C(\alpha)$ - manifold M is a \mathcal{L} - recurrent, then it is either a generalized Ricci recurrent or Ricci recurrent manifold.*

Proof. Let us consider an almost $C(\alpha)$ - manifold M is a \mathcal{L} - recurrent. Then, from (2.11), (2.14) and (2.15) we have

$$\begin{aligned} \eta(X)S(Y, Z) + f\eta(X)g(Y, Z) &= \nabla_X(S(Y, Z) + fg(Y, Z)) \\ &\quad - S(\nabla_X Y, Z) \\ &\quad - fg(\nabla_X Y, Z) \\ &\quad - S(Y, \nabla_X Z) \\ &\quad - fg(Y, \nabla_X Z) \end{aligned} \tag{5.1}$$

for any $X, Y, Z \in \Gamma(TM)$. After a straightforward calculation, we get

$$(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z) + (f\eta(X) - X(f))g(Y, Z). \tag{5.2}$$

On the other hand, taking $Z = \xi$ in (5.1) and using the equalities (2.7), (2.15) one has

$$\begin{aligned} \nabla_X(f + 2n\alpha)\eta(Y) - (f + 2n\alpha)\eta(\nabla_X Y) - S(Y, \nabla_X \xi) \\ - fg(Y, \nabla_X \xi) = (f + 2n\alpha)\eta(X)\eta(Y) \end{aligned}$$

equivalently,

$$(f + 2n\alpha)\eta(X)\eta(Y) = X(f)\eta(Y) - S(Y, \nabla_X \xi) - fg(Y, \nabla_X \xi). \tag{5.3}$$

By setting ξ instead of Y in (5.3) and using (2.2), (2.7), then the equation (5.3) reduces

$$X(f) = (f + 2n\alpha)\eta(X). \tag{5.4}$$

Therefore, by combining (5.2) and (5.4) we derive

$$(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z) - 2n\alpha\eta(X)g(Y, Z). \tag{5.5}$$

If α vanishes identically in (5.5), then we write

$$(\nabla_X S)(Y, Z) = \eta(X)S(Y, Z)$$

which means that an almost $C(\alpha)$ - manifold M is a Ricci recurrent manifold. If $\alpha \neq 0$, then we can take $A(X) = \eta(X)$ and $B(X) = -2n\alpha\eta(X)$. Then, the equation (5.5) becomes

$$(\nabla_X S)(Y, Z) = A(X)S(Y, Z) + B(X)g(Y, Z)$$

which implies that an almost $C(\alpha)$ - manifold M is a generalized Ricci recurrent manifold. Thus, the proof is completed. \square

As an consequence of the Theorem 5.1, we can give the following corollary:

Corollary 5.2. *If an almost $C(0)$ - manifold M is a \mathcal{L} - recurrent, then it is also Ricci recurrent manifold.*

Corollary 5.3. *If an almost $C(\alpha)$ - manifold M ($\alpha \neq 0$) is a \mathcal{L} - recurrent, then it is also generalized Ricci recurrent manifold.*

6. Conclusion

It is well known that almost $C(\alpha)$ - manifolds can be derived from almost contact Riemannian manifolds. In this paper, we investigate almost $C(\alpha)$ - manifolds which have some certain tensor conditions. Also, we prove that if an almost $C(\alpha)$ - manifold M satisfying the condition $R(\xi, X).R = 0$, then M is either a cosymplectic manifold or M is a manifold of constant curvature α . Finally, we study generalized Ricci recurrent, \mathcal{L} - recurrent almost $C(\alpha)$ - manifolds.



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ISSN(P):2319 – 3786

Malaya Journal of Matematik

ISSN(O):2321 – 5666

