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The open hub number of a graph

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Abstract

Let $G=(V,E)$ be a connected graph. A subset H of V is called a hub set of G if for any two distinct vertices $u,v \in V-H$, there exists a u-v path P in G such that all the internal vertices of P are in H .A hub set H of V is called an open hub set if the induced sub graph $<H>$ has no isolated vertices.The minimum cardinality of an open hub set of G is called the open hub number of G and is denoted by $h_O(G)$. In this paper, we present several basic results on the open hub number.

Keywords

Open hub set, Open hub number.

AMS Subject Classification

05C40, 05C99.

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Contents

1. Introduction

By a graph $G=(V,E)$ we mean a finite ordered graph with no loops and no multiple edges.For graph theoretic terminology we refer [1]. Let $G=(V,E)$ be a connected graph.The concept of hub set is introduced by M walsh[3]. A subset H of V is called a hub set of G if for any two distinct vertices $u,v \in V-H$, there exists a u-v path P in G such that all the internal vertices of P are in H. The minimum cardinality of a hub set of G is called the hub number of G and is denoted by $h(G)$. The size of a smallest connected hub set is denoted by $h_c(G)$. A dominating set of a graph G is a sub set D of V such that every vertex not in D is adjacent to at least one member of D. The domination number $\gamma(G)$ is the number of vertices in a smallest dominating set for G. A dominating set D is said to be a connected dominating set if the subgraph $<D>$ induced by D is connected in G. The minimum of the cardinalities of the connected dominating sets of G is called the connected domination number and is denoted by $\gamma_c(G)$. In this paper we introduce the Open hub number of a graph G. A hub set H of V is called an open hub set if the induced subgraph $<H>$ has no isolated vertices. The minimum cardinality of an open hub set of G is called the open hub number of G and is denoted by $h_O(G)$. Since an open hub set has at least two elements we have $h_O(G) \geq 2$.

We use the following results to prove our main results.

Lemma 1.1. *[3] For any graph G, $\gamma(G) \leq h(G)+1$.*

Lemma 1.2. *[3] Let d(G) denote the diameter of G. Then $h(G) \geq d(G)-1$, and the inequality is sharp.*

Theorem 1.3. *[3] If C_n is the cycle with n vertices then, $h(C_n)=n-3$.*

Theorem 1.4. *[3] If P_n is the path with n vertices then, $h(P_n)=n-2$.*

Theorem 1.5. *[4] If G is a connected graph and $n \geq 3$, then $\gamma_c(G)=n-\varepsilon_T(G) \leq n-2$. ,where $\varepsilon_T(G)$ is the maximum number of pendent vertices of underlying spanning tree of G.*

Theorem 1.6. *[5] For any connected graph G, $h(G) \leq h_c(G) \leq \gamma_c(G) \leq h(G)+1$*

2. Main Results

Theorem 2.1. *For every connected graph G , $h(G) \leq h_O(G) \leq 2h(G)$.*

Proof. Since every open hub set is a hub set we have $h(G) \leq h_O(G)$. Let $H=\{v_1,v_2,....,v_k\}$ be a minimum hub set of G. Since G has no isolated vertices ,each open neighborhood $N(v_i) \neq \phi$.Let $u_i \in N(v_i), 1 \leq i \leq k$ and let $H' = \{u_1, u_2, ..., u_k\}$. Therefore the set $H \cup H'$ is an open hub set of G. Hence $h_O(G) \leq |H \cup H'| \leq 2|H| = 2h(G)$. □

Proposition 2.2. *For a connected graph G, if $h_c(G) \geq 2$ then,every connected hub set is an open hub set .*

Proof. Suppose $h_c(G) \geq 2$ and assume H is a connected hub set of G.Then $< H >$ is connected and contains more than one vertex.Hence H is an open hub set. □

Proposition 2.3. *For any connected graph G ,if $\Delta(G) = n-1$, then $h_O(G) = 2$.*

Proof. Suppose $\Delta(G) = n-1$. Let u be a vertex of G having degree n-1.Then $\{u, v\}$ where $v \in N(u)$,the neighborhood of u, forms an open hub set of G. Hence the result. □

Corollary 2.4. $h_O(W_n) = 2$

Proof. The result is obvious from above proposition since $\Delta(W_n) = n-1$. □

Now we characterise a class of graphs having open hub number $n-3$

Theorem 2.5. *Suppose G is a connected graph of order n such that $\Delta(G) \neq n-1$,then $h_O(G) = n-3$ if and only if G isomorphic to one of the following graphs*

1. *The cycle C_n*
2. *A subdivision of $K_{1,3}$*
3. *C_k with a path attached for any k.*
4. *C_3 with two paths attached*
5. *C_3 with three paths attached*
6. *A graph with exactly two cycles C_3 and C_k for any $k \geq 4$, with one edge common ,if G has no pendent vertices.*
7. *A graph with exactly two cycles C_3 and C_k for any $k \geq 3$, with one edge common and a path attached to a vertex of degree 2 in C_3*

Proof. Suppose $h_O(G) = n-3$.Since $h_O(G) \leq n - \Delta(G)$ we have $\Delta(G) \leq 3$.
If $\Delta(G) = 1$,$G \equiv K_2$,a contradiction.
If $\Delta(G) = 2$,G is eiher a cycle or a path.But open hub number of the path P_n is $n-2$.Hence G is a cycle.
If $\Delta(G) = 3$
Case 1 Suppose G is a tree having l leaves.
Since the set of all non leaf vertices of a tree(which is not a star) is a minimum open hub set , the open hub number of tree is $n-l$,we have l=3.That is G is a tree having 3 leaves and $\Delta = 3$.
Therefore G must be isomorphic to a subdivision of $K_{1,3}$
.
Case 2 G is not a tree.
Then G contains cycles.If G contains two disjoint cycles $C_l = (u_1, u_2, ...u_l, u_1)$ and $C_k = (v_1, v_2, ...v_k, v_1)$.Let H_1 and H_2 are minimum open hubsets of C_l and C_k respectively such that $|H_1| = |V(C_l)| - 3$ and $|H_2| = |V(C_k)| - 3$.Then $H = H_1 \cup H_2 \cup T$ where $T = V(G) - (V(C_l) \cup V(C_k))$ is an open hubset of G,a contradiction to $h_O(G) = n-3$.Also since $\Delta = 3$,no cycles have only single vertex in common.Hence any two cycles have common edge.Thus G is either unicyclic or exactly two cycles with a common edge.
Subcase I: G is unicyclilc.
Suppose G contains a cycle $C_k = (v_1, v_2, ...v_k, v_1)$.
Let $S = \{v \in V(C_k) | d(v) = 3\}$.Then $|S| \leq 3$.
If $|S| = 0$ Then G is isomorphic to $C_k, k = n$.
If $|S| = 1$,then G is isomorphic to the graph C_k with a path attached to one vertex .
If $|S| = 2$,then G is isomorphic to C_3 with two paths attached.
If $|S| = 3$,G is isomorphic to C_3 with 3 paths attached.
Subcase II: G is not Unicyclic
Then G contains exactly 2 cycles and atleast one cycle should be C_3.In this case if G has no pendent vertices ,then it is isomorphic to a graph with 2 cycles C_3 and $C_k, k \geq 3$ with one common edge.If G has pendent vertices then it is isomorphic to a graph with 2 cyles C_3 and $C_k, k \geq 4$ with one edge common and a path attached to vertex of degree 2 in C_3
Converse is trivial. □

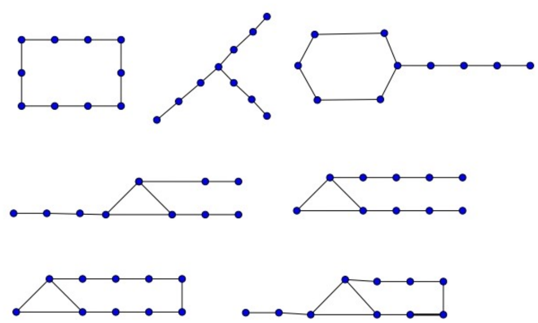

Figure 1. Class of graphs in Theorem 2.5

Theorem 2.6. *Given two integers k and n with $2 \leq k \leq [\frac{n}{2}]$, there exist a connected graph G of order n with $h_O(G) = k$.*

Proof. Let K_{n-k} be the complete graph with $V(K_{n-k}) = \{v_1, v_2, ..., v_{n-k}\}$. Let G be the graph obtained from K_{n-k} by adding k new vertices $u_1, u_2, ...u_k$ and k new edges $u_iv_i, 1 \leq i \leq k$.Then G is a connected graph of order n. The domination number $\gamma(G) = k$.Hence $h(G) \geq k-1$. Let H be a hub set of G. Then either $v_j \in H\ \forall j \quad 1 \leq j \leq k$ or $u_j \in H\ \forall j, 1 \leq j \leq k$. Therefore $h(G) = k$ and $\{v_1, v_2, ...v_k\}$ is a minimum hub set of G and the induced subgraph$< H >$ is the path graph $v_1v_2...v_k$ and hence it is an open hub set.Hence the result. □

Theorem 2.7. *Given two integers k and n with* $2 \leq k \leq n-2$,*there exist a connected graph G of order n with* $h_O(G) = k$.

Proof. Let K_{n-k} be the complete graph with $V(K_{n-k}) = \{v_1, v_2, ..., v_{n-k}\}$.Let P_{k+1} be the path $w_1w_2...w_{k+1}$. Let G be the graph obtaned by identifying the vertices v_1 and w_1.We claim that $h(G) = k$. It follows from Theorem1.5 $\gamma_c(G) = k$ and by Theorem 1.6 $h(G) = k$ or $k-1$.Now suppose $h(G) = k-1$. Let H be a hubset of G with cardinality $k-1$.If both v_1 and w_{k+1} are not in H,then $w_i \in H, \forall i,\ 2 \leq i \leq k$. Since cardinality of H is $k-1$,we have $H = \{w_2, w_3, ..., w_k\}$,a contradiction to H is a hub set of G.Suppose $v_1 \in S$ and $w_{k+1} \notin S$.Then if $w_2 \notin H$ then $w_i \in H, \forall i, 3 \leq i \leq k$.In this case $H = \{v_1, w_3, ...w_k\}$,which is not a hub set,again a contradiction.We must have $w_2 \in H$.By similar argument we have $w_i \in H, \forall i, 3 \leq i \leq k$,a contradiction.We get a similar contradiction if $v_1 \notin H$ and $w_{k+1} \in H$ or if both v_1 and $w_{k+1} \in H$.Thus $h(G) \neq k-1$.Hence $h(G) = k$.Hence $h_O(G) \geq k$.
Now $S = \{v_1, w_2, ...w_k\}$ is a minimum hub set of G and the induced subgraph $< S >$ is the path graph $v_1w_2w_3...w_k$. and hence has no isolated vertices.Hence S is an open hub set of G so that $h_O(G) \leq k$.Hence the result. □

Corollary 2.8. *For each positive integer n,there exist a connected graph G of order n such that* $h_O(G) = n - \Delta(G)$.

Proof. The graph G constructed in the above theorem has $\Delta(G) = n-k$ and $h_O(G) = k$. □

Definition 2.9. *Let* $G_1, G_2, ..., G_r$ *be connected graphs,then the graph G obtained by joining each vertex of* G_i *with each vertex of* G_{i+1}, $\quad 1 \leq i \leq r-1$ *,is called the successive join of* $G_1, G_2, ..., G_r$ *and is denoted by* $G_1 + G_2 + ... + G_r$.

Theorem 2.10. *Let* $G_1, G_2, ..., G_r$ *be connected graphs and let* $G = G_1 + G_2 + ... + G_r$,*then*

$$h_O(G) = \begin{cases} 2 & \text{if } r = 2,\ 3 \\ r-2 & \text{if } r \geq 4 \end{cases}$$

Proof. Case I $\quad r = 2, 3$
Here $H = \{u, v\}$ where $u \in V(G_1)$ and $v \in V(G_2)$, is an open hub set of G.
Case II $\quad r \geq 4$
In this case $H = \{v_2, v_3, ..., v_{r-1}\}$,where $v_i \in V(G_i)$ for $2 \leq i \leq r-1$, is an open hub set of G.
Hence $h_O(G) \leq r-2$.Now the diameter of G,$d(G) = r-1$ and hence by Lemma 1.2, $h_O(G) \geq r-2$.
Hence $h_O(G) = r-2$ □

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