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# Some topological properties of spectrum of intuitionistic fuzzy prime submodules

P. K. Sharma[1]* and Kanchan[2]

**Abstract**

Let $R$ be a commutative ring with a non-zero identity, and let $M$ be an $R$-module. Let $IFSpec(M)$ denotes the collection of all intuitionistic fuzzy prime submodules of $M$. In this regards some basic properties of Zariski topology on $IFSpec(M)$ are investigated. In particular, we prove some equivalent conditions for irreducible subsets of this topological space and it is shown under certain conditions $IFSpec(M)$ is a $T_0$-space or Hausdorff.

**Keywords**

Intuitionistic fuzzy prime submodules, Intuitionistic fuzzy prime Spectrum, top modules, Zariski topology.

**AMS Subject Classification**

54C50, 03F55, 16D10.

[1] *Post-Graduate Department of Mathematics, DAV College, Jalandhar-144008, Punjab, India.*

[2] *Research Scholar, IKGPT University, Jalandhar-144603, Punjab, India.*

***Corresponding author**: [1] pksharma@davjalandhar.com; [2]kanchan4usoh@gmail.com

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## Contents



## 1. Introduction

In the inspection of prime spectrum or that of the Zariski topology introduced on the set of prime submodules of a unitary module $M$, over a commutative ring $R$ with non-zero identity, play a vital role in the study of algebra, geometry and lattices (for example see [6–8, 17]). There has been a consistent development in the intuitionistic fuzzy modules theory and in particular in the area of intuitionistic fuzzy prime (maximal) submodules (for example see [2–4, 9–13]). This leads us to define a suitable topological construction on the collection of all intuitionistic fuzzy prime submodules (IFPSMs) and study their topological properties. The authors in [12] imported and analyzed the concept of intuitionistic $L$-fuzzy prime submodules of an $R$-module $M$, where $R$ is a commutative ring with non-zero identity and $L$ stand for a complete lattice. In [16] they introduced and studied Zariski topology on $IF_LSpec(M)$, the collection of all intuitionistic $L$-fuzzy prime submodules of $M$, which is called intuitionistic $L$-fuzzy prime spectrum of $M$.

In the present paper we follow [11, 12, 16] and find more topological properties of Zariski topology of $X = IFSpec(M)$, the collection of all intuitionistic fuzzy prime submodules (IFPSMs) of $M$, such as irreducibility and separation properties.

In this regards, we extend the results on Zariski topology of prime submodules to intuitionistic fuzzy prime submodules, and obtain some basic results of this topological space.

In Section 2; a couple of definitions, a few results which are to be used in the sequel are given. In Section 3; the irreducible subsets of $X = IFSpec(M)$ are studied. In particular, it is shown every variety, $X(P)$ of $X$ is irreducible closed subset of $X$ for any IFPSM $P$ of $M$. Finally, in Section 4; the separation properties of $IFSpec(M)$ are investigated. In particular, by using the natural mapping some equivalent conditions that $X$ being a $T_0$ or Hausdorff are given. Finally, it is proved that $X$ is homeomorphic to the topological space $Spec(M) \times (0,1) \times (0,1)$.

## 2. Preliminaries

Through out this manuscript, $R$ is going to be a commutative ring with unity, and $M$ is a unitary $R$-module. $L$ is regular if for all $a,b \in L$ such that $a \neq 0, b \neq 0$, then $a \wedge b \neq 0$ and $a \neq 1, b \neq 1$, then $a \vee b \neq 1$. An intuitionistic $L$-fuzzy subset $A$ of a non-empty set $X$ is a function $A = (f_A, g_A) : X \to L \times L$. In the case when $L = [0,1]$, $A$ is called an intuitionistic fuzzy subset (IFS) of $X$. We refer $IFS(X)$ for the set of all intuitionistic fuzzy subsets of $X$. Let $Y \subseteq X$ and $p,q \in (0,1]$ with $p+q \leq 1$. Define $(p,q)_Y \in IFS(X)$ as:

$$(p,q)_Y(x) = \begin{cases} (p,q), & \text{if } x \in Y \\ (0,1), & \text{otherwise} \end{cases}$$

In a particular case when $Y = \{x\}$, we symbolize $(p,q)_{\{x\}}$ by $x_{(p,q)}$ and termed as an intuitionistic fuzzy point (IFP) of $X$. For $A,B \in IFS(X)$ we say $A \subseteq B$ iff $f_A(r) \leq f_B(r)$ and $g_A(r) \geq g_B(r)$ for all $r \in X$.

For $A,B \in IFS(X)$, the intersection and union, $A \cap B, A \cup B \in IFS(X)$ and are defined as

$$f_{A\cap B}(r) = f_A(r) \wedge f_B(r),\ g_{A\cap B}(r) = g_A(r) \vee g_B(r) \text{ and}$$
$$f_{A\cup B}(r) = f_A(r) \vee f_B(r),\ g_{A\cup B}(r) = g_A(r) \wedge g_B(y), \forall r \in X.$$

For better understanding of the subject under discussion, we list a few definitions and important results taken from [1, 2, 11, 13, 14], which are needed for the advancement of the present paper.

**Definition 2.1.** *([2]) Let $A \in IFS(R)$. Then $A$ is called an intuitionistic fuzzy ideal $(IFI)$ of $R$ if for all $r,s \in R$, the following holds*

*(i)* $f_A(r-s) \geq f_A(r) \wedge f_A(s)$;

*(ii)* $f_A(rs) \geq f_A(r) \vee f_A(s)$;

*(iii)* $g_A(r-s) \leq g_A(r) \vee g_A(s)$;

*(iv)*$g_A(rs) \leq g_A(r) \wedge g_A(s)$.

**Definition 2.2.** *([2, 3]) Let $A \in IFS(M)$. Then $A$ is called an intuitionistic fuzzy module $(IFM)$ of $M$ if for all $m,n \in M, r \in R$, the followings are satisfied*

*(i)* $f_A(m-n) \geq f_A(m) \wedge f_A(n)$;

*(ii)* $f_A(rm) \geq f_A(m)$;

*(iii)* $f_A(\theta) = 1$;

*(iv)* $g_A(m-n) \leq g_A(m) \vee g_A(n)$;

*(v)*$g_A(rm) \leq g_A(m)$;

*(vi)*$g_A(\theta) = 0$.

Let $IFM(M)$ $(IFI(R))$ stand for the collection of IF $R$-modules of $M$ ( resp., IF ideals of $R$). It is to be noted that when $R = M$, then $A \in IFM(M)$ iff $f_A(\theta) = 1, g_A(\theta) = 0$ and $A \in IFI(R)$. The trivial IF $R$-modules of $M$ (resp., IF ideals of $R$) are denoted by $\chi_{\{\theta\}}$, $\chi_M$ ( resp., $\chi_{\{0\}}$, $\chi_R$). Further if $A \in IFM(M)$, then the set $A_* = \{m \in M | f_A(m) = f_A(\theta)$ and $g_A(m) = g_A(\theta)\}$ is a submodule of $M$.

**Lemma 2.3.** *([11, 13]) Let $C \in IFI(R), A,B \in IFM(M)$. Then:*

*(i) $CB \subseteq A$ iff $C \circ B \subseteq A$.*

*(ii) If $r_{(s,t)} \in IFP(R), x_{(p,q)} \in IFP(M)$. Then $r_{(s,t)} \circ x_{(p,q)} = (rx)_{(s\wedge p, t\vee q)}$.*

*iii If $f_C(0) = 1, g_C(0) = 0$ then $CA \in IFM(M)$.*

*(iv) Let $r_{(s,t)} \in IFP(R)$. Then for all $m \in M$,*

$$f_{r_{(s,t)}\circ B}(m) = \begin{cases} Sup[s \wedge f_B(x)] & \text{if } m = rx, r \in R, x \in M \\ 0, & \text{if } m \text{ is not expressible as } m = rx \end{cases} \text{ and}$$

$$g_{r_{(s,t)}\circ B}(m) = \begin{cases} Inf[t \vee g_B(x)] & \text{if } m = rx, r \in R, x \in M \\ 1, & \text{if } m \text{ is not expressible as } m = rx. \end{cases}$$

**Definition 2.4.** *([1, 14]) $A \in IFI(R)$ is termed as IF prime ideal (IFPI) of $R$ if $A \neq \chi_{\{0\}}, \chi_R$ and for any $B,C \in IFI(R)$ so that $BC \subseteq A$ implies $B \subseteq A$ or $C \subseteq A$.*

$IFSpec(R)$ denotes the set of all IFPIs of $R$.

**Definition 2.5.** *([11, 13]) For $G,H \in IFS(M)$ and $I \in IFS(R)$, define the residual quotient $(G:H)$ and $(G:I)$ as follows: $(G:H) = \bigcup\{J : J \in ILFS(R)$ such that $J \cdot H \subseteq G\}$ and $(G:I) = \bigcup\{K : K \in ILFS(M)$ such that $I \cdot K \subseteq G\}$.*

In [13] it was proved that if $G \in IFM(M)$, $H \in IFS(M)$, $I \in IFS(R)$ then $(G:H) \in IFI(R)$ and $(G:I) \in IFM(M)$.

**Theorem 2.6.** *([11, 13]) If $G,H \in IFS(M), I \in IFS(R)$. Then*

*(i) $(G:H) \cdot H \subseteq G$ ;*

*(ii) $I \cdot (G:I) \subseteq G$ ;*

*(iii) $I \cdot H \subseteq G$ iff $I \subseteq (G:H)$ iff $H \subseteq (G:I)$.*

**Theorem 2.7.** *([11]) (a) Suppose $N$ is a prime submodule of $M$ and $\alpha, \beta \in (0,1)$ such that $\alpha + \beta < 1$. If $A$ is an IFS of $M$ defined by*

$$f_A(m) = \begin{cases} 1, & \text{if } m \in N \\ \alpha, & \text{if otherwise} \end{cases}; \quad g_A(m) = \begin{cases} 0, & \text{if } m \in N \\ \beta, & \text{otherwise.} \end{cases}$$

*for every $m \in M$. Then $A$ is an IF prime submodule (IFPSM) of $M$.*

*(b)Conversely, any IF prime submodule can be obtained as in (a).*

**Corollary 2.8.** *([13]) If $A \in IFSpec(M)$, then $(A:\chi_M) \in IFSpec(R)$.*

Let $X = IFSpec(M)$ and for any $A \in IFS(M)$, denote the set $V(A) = \{P \in X : A \subseteq P\}$ and $X(A) = \{P \in X : (A:\chi_M) \subseteq (P:\chi_M)\}$ and if $B \in IFS(M)$, by $X(B)$ we mean $X(<B>)$.

Now, we put $\mathscr{C}^*(M) = \{V(A) | A \in IFM(M)\}$; $\mathscr{C}'(M) = \{V(C.\chi_M) | C \in IFI(R)\}$; $\mathscr{C}(M) = \{X(A) | A \in IFM(M)\}$.

Here we analysed three different topologies of $X$ induced by these three sets. In [16], it is shown that there exists a topology $\tau^*$ say, on $X$ that have the family $\mathscr{C}^*(M)$ of closed sets iff $\mathscr{C}^*(M)$ is closed under finite union. In this situation, the topology $\tau^*$ is called quasi-Zariski topology on $X$. As in [16], a module $M$ is termed as IF top module, if $\mathscr{C}^*(M)$ induces the topology $\tau^*$ on $X$.

For $p \in IFSpec(R)$, we symbolize the set $X_p = \{A \in X : (A : \chi_M) = p\}$.

If $A \in IFSpec(M)$, then by corollary (2.8) $(A : \chi_M) \in IFSpec(R)$. Define $\overline{(A : \chi_M)} \in ILFS((R/Ann(M)))$ as follows:

$\overline{(A : \chi_M)}[x] = (f_{\overline{(A:\chi_M)}}([x]), g_{\overline{(A:\chi_M)}}([x]))$, where

$f_{\overline{(A:\chi_M)}}([x]) = \vee\{f_{(A:\chi_M)}(z) | z \in [x]\}$ and
$g_{\overline{(A:\chi_M)}}([x]) = \wedge\{g_{(A:\chi_M)}(z) | z \in [x]\}$.

The map $\varphi : IFSpec(M) \to IFSpec(R/Ann(M))$ defined by

$$\varphi(A) = \overline{(A : \chi_M)}, \text{ for } A \in IFSpec(M)$$

is called the natural map.

In [16], it has been proved that the collection $\mathbb{B} = \{D(x_{(\alpha,\beta)}.\chi_M) | x \in R, \alpha, \beta \in (0,1] \text{ with } \alpha+\beta \leq 1\}$, where that $D(x_{(\alpha,\beta)}.\chi_M) = X \backslash X(x_{(\alpha,\beta)}.\chi_M)$ forms a base for the Zariski topology on $X$.

## 3. Irreducible subsets of $IFSpec(M)$

In the sequel we assume that $M$ is an $R$-module and $X = IFSpec(M)$. For $Y \subseteq X$ we write $\Gamma(Y) = \bigcap_{P \in Y} P$ and $\overline{Y}$ = closure of $Y$ with regard to topology on $X$.

**Lemma 3.1.** *If $A \in IFI(R)$, then $A$ is contained in some intuitionistic fuzzy maximal ideal.*

*Proof.* Let $A \in IFI(R)$. Take $A_* = \{r \in R : f_A(r) = 1, g_A(r) = 0\}$. Since $A_*$ is an ideal of $R$, so there exist a maximal ideal $S$ of $R$ so that $A_* \subseteq S$. Define $B \in IFS(R)$ such that

$$f_B(r) = \begin{cases} 1, & \text{if } r \in S \\ \alpha, & \text{if otherwise} \end{cases}; \quad g_B(r) = \begin{cases} 0, & \text{if } r \in S \\ \beta, & \text{if otherwise.} \end{cases}$$

where $\alpha = sup\{f_A(r) : r \in R\}$ and $\beta = inf\{g_A(r) : r \in R\}$. Clearly, $B$ is an IF maximal ideal (IFMI) of $R$ such that $A \subseteq B$. In other words $\exists$ a IFMI $B$ of $R$ such that $A \subseteq B$. □

**Proposition 3.2.** *Let $B$ be a IFMI of $R$. Then $B.\chi_M$ is an IFPSM of $M$*

*Proof.* Let $B$ be an IFMI of $R$, then $B_*$ is the maximal ideal of $R$.

$$f_B(r) = \begin{cases} 1, & \text{if } r \in B_* \\ \alpha, & \text{if otherwise} \end{cases}; \quad g_B(r) = \begin{cases} 0, & \text{if } r \in B_* \\ \beta, & \text{if otherwise.} \end{cases}$$

where $\alpha, \beta \in (0,1)$ so that $\alpha + \beta < 1$. Since $B_*$ is the maximal ideal of $R$ therefore $B_*M$ is a prime submodule of $M$. Hence by Theorem (2.5), $B.\chi_M$ is an IFPSM of $M$. □

**Definition 3.3.** *An $A \in IFM(M)$ is termed as an IF maximal prime submodule (IFMPSM) of $M$ if $A \in IFSpec(M)$ and there does not exist any $B \in IFSpec(M)$ which contains $A$ properly.*

**Lemma 3.4.** *If $A \in IFSpec(M)$ is maximal prime, then $(A : \chi_M)$ is a IFMI of $R$.*

*Proof.* Let $A \in IFSpec(M)$ is maximal prime. Suppose $C \in IFI(R)$ be such that

$$(A : \chi_M) \subseteq C \tag{3.1}$$

Then from lemma(3.2) $\exists$ a IFMI $B$ of $R$ such that $C \subseteq B$. Since $(A : \chi_M) \subseteq C$, then $A \subseteq C.\chi_M \subseteq B.\chi_M$ also from proposition (3.3) we get, $B.\chi_M$ is an IFPSM of $M$. Therefore we have $A = B.\chi_M$. Since $A$ is maximal prime and so $A = C.\chi_M$. Thus

$$C \subseteq (A : \chi_M) \tag{3.2}$$

Now by equations (1) and (2) we have $(A : \chi_M) = C$ and thus $(A : \chi_M)$ is a IFMI of $R$. □

**Proposition 3.5.** *For any element $P$ of $X$, the subsequent affirmation are satisfied:*

1. *$\overline{\{P\}} = X(P)$;*
2. *For any $Q \in X$, $Q \in \overline{\{P\}}$ iff $(P : \chi_M) \subseteq (Q : \chi_M)$ if and only if $X(Q) \subseteq X(P)$;*
3. *The set $\{P\}$ is closed iff*
   *(a) $P$ is IFMPSM of $M$;*
   *(b) $X_p = \{P\}$, such that $(P : \chi_M) = p$.*

*Proof.* (1) It is an immediate consequences of proposition (3.1)

(2) Follows from (1)

(3) Let $\{P\}$ be a closed set. Then $\{P\} = \overline{\{P\}} = X(P)$. Suppose that $A \in IFSpec(M)$ and $P \subseteq A$, then $(P : \chi_M) \subseteq (A : \chi_M)$, and hence $A \in X(P) = \{P\}$. Thus $A = P$. This means that $P$ is an IFMPSM of $M$.

Now suppose that $A \in X_p$, then $(A : \chi_M) = p = (P : \chi_M)$. So $A \in X(P) = \{P\}$, and hence $A = P$.

Conversely, suppose that $(a)$ and $(b)$ are satisfied. Let $A \in X(P)$, then $(P : \chi_M) \subseteq (A : \chi_M)$. Since $P$ is maximal prime, then by lemma (3.5) it is concluded that $(P : \chi_M) = p$ is a IFMPI of $R$. Then $p = (P : \chi_M) = (A : \chi_M)$. This means that $A \in X_p = \{P\}$. Thus $A = P$., and hence $X(P) = \{P\}$. But $\overline{\{P\}} = X(P) = \{P\}$. It means that $\{P\}$ is closed. □

**Remark 3.6.** *From the last proposition, we conclude that the space $X$ is a $T_1$ space iff every IFPSM of $M$ is maximal prime and $|X_p| \leq 1$ for every $p \in IFSpec(R)$.*

Further, recall that if $A_1$ and $A_2$ be any closed subsets of a space $A$ such that $A = A_1 \cup A_2$, then the space $A$ is said to be irreducible if either $A = A_1$ or $A = A_2$. Also the subspace $A_0$ of $A$ is irreducible if it is irreducible as a subspace of $A$.

In a topological space A, an irreducible component of $A$ is a maximal irreducible subset of $A$.

**Theorem 3.7.** *For any IFPSM $P$ of $M$, the closed set $X(P)$ is an irreducible set in $X$.*

*Proof.* By Proposition (3.6)(i), $X(P) = \overline{\{P\}}$. Let $X(P) = A_1 \cup A_2$ for closed sets $A_1$ and $A_2$, so $\overline{\{P\}} = A_1 \cup A_2$. But $P \in \overline{\{P\}}$, then $P \in A_1$ or $P \in A_2$. Let $P \in A_1$ then $P \in A_1 \in \overline{\{P\}}$, which is a contradiction. Therefore we must have $A_1 = \overline{\{P\}}$ and this mean that $X(P)$ is irreducible. □

**Corollary 3.8.** *Let $Y \subseteq X$. If $\Gamma(Y)$ is a IFPSM of $M$, then $Y$ is irreducible.*

*Proof.* Let $\Gamma(Y) = P$ be a IFPSM of $M$. By proposition (3.1) $X(P) = X(\Gamma(Y)) = \overline{Y}$ is irreducible. Let

$$Y = A_1 \cup A_2 \tag{3.3}$$

for closed subsets $A_1$ and $A_2$. Then $\overline{Y} = \overline{A_1 \cup A_2} = \overline{A_1} \cup \overline{A_2} = A_1 \cup A_2$. Since $\overline{Y}$ is irreducible. then $\overline{Y} = A_1$ or $\overline{Y} = A_2$. Without loss of generality suppose that $\overline{Y} = A_1$. Then $Y \subseteq A_1$ and equation (3) implies that $A_1 \subseteq Y$ and hence $Y = A_1$. This means that $Y$ is irreducible. □

**Corollary 3.9.** *Let $P^* = \bigcap_{P \in X} P$. If $P^*$ is a IFPSM of $M$, then $X$ is irreducible.*

*Proof.* Immediately follows from Corollary (3.8) □

**Corollary 3.10.** *For an R-module $M$ the following holds:*

1. *If $Y = \{P_i : i \in J\}$ is linearly ordered by the set inclusion, then $Y$ is irreducible in $X$;*
2. *$X_p$ is irreducible for $p \in IFSpec(R)$;*
3. *If $p$ is a IFMPI of $R$, then $X_p$ is an irreducible closed subset of $X$.*

*Proof.* For (1) As the members of $Y$ are linearly ordered by the set inclusion, $\Gamma(Y)$ is a IFPSM of $M$. So by corollary (3.9), $Y$ is irreducible.

For (2) We prove that $\Gamma(X_p)$ is a IFPSM of $M$. For this we have

$$(\bigcap_{P \in X} P : \chi_M) = \bigcap_{P \in X} (P : \chi_M) = p$$

Thus $\Gamma(X_p : \chi_M) = p$.

Now suppose that $C \in IFI(R)$, $B \in IFM(M)$ and $C.B \subseteq \Gamma(X_p)$ such that $B \not\subseteq \Gamma(X_p)$. So there exists $P' \in X_p$ such that $B \not\subseteq P'$. Therefore $C \subseteq (P' : \chi_M) = p = (\Gamma(X_p) : \chi_M)$. This means that $\Gamma(X_p)$ is a IFPSM. Then $X_p$ is irreducible by Corollary (3.9).

For (3) Suppose that $p$ is IFMI of $R$. By (2) $X_p$ is irreducible. But because $p$ is maximal, then $((p.\chi_M) : \chi_M) = p$. Also, for $Q \in X(p.\chi_M)$, we have $p = ((p.\chi_M) : \chi_M) \subseteq (Q : \chi_M)$ and since $p$ is maximal then $(Q : \chi_M) = p \Rightarrow Q \in X_p$ implies

$$X(p.\chi_M) \subseteq X_p \tag{3.4}$$

but for $P \in X_p$ it is concluded that $(P : \chi_M) = p = ((p.\chi_M) : \chi_M)$. Thus $P \in X(p.\chi_M)$ implies

$$X_p \subseteq X(p.\chi_M) \tag{3.5}$$

From equations (4) and (5) we obtain $X(p.\chi_M) = X_p$. Therefore $X_p$ is closed as desired. □

**Corollary 3.11.** *Let $Y \subseteq X$ and $(\Gamma(Y) : \chi_M) = p$ be a IFPI of $R$. If $X_p \neq \emptyset$ then $Y$ is irreducible.*

*Proof.* Let $P \in X_p$. Then $(P : \chi_M) = (\Gamma(Y) : \chi_M) = p$. Then $X(\Gamma(Y)) = X(P)$, by proposition (3.3) of [16]. But by proposition (3.1) we have $X(\Gamma(Y)) = \overline{Y}$ and hence $X(P) = \overline{Y}$. Then by Theorem (3.8), $X(P)$ is irreducible. Therefore $\overline{Y}$ and hence $Y$ is irreducible. □

## 4. Separation Properties of IFSpec(M)

**Theorem 4.1.** *For $X$ the subsequent affirmation are adaptable:*

1. *$X$ is $T_0$ space;*
2. *the natural map $\varphi : IFSpec(M) \to IFSpec(R/Ann(M))$ is one-one;*
3. *if $X(P) = X(Q) \Rightarrow P = Q$ for every $P, Q \in X$;*
4. *$|X_p| \leq 1$ for all $p \in IFSpec(R)$.*

*Proof.* By Proposition (5.4) of [16] (2),(3) and (4) are equivalent. Only it reminds to prove $(1) \Leftrightarrow (3)$. It is well-known that a topological space is $T_0$ if and only if closures of distinct points are distinct. Now suppose that $X$ is $T_0$ space and let $X(P) = X(Q)$ for $P, Q \in X$. If $P \neq Q$, then we have $\overline{\{P\}} \neq \overline{\{Q\}}$, but by proposition (3.6)(i) we have $X(P) \neq X(Q)$, a contradiction. Thus $P = Q$.

For the converse, we take $P, Q \in X$ such that $P \neq Q$. By (2) $X(P) \neq X(Q)$, again by proposition (3.6)(i) $\overline{\{P\}} \neq \overline{\{Q\}}$. This means that $X$ is $T_0$ space. □

**Corollary 4.2.** *If $M$ is an intuitionistic fuzzy top module, then $X$ is a $T_0$ space for the Zariski topology $\tau^*$.*

*Proof.* Suppose $P,Q \in X$ such that $P \neq Q$. Then either $P \nsubseteq Q$ or $Q \nsubseteq P$. Suppose $P \nsubseteq Q$, then $Q \notin V(P)$ and $Q \in X \backslash X(P)$, i.e., $Q \in D(P)$ but $P \notin D(P)$ and $D(P)$ is an open set with regard to the topology $\tau^*$. Then from $\tau \subseteq \tau^*$, it concluded that $X$ is $T_0$ space. □

Let $\mathbb{C} = \{p = (P:\chi_M) | P \in IFSpec(M))\}$ and $\mathbb{C}^* = \{p_* | p \in \mathbb{C}\}$

**Lemma 4.3.** $D(x_{(\alpha,\beta)} \cdot \chi_M) = \emptyset$ *if and only if* $x \in \bigcap_{p\in\mathbb{C}}\{p_*\}$.

*Proof.* Let $D(x_{(\alpha,\beta)} \cdot \chi_M) = \emptyset$, then $X(x_{(\alpha,\beta)} \cdot \chi_M) = X$. Suppose $N$ is a prime submodule of $M$ and set $A = \chi_N$. Then $A \in IFSpec(M)$. Let $p = (A:\chi_M)$. Then $((x_{(\alpha,\beta)} \cdot \chi_M) : \chi_M) \subseteq (A:\chi_M) = p$, but $x_{(\alpha,\beta)} \subseteq ((x_{(\alpha,\beta)} \cdot \chi_M) : \chi_M)$, and hence $x_{(\alpha,\beta)} \subseteq p$. Thus $\alpha \leq f_p(x) = 1$, $\beta \geq g_p(x) = 0$ implies that $x \in p_*$ and so $x \in \bigcap_{p\in\mathbb{C}}\{p_*\}$.

Conversely, suppose that $x \in \bigcap_{p\in\mathbb{C}}\{p_*\}$ and $P \in X$. If $p = (P:\chi_M)$, then $x \in p_*$ so $f_p(x) = 1$ and $g_p(x) = 0$
$\Rightarrow f_{(P:\chi_M)}(x) = 1$ and $g_{(P:\chi_M)}(x) = 0$
$\Rightarrow x_{(\alpha,\beta)} \subseteq (P:\chi_M)$
$\Rightarrow x_{(\alpha,\beta)} \cdot \chi_M \subseteq P$
$\Rightarrow (x_{(\alpha,\beta)} \cdot \chi_M : \chi_M) \subseteq (P:\chi_M)$.
Therefore $P \in X(x_{(\alpha,\beta)}).\chi_M)$ and hence $X(x_{(\alpha,\beta)} = X$. Thus $D(x_{(\alpha,\beta)}).\chi_M) = \emptyset$. □

Let $X = IFSpec(M)$ and $\alpha,\beta \in (0,1)$ such that $\alpha+\beta < 1$. We denote the subspace $\{A \in X | Im(A) = \{(1,0),(\alpha,\beta)\}\}$ of $X$ by $X^{(\alpha,\beta)}$.

**Lemma 4.4.** *The subspace $X^{(\alpha,\beta)}$ of $X$ is housdorff when the natural map $\varphi$ is one-one and all the prime ideals of $R$ are maximal ideals.*

*Proof.* Let $A,B \in X^{(\alpha,\beta)}$ be any two distinct elements of $X^{(\alpha,\beta)}$. Then

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A_* \\ \alpha, & \text{if otherwise} \end{cases}; \quad g_A(x) = \begin{cases} 0, & \text{if } x \in A_* \\ \beta, & \text{if otherwise.} \end{cases}$$

and

$$f_B(x) = \begin{cases} 1, & \text{if } x \in B_* \\ \alpha, & \text{if otherwise} \end{cases}; \quad g_B(x) = \begin{cases} 0, & \text{if } x \in B_* \\ \beta, & \text{if otherwise.} \end{cases}$$

Then $A_*$ and $\underline{B_* \text{ are prime submodules}}$ of $M$. Since $\varphi$ is injective then $\overline{(A:\chi_M)} \neq \overline{(B:\chi_M)}$ and then $(A:\chi_M) \neq (B:\chi_M)$. But

$$f_{(A:\chi_M)}(x) = \begin{cases} 1, & \text{if } x \in (A_*:M) \\ \alpha, & \text{if otherwise} \end{cases}$$

$$g_{(A:\chi_M)}(x) = \begin{cases} 0, & \text{if } x \in (A_*:M) \\ \beta, & \text{if otherwise.} \end{cases}$$

and

$$f_{(B:\chi_M)}(x) = \begin{cases} 1, & \text{if } x \in (B_*:M) \\ \alpha, & \text{if otherwise} \end{cases}$$

$$g_{(B:\chi_M)}(x) = \begin{cases} 0, & \text{if } x \in (B_*:M) \\ \beta, & \text{if otherwise.} \end{cases}$$

Therefore, there exists $x \in R$ such that $x \in (A_*:M)$ but $x \notin (B_*:M)$. So we have $f_{(A:\chi_M)}(x) = 1, g_{(A:\chi_M)}(x) = 0$ but $f_{(B:\chi_M)}(x) = \alpha, g_{(B:\chi_M)}(x) = \beta$. Let $\gamma,\delta \in (0,1)$ such that $\alpha < \gamma < 1$ and $0 < \delta < \beta$. Then
$x_{(\gamma,\delta)} = ((x_{(\gamma,\delta)} \cdot \chi_M) : \chi_M) \nsubseteq (B:\chi_M)$.
Thus $B \notin X(x_{(\gamma,\delta)} \cdot \chi_M) \Rightarrow B \in D(x_{(\gamma,\delta)} \cdot \chi_M)$. Since $(B_*:M)$ is a prime ideal of $R$ and $x \notin (B_*:M)$ then $x$ is not nilpotent element of $R$ and hence $< x + C >$ is idempotent, where $C$ is the nilradical of $R$.
Thus $\exists$s $a \in R$ such that $x(1-ax) \in C$ and hence $x(1-ax)$ is nilpotent.

Fix $x \in (A_*:M)$ by hypothesis and the fact that $(A_*:M)$ is prime we have $(A_*:M)$ is maximal and so $(1-ax) \notin (A_*:M)$. Thus $f_{(A_*:M)}((1-ax)) = \gamma$ and $g_{(A_*:M)}((1-ax)) = \delta$.

But $(1-ax)_{(\alpha,\beta)} = (((1-ax)_{(\gamma,\delta)} \cdot \chi_M) : \chi_M) \nsubseteq (A:\chi_M)$. Therefore
$A \notin X((1-ax)_{(\gamma,\delta)} \cdot \chi_M)) \Rightarrow A \in D((1-ax)_{(\gamma,\delta)} \cdot \chi_M))$. On the other hands, we have $D(x_{(\gamma,\delta)} \cdot \chi_M) \cap D((1-ax)_{(\gamma,\delta)} \cdot \chi_M) = D((x(1-ax))_{(\gamma,\delta)} \cdot \chi_M))$. By proposition (5.4) of []. Also $x(1-ax)$ is nilpotent, then by lemma (3.2),
$D((x(1-ax))_{(\gamma,\delta)} \cdot \chi_M)) = \emptyset$ that is $X^{(\alpha,\beta)}$ is Hausdorff. □

**Example 4.5.** *Let $M$ be an arbitrary $R$-module and let $N$ be any prime submodule of $M$. Consider the IFPSMs $A$ and $B$ of $M$ as follows:*

$$f_A(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha_1, & \text{if otherwise} \end{cases}; \quad g_A(x) = \begin{cases} 0, & \text{if } x \in N \\ \beta_1, & \text{if otherwise.} \end{cases}$$

*and*

$$f_B(x) = \begin{cases} 1, & \text{if } x \in N \\ \alpha_2, & \text{if otherwise} \end{cases}; \quad g_B(x) = \begin{cases} 0, & \text{if } x \in N \\ \beta_2, & \text{if otherwise.} \end{cases}$$

*where $\alpha_i,\beta_i \in (0,1)$ such that $\alpha_i + \beta_i < 1, \forall i = 1,2$.*

*Let $D(x_{(\alpha_1,\beta_1)} \cdot \chi_M)$ and $D(y_{(\alpha_1,\beta_1)} \cdot \chi_M)$ be two basic open sets such that $A \in D(x_{(\alpha_1,\beta_1)} \cdot \chi_M)$ and $B \in D(y_{(\alpha_1,\beta_1)} \cdot \chi_M)$ so $x_{(\alpha_1,\beta_1)} = ((x_{(\alpha_1,\beta_1)} \cdot \chi_M) : \chi_M) \nsubseteq (A:\chi_M)$ and $y_{(\alpha_2,\beta_2)} = ((y_{(\alpha_2,\beta_2)} \cdot \chi_M) : \chi_M) \nsubseteq (B:\chi_M)$. Now,*

$$f_{(A:\chi_M)}(x) = \begin{cases} 1, & \text{if } x \in (P:M) \\ \alpha_1, & \text{if otherwise} \end{cases}; \quad g_{(A:\chi_M)}(x) = \begin{cases} 0, & \text{if } x \in (P:M) \\ \beta_1, & \text{if otherwise.} \end{cases}$$

*But $(P:M)$ is a prime ideal of $R$, therefore $xy \notin (P:M)$, and hence $xy \notin \mathbb{C}^*$. Thus $D((xy)_{(\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)} \cdot \chi_M) \neq \emptyset$ and we obtained that*
$D(y_{(\alpha_1,\beta_1)} \cdot \chi_M) \cap D(y_{(\alpha_1,\beta_1)} \cdot \chi_M) = D((xy)_{(\alpha_1\wedge\alpha_2,\beta_1\vee\beta_2)} \cdot \chi_M)$. *This shows that $X$ is not Hausdorff.*

**Proposition 4.6.** *The subspace $X^{(\alpha,\beta)}$ of $X$ is homeomorphic to $Spec(M)$.*

*Proof.* Define the mapping $\phi : X^{(\alpha,\beta)} \to Spec(M)$ by $\phi(A) = A_*$ , $\forall A \in X^{(\alpha,\beta)}$.

Let $D_r$ be the basic opens set in $Spec(M)$. Then

$$D_r = X\backslash X(rM) = \{P \in X : rM \nsubseteq P\} = \{P \in X | ry \notin P \text{ for some } y \in M\}$$

and hence $D(r_1 : \chi_M) \cap X^{(\alpha,\beta)} = \{A \in X | f_A(x) = \alpha, g_A(x) = \beta,$ for some $x, y \in M$ such that $ry = x\}$. Thus

$\phi^{-1}(D_r) = D(r_1 : \chi_M) \cap X^{(\alpha,\beta)}$ and since $D(r_1 : \chi_M) \cap X^{(\alpha,\beta)}$ is an open set in $X^{(\alpha,\beta)}$, then $\phi$ is continuous.

Now we define the map $\eta : Spec(M) \to X^{(\alpha,\beta)}$ as follows:

$$f_{\eta(P)}(x) = \begin{cases} 1, & \text{if } x \in P \\ \alpha, & \text{if } x \notin P \end{cases}; \quad g_{\eta(P)}(x) = \begin{cases} 0, & \text{if } x \in P \\ \beta, & \text{if } x \notin P. \end{cases}$$

Suppose that $D(r_{(\alpha,\beta)} \cdot \chi_M) \cap X^{(\alpha,\beta)}$ is a basic open set in $X^{(\alpha,\beta)}$. Then

$$D(r_{(\alpha,\beta)} \cdot \chi_M) \cap X^{(\alpha,\beta)} = \{A \in X | f_A(x) = \alpha, g_A(x) = \beta, \text{ for some } x, y \in M \text{ such that } ry = x\}.$$

It is easy to verify that $\eta^{-1}(D(r_{(\alpha,\beta)} \cdot \chi_M) \cap X^{(\alpha,\beta)}) = D_r$ and since $D_r$ is an open set in $Spec(M)$, then $\eta$ is continuous. Clearly $\phi$ and $\eta$ are inverse of each other. Then $X^{(\alpha,\beta)}$ and $Spec(M)$ are homeomorphic. □

**Proposition 4.7.** *The spectrum $IFSpec(M)$ is homeomorphic to the space $Spec(M) \times (0,1) \times (0,1)$.*

*Proof.* Define the mapping $\phi : IFSpec(M) \to Spec(M) \times (0,1) \times (0,1)$ as follows:
Let $A \in IFSpec(M)$ such that $Im(A) = \{(1,0), (\alpha, \beta)\}$, then $\phi(A) = (A_*, \alpha, \beta)$.

Suppose that $D_r \times (0,\alpha) \times (\beta, 1)$ is a basic open set in $Spec(M) \times (0,1) \times (0,1)$. Then

$$\begin{aligned} & \phi^{-1}(D_r \times (0,1) \times (0,1)) \\ = & \{A \in X | f_A(x) \in (0,\alpha), g_A(x) \in (\beta,1) \text{ such that } x = ry \text{ for some } x, y \in M\} \\ = & \bigcup\{D(r_{(\gamma,\delta)} \cdot \chi_M), \gamma \in (0,\alpha), \delta \in (\beta,1) \text{ such that } \gamma + \delta \leq 1\}. \end{aligned}$$

which is an open set in $IFSpec(M)$. So $\phi$ is continuous.

Now we define a map $\kappa : Spec(M) \times (0,1) \times (0,1) \to IFSpec(M)$ as follows
for $(P, \alpha, \beta) \in Spec(M) \times (0,1) \times (0,1)$;

$$\kappa((P,\alpha,\beta)) = \begin{cases} (1,0) & \text{if } x \in P \\ (\alpha,\beta) & \text{if } x \notin P \end{cases}$$

Let $D(r_{(\gamma,\delta)} \cdot \chi_M)$ be a basic open set in $IFSpec(M)$ then we can show that $\kappa^{-1}(D(r_{(\gamma,\delta)} \cdot \chi_M)) = D_r \times (0,1) \times (0,1)$ which is an open set in $Spec(M) \times (0,1) \times (0,1)$ so $\kappa$ is continuous. Thus both $\phi$ and $\kappa$ are inverses of each other. Then $IFSpec(M)$ is homeomorphic to $Spec(M) \times (0,1) \times (0,1)$. □

## 5. Conclusion

We have constituted a topology on the collection of all IF-PSMs of an $R$-module $M$, where $R$ is a commutative ring with unity, which is known as Zariski topology, and then the basic topological properties of this space has been investigated. In this regard by finding many results it has been shown that this topological spaces is enough rich in the view point of topological properties. Also, we have tried in this paper to bring the first stones of intuitionistic fuzzy spectral theory based on intuitionistic fuzzy prime submodules, and hence we hope that this paper encourage researchers in the field of intuitionistic fuzzy algebra and intuitionistic fuzzy topology to continue this way for finding further and deep results.

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