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# The majestic edge coloring and the majestic 2-tone edge coloring of some cycle related graphs

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**Abstract**

An edge coloring of a graph G is called a majestic edge coloring if there is the induced proper vertex coloring. If the colors of edges of the G graph have 2-elements sets and the G graph has induced proper vertex coloring then an edge coloring of the G graph is called the majestic 2-tone coloring. The majestic and the majestic 2-tone chromatic indices for some cycle related graphs which are wheel graph, gear graph, helm graph, web graph, and friendship graph are computed.

**Keywords**

Majestic edge coloring, majestic 2-tone edge coloring, cycle related graphs, graph coloring.

**AMS Subject Classification**

05C15, 05C75.

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## Contents



## 1. Introduction

A graph $G$ is a finite nonempty vertex set $V(G)$ together with a edge The coloring problem was propounded by Francis Gutri in 1852 [3]. The coloring of graphs is of great interest in graph theory. Graph coloring is used in the solution of many planning problems. Graph coloring has also applications in many fields such as the industry, industry network ,and security. Various coloring techniques have been proposed for graph coloring. A variety of edge colorings based on vertex colorings was introduced. These edge colorings led to vertex colorings which are defined in terms of sets and multisets of the colors of the edges (see [12], [11]). One of these colorings is the majestic edge coloring.

The majestic edge colorings were defined by the motivation of set irregular edge coloring and adjacent strong edge coloring. The majestic edge colorings were also studied as a general neighbour-distinguishing index which was introduced by E. Gyori, M. Hornak, C. Palmer, and M. Woznick in 2008 [8], [1]. This coloring was examined by I. Hart as the majestic edge coloring in his thesis. In this study, the notations in Hart's thesis will be used.

A proper coloring of $G$ is the assignment of the element $[k] = \{1,...,k\}$, called color, to each the vertex of $V(G)$. Here, two adjacent vertices of $V(G)$ are assigned different colors. If the set of colors of the edges incident to $u$ for any two different vertices $u$ and $v$ in $G$ is different from the set of colors of the edges incident to $v$, it is called the proper edge coloring of the $G$ graph.

For a connected graph $G$ with order 3 or more, let $c : E(G) \rightarrow [k]$ for some positive integer $k$ be an edge coloring of $G$ where adjacent edges may be colored the same. Then the edge coloring $c$ gives rise to a vertex coloring $c'$ of $G$ that is the union of the sets of colors of the edges incident to $v$. Let $\wp^*([k])$ be nonempty subsets of the power set of $[k]$. An edge coloring $c$ of a graph $G$ is called a majestic $k$-edge coloring or majestic edge coloring if there is the induced proper vertex coloring $c' : V(G) \rightarrow \wp^*([k])$ which is $c'(u) \neq c'(v)$ for every pair $u, v$ of adjacent vertices of $G$. The minimum number of the nonempty subsets of $[k]$ for which a graph $G$ has a majestic $k$-edge coloring is the majestic chromatic index of $G$ which is denoted by $maj(G)$ [9], [4], [12]. The majestic edge coloring was introduced and studied in [9] and [4].

Let $[k]_t$ denote the set of $t$-element subsets of $[k]$ for positive integer $t$ with $t < k$. For a connected graph $G$, let $c : E(G) \to [k]_t$ be an edge coloring of $G$ where adjacent edges may be colored the same. Then the edge coloring $c$ gives rise to a vertex coloring $c'$ of $G$ that is the union of the sets of colors of the edges incident to $v$. An edge coloring $c$ of a graph $G$ is called a majestic $t$-tone $k$-edge coloring if there is the induced proper vertex coloring $c'$. The majestic $t$-tone chromatic index is the minimum number of the nonempty subsets of $[k]_t$ and is denoted by $maj_t(G)$ [9].

In this study, the majestic coloring which is a proper vertex coloring of a graph that is induced by an unrestricted edge coloring of the graph is studied. The exact expressions are presented for the majestic 2-tone chromatic indices and the majestic chromatic indices of some cycle related graphs which are wheel graph, gear graph, helm graph, friendship graph, and web graph.

## 2. Preliminaries

Let $G$ be a simple connected graph with a vertex set $V(G)$ and edge set $E(G)$ where $V(G) = \{v_1, v_2, .., v_n\}$. The number of a vertex set and an edge set are defined by $n$ and $m$, respectively. For standard terminology and notations, we follow Buckley and Harary [5].

I. Hart presented the following results [9]:

**Theorem 2.1.** *Let $K_n$ be complete graph. Then, $maj(K_n) = \lceil \log_2 n \rceil + 1$.*

**Theorem 2.2.** *Let $P_n$ be path graph of order 3 or more. Then,*

**i.**

$$maj(P_n) = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 3 & \text{if } n \text{ is even} \end{cases}$$

and

**ii.**

$$maj_2(P_n) = \begin{cases} 3 & \text{if } n \text{ is odd} \\ 4 & \text{if } n \text{ is even} \end{cases}.$$

**Theorem 2.3.** *Let $C_n$ be cycle graph with $n \geq 3$. Then,*

**i.**

$$maj(C_n) = \begin{cases} 2 & \text{if } n \equiv 0 \pmod 4 \\ 3 & \text{if } n \neq 0 \pmod 4 \end{cases}.$$

and

**ii.**

$$maj_2(C_n) = \begin{cases} 3 & \text{if } n \text{ is even} \\ 4 & \text{if } n \text{ is odd} \end{cases}.$$

**Theorem 2.4.** *Let $W_n$ be wheel graph with $n \geq 3$. Then,*

$$maj_2(W_n) = 4.$$

**Theorem 2.5.** *If $G$ is a bipartite graph of order 3 or more, then $maj(G) \leq 3$.*

**Theorem 2.6.** *If $G$ is a connected graph of order 3 or more and $t \geq 2$, then $t+1 \leq maj_t(G) \leq maj(G) + (t-1)$.*

**Theorem 2.7.** *If $G$ is a connected graph with $maj(G) = 2$ and $n \geq 3, t \geq 2$, then $maj_t(G) = t+1$.*

## 3. The Majestic Edge Coloring of Some Cycle Related Graph

In this section, the majestic chromatic indexes of the cycle-related graphs are given. The graphs related to cycle graphs are wheel graph, gear graph, helm graph, friendship graph, and web graph.

**Definition 3.1.** *Wheel $W_n$ for $n \geq 3$ is obtained by joining $n$-cycle and central vertex $v_c$. The wheel graph has $n+1$ vertices and $2n$ edges. The wheel graph consists of a vertex set*

$$V(W_n) = V(C_n) \cup \{v_c\},$$

*where $v_c$ vertex is the center vertex of the wheel graph, $V(C_n)$ vertex set is vertices of the outer cycle of the wheel graph.*

**Theorem 3.2.** *Let $W_n$ be the wheel graph of order $n$. Then*

$$maj(W_n) = 3.$$

*Proof.* Let $V(C_n) = \{v_1, v_2, ..., v_{n+1} = v_1\}$ and $v_c$ be center vertex of $W_n$. The $W_n$ graph has triangles. Let $maj(W_n) = k$.

Let $n$ be even. Suppose that $X = \{v_{2i} \in V(C_n), i = 1, ..., \frac{n}{2}\}$, $Y = \{v_{2i-1} \in V(C_n),\ i = 1, ..., \frac{n}{2}\}$ and $Z = \{v_c \in V(W_n)\}$. Hence, the colors of the vertices of the $W_n$ graph are assigned as follows $C_1 = \{c'(v_j) \,|\, v_j \in X\}$, $C_2 = \{c'(v_j) \,|\, v_j \in Y\}$ and $C_3 = \{c'(v_j) \,|\, v_j \in Z\}$. These color sets are color of the vertices of the $K_3$ graph. Hence, we have 3 color sets. That is $maj(W_n) = 3$.

Let $n$ be odd and $n = 2l+1$. Suppose that $X_1 = \{v_{2i} \in V(C_n), i = 1, ..., l\}$, $X_2 = \{v_{2i-1} \in V(C_n),\ i = 1, ..., l\}, X_3 = \{v_n \in V(C_n)\}$ and $X_4 = \{v_c \in V(W_n)\}$. Hence, colors of the vertices of the $W_n$ graph are assigned as follows $C_1 = \{c'(v_j) \,|\, v_j \in X_1\}, C_2 = \{c'(v_j) \,|\, v_j \in X_2\}, C_3 = \{c'(v_j) \,|\, v_j \in X_3\}$ and $C_4 = \{c'(v_j) \,|\, v_j \in X_4\}$. In this case, we need 4 color sets. Then, we obtain $k = 3$ since $2^k - 1 \geq 4$. ☐

**Definition 3.3.** *Gear graph, $G_n$, is a wheel graph with a vertex added between each pair adjacent vertices of the outer cycle [7]. The gear graph has $2n+1$ vertices and $3n$ edges. Obviously,*

$$V(G_n) = V_1 \cup V_2 \cup V_3$$

*where*

$$V_1 = \{v_i \in V(G_n),\ i = \overline{1,n}\},$$

$$V_2 = \{u_i \in V(G_n),\ i = \overline{1,n}\},$$

$$V_3 = \{v_c \in V(G_n)\},$$

*where $v_c$ vertex is the center vertex of the gear graph, $V_1$ vertex set is vertices of the outer cycle of the wheel graph and $V_2$ is set of added vertices to the outer cycle.*

**Theorem 3.4.** *The majestic chromatic index of the $G_n$ gear graph is* 2.

*Proof.* The gear graph is a bipartite graph. From Theorem 2.5, we can write $maj(G_n) \le 3$. Suppose that $maj(G_n) = 2$. Let $X = \{c'(v_j), v_j \in V_1\}$, $Y = \{c'(v_j), v_j \in V_2\}$, $Z = \{c'(v_j), v_j \in V_3\}$. Clearly, $X \cap Y = \oslash$, $X \cap Z = \oslash$, $Y \cap Z = \oslash$ . Since $\{1\} \cap \{2\} = \oslash$, $c^{'}(u)$ or $c^{'}(v)$ is equal to $\{1,2\}$ for any $uv \in E(G_n)$.

If $X$ is equal to $\{1,2\}$ then $Y$ or $Z$ is equal to $\{1\}$ or $\{2\}$. If $X$ is equal to $\{1\}$ and $\{2\}$, then $Y$ and $Z$ are equal to $\{1,2\}$. Then, we obtain $maj(G_n) = 2$. An example of the majestic coloring of the gear graph of order 6 is given in Figure 1.

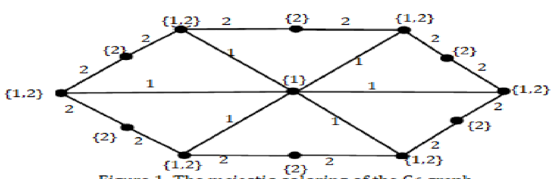


Figure 1. The majestic coloring of the $G_6$ graph

□

**Definition 3.5.** *Helm graph $H_n$, is obtained from a wheel $W_n$ with cycle $C_n$ having a pendant edge attached to each vertex of cycle [7]. The helm graph has $2n+1$ vertices and $3n$ edges. Obviously,*

$$V(H_n) = V_1 \cup V_2 \cup V_3$$

*where*

$$V_1 = \{v_i \in V(H_n),\ i = \overline{1,n}\},$$

$$V_2 = \{u_i \in V(H_n),\ i = \overline{1,n}\},$$

$$V_3 = \{v_c \in V(H_n)\},$$

*where $v_c$ vertex is the center vertex of the helm graph, $V_1$ vertex set is vertices of the outer cycle of the wheel graph and $V_2$ is set vertices of a pendant edge attached to each vertex of cycle.*

**Theorem 3.6.** *One has*

$$maj(H_n) = 3.$$

*Proof.* The $c'(v_i)$ for $v_i \in V_2$ must 1-element sets. Thus, the vertices of the $C_n$ graph in the $W_n$ graph are assigned at least 2-element sets. The vertices of the $W_n$ graph in the $H_n$ graph are assigned by Theorem 3.2. When the majestic chromatic index of the $H_n$ graph is 3, the 4 sets have at least 2-element sets. Hence, the majestic chromatic index of the $H_n$ graph is 3 since the $W_n$ graph in the $H_n$ graph is assigned with maximum of 4 sets. □

**Definition 3.7.** *Friendship graph $F_n$, is obtained from a wheel $W_{2n}$ with cycle $C_{2n}$ by deleting the alternate edges of the cycle [7]. The friendship graph has $2n+1$ vertices and $3n$ edges. The friendship graph consists of*

$$V(F_n) = V_1 \cup V_2 \cup V_3,$$

*where*

$$V_1 = \{v_i \in V(F_n),\ i = \overline{1,n}\},$$

$$V_2 = \{u_i \in V(F_n),\ i = \overline{1,n}\},$$

$$V_3 = \{v_c \in V(F_n)\},$$

*where $v_c$ vertex is the center vertex of the friendship graph.*

**Theorem 3.8.** *The majestic chromatic index of friendship graph $F_n$ is* 3.

*Proof.* The friendship graph has triangles. The majestic chromatic index of the $K_3$ graph is 3. Thus, we obtain that this coloring is 3-majestic edge coloring from the definition of friendship graph. □

**Definition 3.9.** *Web graph, $Web_n$ is known as stacked prism graph. $Y_{n,m} = C_n \times P_m$ , which is obtained by cartesian product of $C_n$ and $P_m$. The web graph consists of*

$$V(Web_n) = V_1 \cup V_2 \cup V_3,$$

*where $V_1$ is set of vertices of the inner cycle of the web graph, $V_2$ is set of vertices of the outer cycle of the web graph, $V_3$ is set of pendant vertices added to the outer cycle of the web graph and*

$$V_1 = \{v_i \in V(Web_n)\ i = \overline{1,n}\},$$

$$V_2 = \{u_i \in V(Web_n)\ i = \overline{1,n}\},$$

$$V_3 = \{x_i \in V(Web_n)\ i = \overline{1,n}\}.$$

**Theorem 3.10.** *Let $Web_n$ be the web graph with $n > 3$. Then*

$$maj(Web_n) = 3.$$

*Proof.* Since the pendant vertices attached to the vertices of the outer cycle of the $Web_n$ graph are assigned with 1-element colors, the color of each vertex of the outer cycle of $Web_n$ is at least 2-element color sets. So, the majestic edge chromatic index of $Web_n$ is not 2. Assume that $maj(Web_n) = 3$. The $C_n$ graphs of the $Web_n$ graph are assigned with maximum 3 colors by Theorem 2.3 (i). Hence, the proof is completed. □

**Definition 3.11.** *The graphs $(C_n \times P_2) + K_1$ are like double wheel graphs, but the vertices of the two wheels are joined pair-wise. They could alternatively be thought of like a prism $C_n \times P_2$, with every vertex joined to a common point [6].*

Let $V(C_n) = \{v_1, ..., v_n, v_{n+1} = v_1\}$ be the vertices of the $C_n$ graph, $V(P_2) = \{x_1, x_2\}$ be the vertices of the $P_2$ graph and $V(K_1) = \{v_0\}$ be the vertex of the $K_1$ graph. We can partition the vertices of the $(C_n \times P_2) + K_1$ as follows. $V_1 = \{y_j = v_j x_1 : v_j x_1 \in V(C_n) \times V(P_2), j = 1, ..., n\}$, $V_2 = \{u_i = v_i x_2 : v_i x_2 \in V(C_n) \times V(P_2), i = 1, ..., n\}$, $V_3 = \{v_0 : v_0 \in V(K_1)\}$, where $y_j$ is adjacent to $u_i$ if $v_j = v_i$.

**Theorem 3.12.** *Let $G$ be the $(C_n \times P_2) + K_1$ graph of $2n+1$. Then, $maj(G) = 3$.*

*Proof.* From the Definition 3.11, the $v_0$ vertex is the central vertex of the two-wheel graphs. We can color of $(C_n \times P_2) + K_1$ as follows. These wheel graphs are assigned such that $c^{'}(y_j) = c^{'}(u_{i+1})$ for $1 \leq i, j \leq n$. By Theorem 3.2, we obtain that $maj(G)$ is equal to 3. □

## 4. 2-Tone Majestic Coloring of Some Cycle Related Graph

In this section, the 2-tone majestic coloring is studied. The 2-tone majestic chromatic indices of gear graph, helm graph, friendship graph, and web graph are computed.

**Theorem 4.1.** *The 2-tone majestic chromatic index of the $G_n$ gear graph is 3.*

*Proof.* By using Theorem 3.4 and Theorem 2.7, this proof is completed.
An example of the 2-tone majestic coloring of the gear graph of order 6 is given in Figure 2.

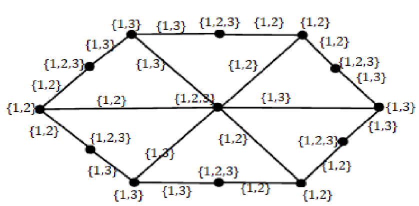

Figure 2. The 2-tone majestic colorings of the G6 graph

□

**Theorem 4.2.** *The 2-tone majestic chromatic index of the $H_n$ helm graph is 4.*

*Proof.* From Theorem 2.6 and Theorem 3.6, it can be said that

$$3 \leq maj_2(H_n) \leq 4. \tag{4.1}$$

Assume that $maj_2(H_n) = 3$. Since colors of the pendant vertices attached to the vertices of the outer cycle are assigned by 2-elements sets from the definition the 2-tone majestic coloring, the colors of the vertices of the outer cycle in the helm graph are at least 3-elements sets. When the 2-tone majestic chromatic index is 3, the number of 3-elements sets is 1. So, $maj_2(H_n)$ is not 3 and $maj_2(H_n)$ is equal to 4 from the equation (4.1). □

**Theorem 4.3.** *Let $F_n$ be the friendship graph of order $n$. Then*

$$maj_2(F_n) = 4.$$

*Proof.* From Theorem 2.6 and Theorem 3.8, we have

$$3 \leq maj_2(F_n) \leq 4. \tag{4.2}$$

We can partiton the colors sets of the vertices as follows: $X = \{c'(v_i) : v_i \in V_1\}, Y = \{c'(v_i) : v_i \in V_2\}$ and $Z = \{c'(v_c) : v_c \in V_3\}$. We need 3 color sets. Assume that $maj_2(F_n) = 3$. If the colors of edge are {1,2},{1,3}, {2,3} then the colors of two of $X, Y, Z$ sets are {1,2,3}. Hence $maj_2(F_n)$ is not equal to 3. We obtain that $maj_2(F_n)$ is equal to 4 from the equation 4.2. □

**Theorem 4.4.** *Let $Web_n$ be the web graph with $n > 3$. Then*

$$maj_2(Web_n) = 4.$$

*Proof.* From Theorem 2.6 and Theorem 3.10, we have

$$3 \leq maj_2(Web_n) \leq 4. \tag{4.3}$$

The colors of the pendant vertices are 2-element sets. Thus, the vertices of the inner cycle the web graph must be at least 3-element sets. Assume that $maj_2(Web_n) = 3$. Since the color of each vertex of the $Web_n$ graph is $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, $\{1,2,3\}$, the majestic chromatic index of $Web_n$ can not be equal to 3. Hence, $maj_2(Web_n)$ is equal to 4 from the equation 4.3. □

**Theorem 4.5.** *Let $G$ be the $(C_n \times P_2) + K_1$ graph of $2n+1$. Then, $maj_2(G) = 4$.*

*Proof.* We can colors of $(C_n \times P_2) + K_1$ as the proof of Theorem 3.12. These wheel graphs are assigned such that $c^{'}(y_j) = c^{'}(u_{i+1})$ for $1 \leq i, j \leq n$. By Theorem 2.4, we obtain that $maj_2(G)$ is equal to 4. □

## 5. Conclusion

The cycle related graphs are wheel, helm, gear, friendship, and web graph. These graphs are especially used in the network [2], [6], [10]. Graph coloring is important to strengthen a network or to find faulty computers.

In this paper, the majestic edge coloring which is an edge coloring based on vertex colorings was studied. If the following conditions are provided, this coloring is called a majestic edge coloring:

i. The color of a vertex in the $G$ graph is the union of the sets of colors of the edges incident to a vertex in the $G$ graph ii.No two adjacent vertices receives the same color,

iii. Edges of the incident of a vertex can receive the same color.

If the rules of majestic edge coloring and 2-tone coloring are provided, this coloring is called majestic 2-tone coloring.

The formulas for the 2-tone majestic chromatic indices and the majestic chromatic indexes of cycle-related graphs were given.

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