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New three step derivative free iterative method for solving nonlinear equations

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Abstract
In this paper, we present a three step derivative free iterative method for solving nonlinear equations $f(x) = 0$. We discuss the convergence criteria of this new derivative free iterative method. A comparison with other existing methods is also given. The aim of this paper is to develop a new derivative free iterative method to find the approximation of the root α is nonlinear equations $f(x) = 0$, without the evaluation of the derivatives. This new method is based on Steffensen's method [11]. It is prove that the new method has cubic convergence. The benefit of this method is that it does not need to calculate any derivative. Numerical comparisons are made with other existing methods to show the better performance of the presented method.

Keywords
Nonlinear equations, convergence analysis, iterative methods, derivative free, three step.

AMS Subject Classification
65H05.

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Contents

1. Introduction

Solving nonlinear equations is one of the most important and challenging problem in scientific and engineering applications. In this paper, we consider an iterative method to solve nonlinear equation [2]

$$f(x) = 0. \tag{1.1}$$

Recently, some method have been proposed and analyzed for solving nonlinear equations. These method have been suggested by using quadrature formulas, decomposition and Taylor's series [10,14-18].

The well known Newton Raphson's method is largely used to solve nonlinear equation (1.1) and written as [11]

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}. \tag{1.2}$$

The Newton's method was modified by Steffensen's who replaced the first derivative $f'(x)$ in Newton's method by forward difference approximation [11,13]

$$f'(x) = \frac{f(x_n + f(x_n)) - f(x_n)}{f(x_n)} = P_0(x_n) \tag{1.3}$$

and obtained the famous Steffensen's method (SM)

$$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n + f(x_n)) - f(x_n)}. \tag{1.4}$$

Newton's method and Steffensen's method are of second order convergence.[11,13] We use Predictor – corrector methods, we shall now discuss the application of the explicit and implicit multistep method, for solution of the initial value problems. We use explicit (predictor) method for predicting a value and then use the implicit (corrector) method iteratively until the convergence is obtained. [1, 5-7]

(Computational order of convergence)

Let x^* be a root of the function $f(x)$ and suppose that x_{n+1}, x_n, x_{n-1} are three consecutive iterates closer to the root x^*. Then, the computational order of convergence ρ can be approximated using the following formula:[19]

$$\rho \approx \frac{ln|(x_{n+1}-x^*)/(x_n-x^*)|}{ln|(x_n-x^*)/(x_{n-1}-x^*)|} \tag{1.5}$$

2. Description of the method

A problem of great importance in applied mathematics and engineering is that of determining the roots of an equation of the form $f(x)=0$.

A significant part in developing our new iterative methods free from first derivative with respect to y. To be more precise, we now approximate $f'(y_n)$, to reduce the number of evaluations per iteration by a combination. Toward this end, an estimation of the function $P_1(t)$ is taken into consideration as follows: [3]

$$P_1(t)=a+b(t-x_n)+c(t-x_n)^2,$$
$$P_1'(t)=b+2c(t-x_n),$$

By substituting the known values, we have

$$P_1(y_n)=f(y_n)=a+b(y_n-x_n)+c(y_n-x_n)^2,$$
$$P_1'(y_n)=f'(y_n)=b+2c(y_n-x_n),$$
$$P_1(x_n)=f(x_n)=a, P_1'(x_n)=f'(x_n)=b,$$

we could easily obtain the unknown parameters. Thus we have

$$f'(y_n)=2(\frac{f(y_n)-f(x_n)}{(y_n-x_n)})-f'(x_n)=P_1(x_n,y_n), \tag{2.1}$$

$$f'(y_n)=2(\frac{f(y_n)-f(x_n)}{(y_n-x_n)})-\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)}=P_1(x_n,y_n),$$

$$f'(y_n)=2(\frac{f(y_n)-f(x_n)}{(y_n-x_n)})-\frac{f(x_n)}{(y_n-x_n)}=P_1(x_n,y_n), \tag{2.2}$$

Now using the trapezoidal rule and fundamental theorem of calculus, one can show that the function $f(x)$ can be approximated by the series [3].

$$f(x)=f(\gamma)+\frac{x-\gamma}{2}[f'(x)+f'(\gamma)] \tag{2.3}$$

Where $f'(x)$ is the differential of x.

Theorem 2.1

From equation (1.1) and equation (2.3), we can have

$$x_{(n+1)}=x_n-\frac{(2f(x_n))}{[f'(x_{(n+1)})+f'(x_n)]}$$
$$n=0,1,2,3,\dots.$$

Theorem 2.2

From equation (1.1) and equation (2.1), we can have

$$x_{(n+1)}=x_n-\frac{(f(x_n)(y_n-x_n))}{[f(y_n)-f(x_n)]}$$
$$n=0,1,2,3,\dots\dots\dots..$$

Theorem 2.3

From equation (1.3) and equation (2.2), we can have

$$x_{n+1}=x_n-\frac{(f(x_n))}{2\frac{(f(y_n)-f(x_n))}{(y_n-x_n)}-\frac{f(x_n)}{(y_n-x_n)}}$$
$$n=0,1,2,3,\dots\dots\dots..$$

Now using equations (1.4), theorem (2.2) and theorem (2.3) to suggest the following new derivative free iterative method for solving nonlinear equation, It is established that the following new method has convergence of order three, which will denote by THREE STEP DERIVATIVE FREE METHOD (TSDFM). Then it can be written in the following form

TSDFM

For a given initial choice x_0, compute approximate solution x_{n+1}, by the iterative schemes

$$a_n=x_n-\frac{[f(x_n)]^2}{f(x_n+f(x_n))-f(x_n)}$$

$$b_n=x_n-\frac{(f(x_n))}{2\frac{(f(a_n)-f(x_n))}{(a_n-x_n)}-\frac{f(x_n)}{(a_n-x_n)}}$$

$$x_{n+1}=x_n-\frac{(f(x_n)(b_n-x_n))}{[f(b_n)-f(x_n)]}$$
$$n=0,1,2,3,\dots.$$

We use a_n as a predictor, b_n and x_{n+1} as a corrector then we have the following three step derivative free method third order convergence.

3. Convergence Analysis

Let us now discuss the convergence analysis of the TSDFM method for numerical solution of nonlinear equations.

Theorem 3.1

let $\alpha \in I$ be a simple zero of sufficiently differential function $f: I \subseteq R \to R$ for an open interval I, if x_0 is sufficiently close to α then the three step derivative free method denoted by TSDFM of third order convergence.

Proof. Let α be a simple zero of f. Then by expanding $f(x_n)$ about α we have

$$f(x_n) = e_n c_1 + e_n^2 c_2 + e_n^3 c_3 + \ldots\ldots\ldots \quad (3.1)$$

Where $c_k = \frac{1}{k!} f^{(k)}(\alpha)$
$k = 1, 2, 3, \ldots\ldots\ldots$
and $e_n = x_n - \alpha$

$$[f(x_n)]^2 = c_1^2 e_n^2 + 2c_1 c_2 e_n^3 + c_2^2 e_n^4 + \ldots\ldots\ldots \quad (3.2)$$

$$f(x_n + f(x_n)) = c_1(1 + c_1)e_n + (3c_1 c_2 + c_1^2 c_2 + 2c_2^2) \quad (3.3)$$

$$(\times)e_n^2 + \ldots$$

Equation (3.1) and (3.3) yield,

$$f(x_n + f(x_n)) - f(x_n) = c_1^2 e_n + (3c_1 c_2 + c_1^2 c_2 + 2c_2^2)e_n^2 + \ldots \quad (3.4)$$

From (3.2) and (3.4), we have

$$\frac{[f(x_n)]^2}{(f(x_n + f(x_n)) - f(x_n))} = e_n - (\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2})e_n^2 + \ldots. \quad (3.5)$$

From (3.5), we have

$$a_n = \alpha + (\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2})e_n^2 \quad (3.6)$$

Let us set,A = a_n - α. Then the equation (3.6) can be re - written in the form

$$A = (\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2})e_n^2 \ldots\ldots\ldots \quad (3.7)$$

Now, expanding (a_n) about α and using (3.6), we have

$$f(a_n) = Ac_1 + A^2 c_2 + A^3 c_3 + \ldots \quad (3.8)$$

$$a_n - x_n = -e_n + (\frac{c_2}{c_1} + c_2 + 2\frac{c_2^2}{c_1^2})e_n^2 + \ldots \quad (3.9)$$

From (3.1) and (3.8), we have

$$f(a_n) - f(x_n) = -c_1 e_n + (c_1 c_2 + \frac{2c_2^2}{c_1})e_n^2 - c_3 e_n^3 + \ldots\ldots \quad (3.10)$$

From (3.9) and (3.10), we have

$$2[\frac{(f(a_n) - f(x_n))}{(a_n - x_n)}] = 2c_1 + 2c_2 e_n + 2(c_2(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}) + c_3)e_n^2 + \ldots \quad (3.11)$$

From (3.1) and (3.9), we have

$$\frac{(f(x_n))}{(a_n - x_n)} = -(c_1 + (2c_2 + c_1 c_2 + \frac{2c_2^2}{c_1})e_n + (c_1(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2})^2 + c_2(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}) + c_3)e_n^2 + \ldots\ldots\ldots \quad (3.12)$$

From (3.11) and (3.12), we have

$$2[\frac{(f(a_n) - f(x_n))}{(a_n - x_n)}] - \frac{(f(x_n))}{(a_n - x_n)} = 3c_1 + (4c_2 + c_1 c_2 + \frac{2c_2^2}{c_1})e_n + (c_1(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2})^2 + 3c_2(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}) + 3c_3)e_n^2 + \ldots \quad (3.13)$$

From (3.8) and (3.13), we have

$$\frac{f(a_n)}{2[\frac{(f(a_n) - f(x_n))}{(a_n - x_n)}] - \frac{(f(x_n))}{(a_n - x_n)}} = \frac{1}{3}(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2})e_n^2 - \frac{1}{9}(\frac{4c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2})(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2})e_n^3 + \ldots\ldots \quad (3.14)$$

From (3.14), we have

$$b_n = \alpha + \frac{2}{3}(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2})e_n^2 + \ldots\ldots \quad (3.15)$$

Let us set, $B = b_n - \alpha$. Then the equation (3.15) can be re - written in the form

$$B = \frac{2}{3}(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2})e_n^2 + \ldots\ldots\ldots \quad (3.16)$$

Now expanding $f(b_n)$ about α

$$f(b_n) = Bc_1 + B^2 c_2 + B^3 c_3 + \ldots\ldots \quad (3.17)$$

From (3.1) and (3.17), we have

$$f(b_n) - f(x_n) = -c_1 e_n + (\tfrac{2}{3}(\tfrac{c_2}{c_1} + c_2 + \tfrac{2c_2^2}{c_1^2}) - c_2)e_n^2 - c_3 e_n^3 + (\frac{2}{3}(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}) - c_4)e_n^4 + \ldots\ldots\ldots \quad (3.18)$$

$$b_n - x_n = -e_n + (\frac{2}{3}(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}) - c_2)e_n^2 + \ldots\ldots \quad (3.19)$$

From (3.1) and (3.19), we have

$$f(x_n)(b_n - x_n) = -c_1 e_n^2 + (\tfrac{2}{3}c_1 c_2 - \tfrac{c_2}{3} + \tfrac{4}{3}\tfrac{c_2^2}{c_1})e_n^3 + (\frac{2}{3}c_2(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}) - c_3)e_n^4 + \ldots\ldots\ldots \quad (3.20)$$

From (3.18) and (3.20), we have

$$\frac{f(x_n)(b_n - x_n)}{f(b_n) - f(x_n)} = e_n - (\frac{2}{3}\frac{c_2}{c_1}(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}) - \frac{c_3}{c_1})e_n^3 + \ldots$$

(3.21)

Now

$$x_{n+1} = \alpha + (\frac{2}{3}\frac{c_2}{c_1}(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}) - \frac{c_3}{c_1})e_n^3 + O(e_n^4), \tag{3.22}$$

$$e_{n+1} = (\frac{2}{3}\frac{c_2}{c_1}(\frac{c_2}{c_1} + c_2 + \frac{2c_2^2}{c_1^2}) - \frac{c_3}{c_1})e_n^3 + O(e_n^4), \tag{3.23}$$

This shows that TSDFM of third order convergence. □

4. Numerical Examples

We present some example to illustrate the roots of nonlinear equations by the newly developed three step derivative free iterative method. All computation are performed using MATLAB. The following examples are used for numerical testing Table 1.

In Table 2. For comparisons, we have used Jain method (JM) [9], Jarratt method (JaM) [4], Ostrowski's method (OM) [4, 18], Householder iterative method (HHM) [4, 8], Improvement of Super–Halley method (ISHM) [12] and the newly developed Three Step Derivative Free Method (TSDFM). The methods are compared in the terms of number of iteration (IT), the total number of function evaluation (NFE) [19], the computational order of convergence (COC) [19] and the absolute values of the function $|f(x)|$.

Table 1. Test function and their roots:

$Functions$	$Roots$
$f_1(x) = e^x - 3x$	0.619061286735945
$f_2(x) = cos(x) - x$	0.739085633215161
$f_3(x) = xtan(x) + 1$	2.79838604578389
$f_4(x) = sin(x) - 1 + x$	0.510973429388569
$f_5(x) = 2sin(x) - x$	1.89549426703398
$f_6(x) = x^2 - 9$	3
$f_7(x) = cos(x) - xe^x$	0.51775736382458
$f_8(x) = x^2 + 4sin(x)$	−1.93375376282702
$f_9(x) = cos(x) - \sqrt{x} + 1$	1.39058983057821
$f_{10}(x) = xtan(x) - 1.28$	−3.49285716965564

5. Conclusion and Discussion

In this paper, new method tested for almost all types of nonlinear equations. Table 2 shows that the newly developed three step derivative free iterative method of third order convergence is comparable with the existing methods of this domain in terms of number of iterations and number of functions evaluations per iteration. In almost all examples the newly developed methods perform better than the existing method and the comparable methods diverges for the functions but the newly method (TSDFM) converges for all functions.

Table 2. Comparison of various iterative methods

$method$	JM	JaM	OM	HHM	$ISHM$	$TSDFM$
$f_1, x_0 = -10$						
IT	7	4	4	5	4	4
nfe	21	12	12	15	16	16
COC	3.50	3.55	3.81	2.15	2.65	3.45
$\|f(x_n)\|$	$2.2204e^{-016}$	$6.0267e^{-009}$	$2.1303e^{012}$	$1.9631e^{011}$	$2.6225e^{-011}$	$3.7050e^{-006}$
$f_2, x_0 = -2$						
IT	4	6	11	–	–	4
nfe	12	18	33	–	–	16
COC	2.92	3.67	4.45	–	–	2.82
$\|f(x_n)\|$	$1.8491e^{-006}$	$2.4439e^{-012}$	$2.5421e^{007}$	D	D	$1.9473e^{-007}$
$f_3, x_0 = 1.5$						
IT	4	–	–	–	–	4
nfe	12	–	–	–	–	16
COC	5.12	–	–	–	–	4.52
$\|f(x_n)\|$	$2.4144e^{-004}$	D	D	D	D	$9.1385e^{-008}$
$f_4, x_0 = 3$						
IT	5	–	–	–	–	4
nfe	15	–	–	–	–	16
COC	3.04	–	–	–	–	3.33
$\|f(x_n)\|$	$7.3275e^{-015}$	D	D	D	D	$2.9618e^{-007}$
$f_5, x_0 = 0.9$						
IT	4	–	–	–	–	4
nfe	12	–	–	–	–	16
COC	2.74	–	–	–	–	2.94
$\|f(x_n)\|$	$7.5403e^{-007}$	D	D	D	D	$2.2032e^{-009}$
$f_6, x_0 = 0$						
IT	7	–	–	–	–	5
nfe	21	–	–	–	–	20
COC	2.96	–	–	–	–	3.10
$\|f(x_n)\|$	$1.8212e^{-008}$	D	D	D	D	$4.0940e^{-007}$
$f_7, x_0 = 0.5$						
IT	7	–	–	–	–	4
nfe	21	–	–	–	–	16
COC	3.00	–	–	–	–	3.02
$\|f(x_n)\|$	$1.0136e^{-013}$	D	D	D	D	$6.8392e^{-011}$
$f_8, x_0 = -1$						
IT	5	–	–	13	–	6
nfe	15	–	–	39	–	24
COC	3.10	–	–	3.12	–	2.98
$\|f(x_n)\|$	$2.4012e^{-007}$	D	D	$2.5111e^{007}$	D	$3.3213e^{-011}$
$f_9, x_0 = 5$						
IT	–	–	–	–	6	4
nfe	–	–	–		24	16
COC	–	–	–		3.81	3.47
$\|f(x_n)\|$	D	D	D	D	$5.2464e^{010}$	$3.7857e^{-006}$
$f_{10}, x_0 = -2$						
IT	4	–	–	4	–	4
nfe	12	–	–	12	–	16
COC	2.97	–	–	2.89	–	2.99
$\|f(x_n)\|$	$7.7107e^{-007}$	D	D	$9.6185e^{012}$	D	$7.7116e^{-010}$

$^*D = divergence$

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